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Minimal skew energy of oriented unicyclic graphs with fixed diameter

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Abstract

Let $S(G^\sigma)$ be the skew adjacency matrix of the oriented graph G^σ, which is obtained from a simple undirected graph G by assigning an orientation σ to each of its edges. The skew energy of an oriented graph G^σ is defined as the sum of all singular values of $S(G^\sigma)$. For any positive integer d with $3 \le d \le n-2$, we in this paper, determine the graph with minimal skew energy among all oriented unicyclic graphs on n vertices with fixed diameter d.

MSC: 05C50; 15A18

Keywords: oriented graph; unicyclic graph; skew energy; diameter

1 Introduction

Research on the energy of a matrix in terms of a related graph can be traced back to 1970s [1] when Gutman investigated the energy with respect to the adjacency matrix of an (undirected) graph, which has a still older chemical origin; see [2]. Then much attention has been devoted to the energy of the adjacency matrix of a graph; see [1, 3–8], and the references cited therein. Recently, in analogy to the energy of the adjacency matrix, a few other versions of graph energy were introduced in the mathematical literature, such as Laplacian energy [9], signless Laplacian energy [10] and skew energy [11].

Let G be a simple undirected graph with an orientation σ, which assigns to each edge a direction so that G^σ becomes an *oriented graph*. Then G is usually called the *underlying graph* of G^σ. The *skew-adjacency matrix* associated to the oriented graph G^σ with vertex set $\{1,2,\dots,n\}$ is defined as the $n \times n$ matrix $S(G^\sigma) = [s_{ij}]$ whose (i,j)th entry satisfies:

$$s_{ij} = \begin{cases} 1 & \text{if there is an arc with head } i \text{ and tail } j; \\ -1 & \text{if there is an arc with head } j \text{ and tail } i; \\ 0 & \text{otherwise.} \end{cases}$$

Then $S(G^\sigma)$ is a skew-symmetric matrix, and thus the eigenvalues of $S(G^\sigma)$ are all purely imaginary numbers.

In [11], Adiga *et al.* introduced the concept the *skew energy* of an oriented graph G^σ, denoted by $\mathcal{E}(G^\sigma)$, which is defined as

$$\mathcal{E}_s(G^\sigma) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ are all eigenvalues of the skew adjacency matrix $S(G^\sigma)$. Adiga *et al.* [11] showed that the skew energy of an oriented tree is independent of its orientation, which is equal to the energy of its underlying tree. Moreover, Adiga *et al.* [11] investigated the skew energies of oriented cycles by computing. Then Hou *et al.* [12] determined the oriented unicyclic graphs with minimal and maximal skew energy, respectively; Gong *et al.* determined all oriented graphs with minimal skew energy among all connected oriented graphs on n vertices with m $(n-1 \leq m \leq 2(n-2))$ edges [13], and all 3-regular connected oriented graphs with optimal skew energies [14].

In this paper, we continue to investigate the skew energy of oriented unicyclic graphs. Below, we focus on the graphs with order at least 5, since the skew energy of an oriented graph with a small order can be calculated directly by mathematical software such as Matlab 7.0. As we know that $2 \leq d(G^\sigma) \leq n-2$ for any oriented unicyclic graph G^σ with order n $(n \geq 6)$. If $d(G^\sigma) = 2$, then, up to isomorphism, G^σ must be the graph obtained from the oriented star S_n by adding one arc between arbitrary two pendent vertices of it. Therefore, we in the following always assume that $3 \leq d \leq n-2$.

The rest of this paper is organized as follows: In Section 2, we give some notation and preliminary results, which will be used in the following discussion. The graph with minimal skew energy among all oriented unicyclic graphs on n (≥ 6) vertices with diameter d $(3 \leq d \leq n-2)$ will be determined in Section 3.

2 Preliminary results

Let $G = (V(G), E(G))$ be a simple graph. Denote by $G - e$ the graph obtained from G by deleting the edge e and by $G - v$ the graph obtained from G by deleting the vertex v together with all edges incident to it and by $d(G)$ the diameter of G, which is defined as the greatest distance between any two vertices in G. The *union* of the graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, denoted by $G_1 \cup G_2$, is the graph with vertex-set $V(G_1) \cup V(G_2)$ and edge-set $E(G_1) \cup E(G_2)$. An *r-matching* of G is a subset with r edges such that every vertex of $V(G)$ is incident with at most one edge in it. Denote by $m(G, r)$ the number of r-matchings contained in G. We refer to Cvetković *et al.* [15] for more terminology and notation not defined here.

For convenience, in terms of defining subgraph, matchings, degree, diameter, *etc.*, of an oriented graph, we focus only on its underlying graph. Moreover, we will briefly use the notations C_n, S_n and P_n to denote the oriented cycle, the oriented star and the oriented path on n vertices, respectively, if no conflict exists there.

An even cycle C is called *oddly oriented* if for either choice of direction of traversing around C, the number of edges of C directed in the direction of traversal is odd. Since C is even, this is clearly independent of the initial choice of direction of traversal. Otherwise, such an even cycle C is called *evenly oriented*. (We here do not involve the parity of the cycle with length odd, the reason is it dependents on the initial choice of direction of traversal.)

An oriented graph H is called a 'basic oriented graph' if each component of H is K_2 or a cycle with length is even.

Denote by $\phi(G^\sigma,\lambda)$ the *skew characteristic polynomial* of the oriented graph G^σ, which is defined as

$$\phi(G^\sigma,\lambda) = \det(\lambda I_n - S(G^\sigma)) = \sum_{i=0}^{n} (-1)^i a_i(G^\sigma)\lambda^{n-i},$$

where I_n denotes the identity matrix of order n.

The following result is a cornerstone of our discussion below, which gives an interpretation of all coefficients of the skew characteristic polynomial of an oriented graph in terms of its basic oriented subgraphs.

Proposition 2.1 ([16, Corollary 2.4], [17, Theorem 2.3]) *Let G^σ be an oriented graph on n vertices with skew characteristic polynomial*

$$\phi(G^\sigma,\lambda) = \sum_{i=0}^{n} (-1)^i a_i \lambda^{n-i} = \lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + (-1)^{n-1}a_{n-1}\lambda + (-1)^n a_n.$$

Then $a_i = 0$ if i is odd; and

$$a_i = \sum_{\mathcal{H}} (-1)^{c^+(\mathcal{H})} 2^{c(\mathcal{H})} \quad \text{if } i \text{ is even},$$

where the summation is over all basic oriented graphs $\mathcal{H}$, of G^σ, having i vertices, and $c^+(\mathcal{H})$ and $c(\mathcal{H})$ are respectively the number of evenly oriented even cycles and even cycles contained in $\mathcal{H}$.

As an analogy to the Coulson integral formula for the energy of an undirected graph with respect to its adjacency matrix, Hou *et al.* [12] deduce an integral formula for the skew energy of an oriented graph in terms of the coefficients of its skew characteristic polynomial.

Lemma 2.2 [12, Theorem 2.6] *Let G^σ be an oriented graph with order n. Then*

$$\mathcal{E}_s(G^\sigma) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \lambda^{-2} \ln \psi(G^\sigma,\lambda)\, d\lambda, \tag{1}$$

where

$$\psi(G^\sigma,\lambda) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a_{2i}(G^\sigma)\lambda^{2i}$$

and $a_{2i}(G^\sigma)$ denotes the coefficient of λ^{n-2i} in $\phi(G^\sigma,\lambda)$.

From Lemma 2.2, for an oriented graph G^σ, the skew energy $\mathcal{E}_s(G^\sigma)$ is a strictly monotonically increasing function of the coefficients $a_{2k}(G^\sigma)$ ($k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$). Thus, similar to comparing the energies of two undirected graphs with respect to their adjacency matrices, we define the quasi-ordering relation '$\preceq$' for oriented graphs as follows.

Let $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ be two oriented graphs of order n. (G_1 is not necessary different from G_2.) If $a_{2i}(G_1^{\sigma_1}) \le a_{2i}(G_2^{\sigma_2})$ for all i with $0 \le i \le \lfloor \frac{n}{2} \rfloor$, then we write that $G_1^{\sigma_1} \preceq G_2^{\sigma_2}$.

Furthermore, if $G_1^{\sigma_1} \preceq G_2^{\sigma_2}$ and there exists at least one index i such that $a_{2i}(G_1^{\sigma_1}) < a_{2i}(G_2^{\sigma_2})$, then we write that $G_1^{\sigma_1} \prec G_2^{\sigma_2}$. If $a_{2i}(G_1^{\sigma_1}) = a_{2i}(G_2^{\sigma_2})$ for all i, we write that $G_1^{\sigma_1} \sim G_2^{\sigma_2}$. Note that there are non-isomorphic oriented graphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ with $G_1^{\sigma_1} \sim G_2^{\sigma_2}$, which implies that '$\preceq$' is not a partial order in general.

According to the integral formula (1), we have for two oriented graphs $G_1^{\sigma_1}$ and $G_2^{\sigma_2}$ of order n that

$$G_1^{\sigma_1} \preceq G_2^{\sigma_2} \quad \Longrightarrow \quad \mathcal{E}_s(G_1^{\sigma_1}) \leq \mathcal{E}_s(G_2^{\sigma_2})$$

and

$$G_1^{\sigma_1} \prec G_2^{\sigma_2} \quad \Longrightarrow \quad \mathcal{E}_s(G_1^{\sigma_1}) < \mathcal{E}_s(G_2^{\sigma_2}). \tag{2}$$

Proposition 2.1 also implies that to study the spectral properties, as well as the skew energy, of an oriented graph, we need only consider the orientations of those arcs lying on even cycles. Let G be a connected unicyclic graph whose unique cycle is even. Denote by G^+ and G^- the oriented graph with underlying graph G and the unique oriented cycle is evenly oriented and oddly oriented, respectively. Combining with Proposition 2.1 and Lemma 2.2, we have the following.

Theorem 2.3 *Let G be a connected unicyclic graph whose unique cycle C is even. Then*

$$G^- \succ G^+.$$

Proof Let C_l be the unique even cycle of G with length l. By Proposition 2.1, we have

$$a_{2i}(G^+) = m(G,i) - 2m\left(G - C_l, i - \frac{l}{2}\right);$$

$$a_{2i}(G^-) = m(G,i) + 2m\left(G - C_l, i - \frac{l}{2}\right).$$

Then $a_{2i}(G^+) \leq a_{2i}(G^-)$ and $a_l(G^+) < a_l(G^-)$. Thus, the result holds. □

For convenience, denote by $U(n,d)$ the set of all oriented unicyclic graphs on n vertices with diameter d and by $T(n,d)$ the set of all undirected or oriented trees on n vertices with diameter d. From Theorem 2.3, we can narrow down the possibility of the graph with minimum skew energy among all oriented unicyclic graphs as follows.

Lemma 2.4 *Let G^σ be an oriented graph with minimum skew energy among all graphs of $U(n,d)$. Denote by C the unique oriented cycle of G^σ. Then C is either an odd cycle or an evenly oriented even cycle.*

Moreover, the following recursions concerning skew characteristic polynomials of oriented graphs are needed.

Lemma 2.5 *Let G^σ be an oriented unicyclic graph with skew characteristic polynomial*

$$\phi(G^\sigma, \lambda) = \sum_{i=0}^{n} (-1)^i a_i \lambda^{n-i},$$

C be the unique cycle of G^σ and $e = (u, v)$ be an arc of G^σ. Suppose that $|C| = l$. Then

(a) $a_i(G^\sigma) = a_i(G^\sigma - e) + a_{i-2}(G^\sigma - u - v) + 2a_{i-l}(G^\sigma - V(C))$ *if C is oddly oriented and $e \in C$;*

(b) $a_i(G^\sigma) = a_i(G^\sigma - e) + a_{i-2}(G^\sigma - u - v) - 2a_{i-l}(G^\sigma - V(C))$ *if C is evenly oriented and $e \in C$;*

(c) $a_i(G^\sigma) = a_i(G^\sigma - e) + a_{i-2}(G^\sigma - u - v)$ *otherwise.*

Proof (a) We divide all basic subgraphs of G^σ having i vertices into three parts: those that do not contain the arc e; those that contain e as the elementary subgraph K_2, and those that contain the evenly oriented cycle C. Then the former is the coefficient of λ^{n-i} in $\phi(G^\sigma - e, \lambda)$, the second part can be considered as the coefficient of λ^{n-i-2} in $\phi(G^\sigma - u - v, \lambda)$ and the latter can be considered as the coefficient of λ^{n-i-l} in $\phi(G^\sigma - V(C), \lambda)$ multiplied by 2. Hence, the equality holds.

Similarly, (b) and (c) can be proved. □

Combining with Lemma 2.5(c) and (2), we have

Lemma 2.6 *Let G^σ be an oriented graph containing no even cycles and G_1^σ a spanning subgraph* (*resp. proper spanning subgraph*) *of G^σ. Then $G^\sigma \succeq G_1^\sigma$* (*resp. $G^\sigma \succ G_1^\sigma$*).

3 The graphs with minimum skew energy among all oriented unicyclic graphs

Denote by $T_{n,d}$ the tree obtained from the path P_{d-1} and the star S_{n-d+2} by identifying one pendent vertex of them, and by $U_{n,d}$ the undirected unicyclic graph obtained from the cycle C_4 by attaching a pendent vertex of the path P_{d-2} and $n - d - 1$ pendent edges to its two non-adjacent vertices, respectively; see Figure 1. Note that both $T_{n,d}$ and $U_{n,d}$ contain n vertices and have diameter d.

Let v be an arbitrary vertex of the oriented graph G^σ. The operation by reversing the orientations of all arcs incident with v and preserving the orientations of all its other arcs is called a *reversal of G^σ at v*, the resultant graph is denoted by G_v^σ. Let $S(G^\sigma)$ and $S(G_v^\sigma)$ be the skew adjacency matrices of the oriented graphs G^σ and G_v^σ, respectively. One can find that $S(G^\sigma)$ is similar to $S(G_v^\sigma)$, thus G_v^σ has the same skew energy as that of G^σ. Especially, Adiga *et al.* [11] showed that the skew energy of a directed tree is independent of its orientation, which is equal to the energy of its underlying tree. Hence, the following results for undirected trees apply equally well to oriented trees, which will be cited in the following discussion directly.

Lemma 3.1 [18] *For $2 \le i \le \lfloor \frac{n}{2} \rfloor$ and $n \ge 4$,*

$$P_n \succ P_i \cup P_{n-i} \succ P_1 \cup P_{n-1}.$$

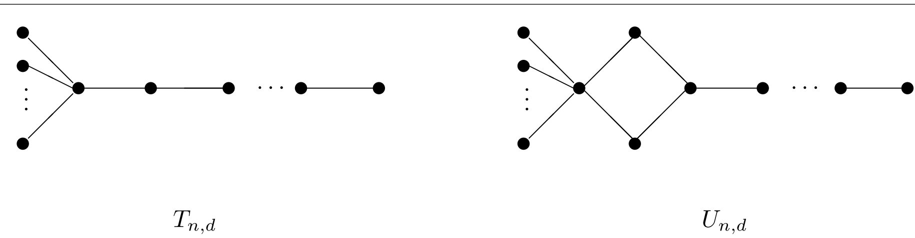

Figure 1 The tree $T_{n,d}$ and the unicyclic graph $U_{n,d}$.

Lemma 3.2 [19] *Let $n \geq 5$, T_n denote any tree with order n and $T_n \neq P_n, S_n$. Then*

$$P_n \succ T_n \succ S_n.$$

Lemma 3.3 [20] *Let $T \in T(n,d)$ and $T \neq T_{n,d}$. Then*

$$T \succ T_{n,d}.$$

Lemma 3.4 [21] *If $d > d_0 \geq 3$, then*

$$T_{n,d} \succ T_{n,d_0}.$$

Denote by $U^+_{n,d}$ the set of oriented graphs with underlying graph $U_{n,d}$ and the unique oriented cycle C_4 is evenly oriented. Because all skew adjacency matrices of $U^+_{n,d}$ are similar by the method reversal above, and thus each of them has the same skew energy, we in the following do not concern the orientation of each graph of $U^+_{n,d}$. To obtain the main result of this paper, we first establish the following two lemmas.

Lemma 3.5 *Let $n \geq 5$ and $G^\sigma \in U(n, n-2)$. If $G^\sigma \neq U^+_{n,n-2}$, then*

$$G^\sigma \succ U^+_{n,n-2}.$$

Proof By Lemma 2.4, it suffices to consider those oriented graphs whose unique cycle is either odd or evenly oriented.

We prove the result by induction on n. If $G^\sigma \in U(5,3)$ and $G^\sigma \neq U^+_{5,3}$, then G^σ is isomorphic to either G^σ_1 or G^σ_2; if $G^\sigma \in U(6,4)$ and $G^\sigma \neq U^+_{6,4}$, then G^σ is isomorphic to either G^σ_3 or G^σ_4 or G^σ_5; see Figure 2. (Here and in the sequel we do not consider the oriented graphs with oddly oriented cycle.) By a directly calculation, we have that

$$\phi(G^\sigma_1, \lambda) = \lambda^5 + 5\lambda^3 + 4\lambda, \qquad \phi(G^\sigma_2, \lambda) = \lambda^5 + 5\lambda^3 + 3\lambda,$$
$$\phi(G^\sigma_3, \lambda) = \lambda^6 + 6\lambda^4 + 8\lambda^2 + \lambda, \qquad \phi(G^\sigma_4, \lambda) = \lambda^6 + 6\lambda^4 + 7\lambda^2 + \lambda,$$
$$\phi(G^\sigma_5, \lambda) = \lambda^6 + 6\lambda^4 + 6\lambda^2$$

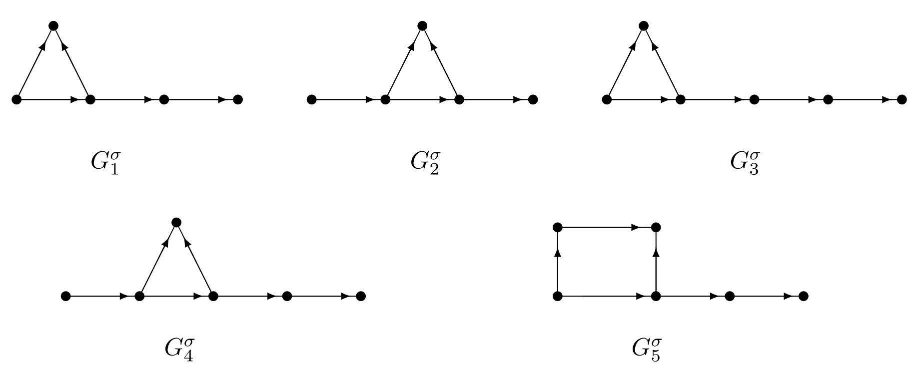

Figure 2 Some oriented graphs for Lemma 3.5.

and

$$\phi(U_{5,3}^+,\lambda) = \lambda^5 + 5\lambda^3 + 2\lambda, \qquad \phi(U_{6,4}^+,\lambda) = \lambda^6 + 6\lambda^4 + 5\lambda^2.$$

Then by (2) the result follows for $n = 5$ and 6.

Suppose that the result holds for graphs of $U(n-1,n-3)$ and $U(n-2,n-4)$ with $n \geq 7$. For $G^\sigma \in U(n,n-2)$, there exists pendent arc $e = (u,v)$ such that v is the pendent vertex of G^σ and u has degree 2, since $d(G^\sigma) = n-2$, thus, the length of its unique cycle is not more than 4. Then $G^\sigma - v \in U(n-1,n-3)$ and $G^\sigma - v - u \in U(n-2,n-4)$. By Lemma 2.5(c), we have

$$a_i(U_{n,n-2}^+) = a_i(U_{n-1,n-3}^+) + a_{i-2}(U_{n-2,n-4}^+),$$
$$a_i(G^\sigma) = a_i(G^\sigma - v) + a_{i-2}(G^\sigma - v - u).$$

Combining with the induction hypothesis, $G^\sigma - v \succeq U_{n-1,n-3}^+$ and $G^\sigma - v - u \succeq U_{n-2,n-4}^+$ with equality if and only if $G^\sigma - v = U_{n-1,n-3}^+$ and $G^\sigma - v - u = U_{n-2,n-4}^+$, respectively. The proof is complete. □

Lemma 3.6 *Let $n \geq 6$ and $G^\sigma \in U(n,d)$ with $3 \leq d \leq n-3$. If $G^\sigma \neq U_{n,d}^+$, then*

$$G^\sigma \succ U_{n,d}^+.$$

Proof Similarly to Lemma 3.5, it suffices to consider those oriented graphs whose unique cycle is either odd or evenly oriented. Let $n - d = p$. We prove the lemma by induction on p. By Lemma 3.5, the result follows for $p = 2$. Suppose now that $p \geq 3$ and the result holds for $n - d < p$.

Because $n - d = p \geq 3$, there is exactly one vertex in $U_{n,d}^+$ with degree 3. Let u be such a vertex and v, adjacent to u, lie on the cycle C_4. Applying Lemma 2.5(b) to the arc (u,v), we have

$$a_i(U_{n,d}^+) = a_i(T_{n,d}) + a_{i-2}(P_{d-3} \cup S_{n-d+1}) - 2a_{i-4}(P_{d-3}). \tag{3}$$

Applying the recursion of Lemma 2.5(c) to $T_{n,d}$ and $P_{d-3} \cup S_{n-d+1}$ respectively, we have

$$\begin{aligned} a_i(T_{n,d}) &= a_i(T_{n-1,d}) + a_{i-2}(P_{d-1}) \\ &= a_i(T_{n-1,d}) + a_{i-2}(P_{d-2}) + a_{i-4}(P_{d-3}); \\ a_{i-2}(P_{d-3} \cup S_{n-d+1}) &= a_{i-2}(P_{d-3} \cup S_{n-d}) + a_{i-4}(P_{d-3} \cup (n-d-1)K_1) \\ &= a_{i-2}(P_{d-3} \cup S_{n-d}) + a_{i-4}(P_{d-3}). \end{aligned}$$

Then another recursion for $a_i(U_{n,d}^+)$ can be obtained as follows

$$a_i(U_{n,d}^+) = a_i(T_{n-1,d}) + a_{i-2}(P_{d-2}) + a_{i-2}(P_{d-3} \cup S_{n-d}). \tag{4}$$

Let $G^\sigma \in U(n,d)$ $(3 \leq d \leq n-3, n \geq 6)$ and $G^\sigma \neq U_{n,d}^+$. To show $G^\sigma \succ U_{n,d}^+$, we divide G^σ into the following three cases.

Case 1. G^σ contains no pendent vertices.

Then $G^\sigma = C_n$. Let $e = (u, v)$ be any arc of C_n. By Lemma 2.5,

$$a_i(C_n) = a_i(C_n - e) + a_{i-2}(C_n - u - v) - 2 = a_i(P_n) + a_{i-2}(P_{n-2}) - 2, \tag{5}$$

if $i = n$ and C_n is evenly oriented; otherwise

$$a_i(C_n) = a_i(C_n - e) + a_{i-2}(C_n - u - v) = a_i(P_n) + a_{i-2}(P_{n-2}). \tag{6}$$

By Lemma 3.2, $a_i(P_n) > a_i(T_{n,d})$ and $a_{i-4}(P_{d-3}) \geq 1$. By Lemma 3.1, $a_{i-2}(P_{n-2}) \geq a_{i-2}(P_{d-3} \cup S_{n-d+1})$. Then combining with (3), (5) and (6), $G^\sigma \succ U^+_{n,d}$.

Case 2. All pendent vertices are contained in the longest path of G^σ.

Let, in G^σ, C_r $(r < n)$ be the unique cycle and $P(G^\sigma) = v_0 v_1 \cdots v_k$ be the longest path. Then $k = d$, and either v_0 or v_d is the pendent vertex of G^σ. Recall that $p \geq 3$, then there exist two adjacent vertices, say u and v, are not contained in $P(G^\sigma)$. Hence, both u and v, are contained in C_r. Consequently, $G^\sigma - (u, v) \in T(n, d_1)$, $G^\sigma - v \in T(n-1, d_2)$ and $G^\sigma - v - u \in T(n-2, d_3)$. Since $P(G^\sigma)$ still is contained in $G^\sigma - (u, v)$, $G^\sigma - v$ and $G^\sigma - v - u$, we have $d_i \geq d$ for $i = 1, 2, 3$. Then by Lemmas 3.3 and 3.4,

$$G^\sigma - (u, v) \succeq T_{n,d_1} \succeq T_{n,d}, \qquad G^\sigma - v \succeq T_{n-1,d_2} \succeq T_{n-1,d} \tag{7}$$

and

$$G^\sigma - v - u \succeq T_{n-2,d_3} \succeq T_{n-2,d} \succ P_{d-3} \cup P_{n-d+1} \succ P_{d-3} \cup S_{n-d+1}. \tag{8}$$

Subcase 2.1. r is odd.

Combining with (3), (7), (8) and Lemma 2.5(c), we have

$$\begin{aligned} a_i(G^\sigma) &= a_i\bigl(G^\sigma - (u, v)\bigr) + a_{i-2}\bigl(G^\sigma - u - v\bigr) \\ &> a_i(T_{n,d}) + a_{i-2}(P_{d-3} \cup S_{n-d+1}) > a_i\bigl(U^+_{n,d}\bigr). \end{aligned}$$

Hence, $G^\sigma \succ U^+_{n,d}$.

Subcase 2.2. r is even and $r > 4$.

Then there exist three vertices, say u, v and w, such that each of them is not contained in $P(G^\sigma)$ and v adjacent to both u and w. Further, let u_1 ($\neq v$) adjacent to u and w_1 ($\neq v$) adjacent to w. Then $G^\sigma - v - u - w - w_1 \succ G^\sigma - C_r$, $G^\sigma - v - u - w - u_1 \succ G^\sigma - C_r$ and $G^\sigma - w - u - v \in T(n-3, d_4)$ with $d_4 \geq d$. By Lemma 2.5(b), we have

$$a_i\bigl(G^\sigma\bigr) = a_i\bigl(G^\sigma - (u, v)\bigr) + a_{i-2}\bigl(G^\sigma - u - v\bigr) - 2a_{i-r}\bigl(G^\sigma - C_r\bigr). \tag{9}$$

Applying Lemma 2.5(c) repeatedly, we have

$$\begin{aligned} a_i\bigl(G^\sigma - (u, v)\bigr) &= a_i\bigl(G^\sigma - v\bigr) + a_{i-2}\bigl(G^\sigma - v - w\bigr) \\ &= a_i\bigl(G^\sigma - v\bigr) + a_{i-2}\bigl(G^\sigma - u - v - w\bigr) + a_{i-4}\bigl(G^\sigma - u_1 - u - v - w\bigr) \\ &\geq a_i\bigl(G^\sigma - v\bigr) + a_{i-2}\bigl(G^\sigma - u - v - w\bigr) + a_{i-r}\bigl(G^\sigma - C_r\bigr), \end{aligned}$$

$$a_{i-2}(G^\sigma - u - v) = a_{i-2}(G^\sigma - w - u - v) + a_{i-4}(G^\sigma - w_1 - w - u - v)$$
$$\geq a_{i-2}(G^\sigma - w - u - v) + a_{i-r}(G^\sigma - C_r).$$

Then combining with (7), (9) and Lemma 3.4

$$a_i(G^\sigma) \geq a_i(G^\sigma - v) + 2a_{i-2}(G^\sigma - u - v - w) \geq a_i(T_{n-1,d}) + 2a_{i-2}(T_{n-3,d}).$$

From Lemma 2.6, we have $a_{i-2}(T_{n-3,d}) > a_{i-2}(P_{d-3} \cup S_{n-d})$ and $a_{i-2}(T_{n-3,d}) > a_{i-2}(P_{d-2})$. Consequently, combining with (4), $G^\sigma \succ U^+_{n,d}$.

Subcase 2.3. $r = 4$.

Recall that $p \geq 3$, then C_r adjoint to $P(G^\sigma)$ at most one arc. If C_r and $P(G^\sigma)$ have no common arc, then there exist three vertices, say u, v and w, such that each of them is not contained in $P(G^\sigma)$ and v adjacent to both u and w. Then $G^\sigma \succ U^+_{n,d}$ by the discussion similar to Subcase 2.2. If C_r and $P(G^\sigma)$ have exactly one common arc, then $d = n - 3$. Let $e = (u, v)$ be the unique arc of C_4 which has no common vertices with $P(G^\sigma)$. Applying Lemmas 2.5, 3.1, 3.3 and (4), we have for some j with $1 \leq j \leq d - 1$

$$\begin{aligned}
a_i(G^\sigma) &= a_i(G^\sigma - (u,v)) + a_{i-2}(G^\sigma - u - v) - 2a_{i-4}(G^\sigma - C_4) \\
&= a_i(G^\sigma - v) + a_{i-2}(P_j \cup P_{n-2-j}) + a_{i-2}(P_{n-2}) - 2a_{i-4}(P_j \cup P_{n-4-j}) \\
&= a_i(G^\sigma - v) + a_{i-2}(P_j \cup P_{n-3-j}) + a_{i-4}(P_j \cup P_{n-4-j}) \\
&\quad + a_{i-2}(P_{n-3}) + a_{i-4}(P_{n-4}) - 2a_{i-4}(P_j \cup P_{n-4-j}) \\
&= a_i(G^\sigma - v) + a_{i-2}(P_j \cup P_{n-3-j}) + a_{i-2}(P_{n-3}) + a_{i-4}(P_{n-4}) - a_{i-4}(P_j \cup P_{n-4-j}) \\
&> a_i(G^\sigma - v) + a_{i-2}(P_j \cup P_{n-3-j}) + a_{i-2}(P_{n-3}) \\
&> a_i(U^+_{n-1,n-3}) + a_{i-2}(P_{n-4}) + a_{i-2}(P_{n-3}).
\end{aligned}$$

On the other hand, we have

$$a_i(U^+_{n,n-3}) = a_i(U^+_{n-1,n-3}) + a_{i-2}(P_1 \cup T_{n-3,n-5}) \leq a_i(U^+_{n-1,n-3}) + 2a_{i-2}(P_{n-4})$$
$$< a_i(U^+_{n-1,n-3}) + a_{i-2}(P_{n-4}) + a_{i-2}(P_{n-3}).$$

Consequently, $G^\sigma \succ U^+_{n,d}$.

Case 3. There exist pendent vertices not contained in the longest path of G^σ.

Let $P(G^\sigma) = v_0 v_1 \cdots v_k$ be the longest path of G^σ, which is defined as Case 2, and $u \notin P(G^\sigma)$, a pendent vertex of G^σ. Suppose that v is the unique neighbor of u. By Lemma 2.5(c),

$$a_i(G^\sigma) = a_i(G^\sigma - u) + a_{i-2}(G^\sigma - u - v). \tag{10}$$

Let v_1 be the vertex of $U^+_{n,d}$ with degree $n - d - 1$ and u_1, adjacent to v_1, the pendent vertex. Then

$$\begin{aligned}
a_i(U^+_{n,d}) &= a_i(U^+_{n,d} - u_1) + a_{i-2}(U^+_{n,d} - v_1 - u_1) \\
&= a_i(U^+_{n-1,d}) + a_{i-2}((n-d-2)P_1 \cup T_{d,d-2}).
\end{aligned} \tag{11}$$

Subcase 3.1. There exists a pendent vertex such that its neighbor is not contained in $P(G^\sigma)$.

Say u is such a pendent vertex. Then $G^\sigma - u \in U(n-1,d)$, and thus $a_i(G^\sigma - u) \geq a_i(U^+_{n-1,d})$ by the induction hypothesis; $G^\sigma - u - v \succeq (n-d-3)P_1 \cup P_{d+1} \succ (n-d-2)P_1 \cup T_{d,d-2}$ by Lemmas 3.3 and 2.6. Consequently, $G^\sigma \succ U^+_{n,d}$.

Subcase 3.2. The neighbor of each pendent vertex is contained in $P(G^\sigma)$, and there exist no pendent vertices whose neighbor lies on C_r.

Without loss of generality, let j $(1 \leq j \leq d-1)$ be the least index such that each vertex of $\{v_1, v_2, \dots, v_{j-1}\}$ has degree 2 and v, a pendent vertex, adjacent to v_j. Then $G^\sigma - u - v_j \supseteq P_j \cup G^\sigma_1$, where $G^\sigma_1 \in U(s, d^*)$ with $j + s \leq n-2$ and $d^* \geq d - j - 1$. Then by Lemma 3.3 and the induction hypothesis, we have $G^\sigma_1 \succeq U^+_{d-j+1,d-j-1}$. Moreover,

$$\begin{aligned}
a_i\bigl(P_j \cup U^+_{d-j+1,d-j-1}\bigr) &= a_i(P_j \cup T_{d-j+1,d-j-1}) + a_{i-2}(P_j \cup P_3 \cup P_{d-j-4}) - 2a_{i-4}(P_j \cup P_{d-j-4}) \\
&= a_i(P_j \cup P_{d-j}) + a_{i-2}(P_j \cup P_{d-j-2}) + a_{i-2}(P_j \cup P_2 \cup P_{d-j-4}) \\
&\quad + a_{i-4}(P_j \cup P_1 \cup P_{d-j-4}) - 2a_{i-4}(P_j \cup P_{d-j-4}) \\
&= a_i(P_j \cup P_{d-j}) + a_{i-2}(P_j \cup P_{d-j-2}) + a_{i-2}(P_j \cup P_{d-j-4}) \\
&\geq a_i(P_j \cup P_{d-j}) + a_{i-2}(P_j \cup P_{d-j-2}) \\
&> a_i(P_{d-1}) + a_{i-2}(P_{d-3}) \quad \text{(By Lemma 3.1)} \\
&= a_i(T_{d,d-2}).
\end{aligned}$$

Thus, $G^\sigma - u - v \succ T_{d,d-2}$. Consequently, $G^\sigma \succ U^+_{n,d}$ by (11).

Subcase 3.3. The neighbor of each pendent vertex is contained in $P(G^\sigma)$, and there exist pendent vertices whose neighbor lies on C_r.

Let now $v_l, v_{l+1}, \dots, v_{l+s}$ $(s \geq 0)$ be all common vertices of $P(G^\sigma)$ and C_r. If $s = 0$, then $v = v_l$, *i.e.*, v is the unique common vertex of $P(G^\sigma)$ and C_r. Then $G^\sigma - u - v \supseteq P_l \cup P_{d-l} \cup P_{r-1}$ and thus

$$\begin{aligned}
a_i\bigl(G^\sigma - u - v\bigr) &\geq a_i(P_l \cup P_{d-l} \cup P_{r-1}) \\
&= a_i(P_l \cup P_{d-l} \cup P_{r-2}) + a_{i-2}(P_l \cup P_{d-l} \cup P_{r-2}) \\
&> a_i(P_{d-1}) + a_{i-2}(P_{d-3}) \\
&= a_i(T_{d,d-2}).
\end{aligned}$$

Consequently, $G^\sigma \succ U^+_{n,d}$. If $s > 0$ and $v \neq v_l, v_{l+s}$, then $G^\sigma - u - v \supseteq P_{d+l}$, and thus $G^\sigma - u - v \succ T_{d,d-2}$. Hence, $G^\sigma \succ U^+_{n,d}$. If $s > 0$, and $v = v_l$ or $v = v_{l+s}$. (Without loss of generality, suppose that $v = v_l$.) Then $G^\sigma - u - v \supseteq P_l \cup T_1$, where T_1 denotes the tree obtained from the path $v_{l+1} \cdots v_{l+s}$ by adding a pendent vertex. Similarly, we have

$$\begin{aligned}
a_i(P_l \cup T_1) &= a_i(P_l \cup P_{d-l}) + a_{i-2}(P_l \cup P_{s-1} \cup P_{d-l-s}) \\
&> a_i(P_{d-1}) + a_{i-2}(P_{d-3}) \\
&= a_i(T_{d,d-2}).
\end{aligned}$$

Hence, $G^\sigma \succ U^+_{n,d}$.

Thus the result follows. □

Putting Lemma 3.5 together with Lemma 3.6, the main result of this paper can be obtained.

Theorem 3.7 *Let $n \geq 6$ and $3 \leq d \leq n-2$. Then the oriented graph with minimum skew energy among all graphs of $U(n,d)$ is of $U^+_{n,d}$.*

Proof Let $n \geq 6$ and $3 \leq d \leq n-2$. Then $G^\sigma \succeq U^+_{n,d}$ with equality if and only if $G^\sigma = U^+_{n,d}$ for any oriented unicyclic graph $G^\sigma \in U(n,d)$ by Lemma 3.5 and Lemma 3.6. Then $\mathcal{E}_s(G^\sigma) \geq \mathcal{E}_s(U^+_{n,d})$ with equality if and only if $G^\sigma = U^+_{n,d}$ by (2), and thus the result follows. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
S-CG wrote and reformed the article. All authors read and approved the final manuscript.

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