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Composite schemes for variational inequalities over equilibrium problems and variational inclusions

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Abstract

Let C be a nonempty closed convex subset of a Hilbert space H, and let $T : H \to H$ be a nonlinear mapping. It is well known that the following classical *variational inequality* has been applied in many areas of applied mathematics, modern physical sciences, computerized tomography and many others. Find a point $x^* \in C$ such that

$$\langle Tx^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{A}$$

In this paper, we consider the following variational inequality. Find a point $x^* \in C$ such that

$$\langle (F - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{B}$$

and, for solutions of the variational inequality (B) with the feasibility set C, which is the intersection of the set of solutions of an equilibrium problem and the set of a solutions of a variational inclusion, construct the two composite schemes, that is, the implicit and explicit schemes to converge strongly to the unique solution of the variational inequality (B).

Recently, many authors introduced some kinds of algorithms for solving the variational inequality problems, but, in fact, our two schemes are more simple for finding solutions of the variational inequality (B) than others.

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1 Introduction

A very common problem in areas of mathematics and physical sciences consists of trying to find a point in a nonempty closed convex subset C of a Hilbert space H. This problem is related to the variational inequality problem (A). One frequently employed approach in solving the variational inequality problems is the approximation methods. Some approximation methods for solving variational inequality problems and the related optimization problems can be found in [1–16].

In this paper, we consider the following *variational inequality*. Find a point $x^* \in C$ such that

$$\langle (F - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{B}$$

where C is the intersection of the set of solutions of an equilibrium problem and the set of a variational inclusion. In fact, the reason that we focus on the set C in the equilibrium problems and the variational inclusion problems, plays a very important role in many practical applications.

For this purpose, we construct the following composite schemes, that is, the *implicit scheme* $\{x_t\}$ and the *explicit scheme* $\{x_n\}$, respectively,

$$x_t = \bigl[I - t(F - \gamma f)\bigr] J_{R,\lambda}(I - \lambda A) S_\mu (I - \mu B) x_t, \quad \forall t \in \left(0, \frac{1}{\rho - \gamma\tau}\right) \tag{1.1}$$

and

$$x_{n+1} = \bigl[I - \alpha_n(F - \gamma f)\bigr] J_{R,\lambda}(I - \lambda A) S_\mu (I - \mu B) x_n, \quad \forall n \geq 0. \tag{1.2}$$

Our idea is to involve directly the operator $F - \gamma f$ to generate the two composite schemes (1.1) and (1.2) that converge strongly to solutions of the variational inequality problem (B). In fact, our two schemes are very simple.

2 Preliminaries

In this section, we introduce some notations and useful conclusions for our main results.

Let H be a real Hilbert space. Let $B : H \to H$ be a nonlinear mapping, let $\varphi : H \to R$ be a function, and let $\Theta : H \times H \to R$ be a bifunction.

Now, we consider the following *equilibrium problem*. Find a point $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Bx, y - x \rangle \geq 0, \quad \forall y \in C. \tag{2.1}$$

The set of solutions of problem (2.1) is denoted by EP. The equilibrium problems include fixed point problems, optimization problems and variational inequality problems as special cases. For the related works, see [17–30].

Let $f : H \to H$ be a τ*-contraction*, that is, there exists a constant $\tau \in [0, 1)$ such that

$$\bigl\| f(x) - f(y) \bigr\| \leq \tau \| x - y \|, \quad \forall x, y \in H,$$

and let $S : H \to H$ be a *nonexpansive mapping*, that is,

$$\| Sx - Sy \| \leq \| x - y \|, \quad \forall x, y \in H.$$

Recall that a mapping $A : H \to H$ is said to be α*-inverse strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \| Ax - Ay \|^2, \quad \forall x, y \in C.$$

A mapping $F : H \to H$ is said to be *strongly positive* if there exists a constant $\rho > 0$ such that $\langle Fx, x\rangle \geq \rho\|x\|^2$ for all $x \in H$.

Let $A : H \to H$ be a single-valued nonlinear mapping, and let $R : H \to 2^H$ be a set-valued mapping.

Now, we consider the following *variational inclusion*. Find a point $x \in H$ such that

$$\theta \in A(x) + R(x), \tag{2.2}$$

where θ is the zero element in H. The set of solutions of problem (2.2) is denoted by $I(A, R)$. The variational inclusion problems have been considered extensively in [31–38] and the references therein.

A set-valued mapping $T : H \to 2^H$ is said to be *monotone* if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is said to be *maximal* if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$.

Let $R : H \to 2^H$ be a maximal monotone set-valued mapping. We define the resolvent operator $J_{R,\lambda}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}(x), \quad \forall x \in H,$$

where λ is a positive number. It is worth mentioning that the resolvent operator $J_{R,\lambda}$ is single-valued, nonexpansive and 1-inverse strongly monotone and, further, a solution of problem (2.2) is a fixed point of the operator $J_{R,\lambda}(I - \lambda A)$ for all $\lambda > 0$.

Throughout this paper, we assume that a bifunction $\Theta : H \times H \to \mathbf{R}$ and a convex function $\varphi : H \to R$ satisfy the following conditions:

(H1) $\Theta(x, x) = 0$ for all $x \in H$;

(H2) Θ is monotone, *i.e.*, $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in H$;

(H3) for all $y \in H$, $x \mapsto \Theta(x, y)$ is weakly upper semi-continuous;

(H4) for all $x \in H$, $y \mapsto \Theta(x, y)$ is convex and lower semi-continuous;

(H5) for all $x \in H$ and $\mu > 0$, there exists a bounded subset $D_x \subset H$ and $y_x \in H$ such that, for any $z \in H \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{\mu}\langle y_x - z, z - x\rangle < 0.$$

Lemma 2.1 [39] *Let H be a real Hilbert space. Let $\Theta : H \times H \to R$ be a bifunction, and let $\varphi : H \to R$ be a proper lower semi-continuous and convex function. For any $\mu > 0$ and $x \in H$, define a mapping $S_\mu : H \to H$ as follows*:

$$S_\mu(x) = \left\{z \in H : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{\mu}\langle y - z, z - x\rangle \geq 0, \forall y \in H\right\}, \quad \forall x \in H.$$

Assume that conditions (H1)-(H5) *hold. Then we have the following results*:

(1) *For each $x \in H$, $S_\mu(x) \neq \emptyset$ and S_μ is single-valued.*

(2) *S_μ is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|S_\mu x - S_\mu y\|^2 \leq \langle S_\mu x - S_\mu y, x - y\rangle.$$

(3) $\mathrm{Fix}(S_\mu(I-\mu B)) = EP$.
(4) *EP is closed and convex.*

Lemma 2.2 [40] *Let $R : H \to 2^H$ be a maximal monotone mapping, and let $A : H \to H$ be a Lipschitz-continuous mapping. Then the mapping $(R + A) : H \to 2^H$ is maximal monotone.*

Lemma 2.3 [8] *Let H be a real Hilbert space. Let the mapping $A : H \to H$ be α-inverse strongly monotone, and let $\lambda > 0$ be a constant. Then, we have*

$$\|(I-\lambda A)x-(I-\lambda A)y\|^2 \leq \|x-y\|^2 + \lambda(\lambda-2\alpha)\|Ax-Ay\|^2, \quad \forall x, y \in H.$$

In particular, if $0 \leq \lambda \leq 2\alpha$, then $I - \lambda A$ is nonexpansive.

Lemma 2.4 [41] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers satisfying $a_{n+1} \leq (1-\gamma_n)a_n + \delta_n$, where $\{\gamma_n\}$ is a sequence in $(0,1)$, and $\{\delta_n\}$ is a sequence such that*
(a) $\sum_{n=1}^\infty \gamma_n = \infty$;
(b) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^\infty |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section, we give our main results. In the sequel, we assume the following conditions are satisfied.

Condition 3.1 H is a real Hilbert space. $\varphi : H \to R$ is a lower semi-continuous and convex function, and $\Theta : H \times H \to R$ is a bifunction satisfying conditions (H1)-(H5).

Condition 3.2 F is a strongly positive bounded linear operator with coefficient $0 < \rho < 1$, $f : H \to H$ is a τ-contraction satisfying $\rho > \gamma\tau$, where $\gamma > 0$ is a constant, and $R : H \to 2^H$ is a maximal monotone mapping.

Condition 3.3 $A, B : C \to C$ are an α-inverse strongly monotone operator and a β-inverse strongly monotone operator, respectively.

Condition 3.4 λ and μ are two constants such that $0 < \lambda < 2\alpha$ and $0 < \mu < 2\beta$.

Condition 3.5 $\Omega := EP \cap I(A, R)$ is nonempty.

Now, we first consider the following scheme.

Algorithm 3.1 For any $t \in (0, \frac{1}{\rho-\gamma\tau})$, define a net $\{x_t\}$ as follows:

$$x_t = [I - t(F-\gamma f)]J_{R,\lambda}(I-\lambda A)S_\mu(I-\mu B)x_t. \tag{3.1}$$

Remark 3.2 The net $\{x_t\}$ defined by (3.1) is well-defined. In fact, from Lemmas 2.1 and 2.3, we know that the mappings $I-\lambda A$ and $I-\mu B$ and S_μ are nonexpansive. For any $x \in H$, we define a mapping $W_t x = [I - t(F-\gamma f)]J_{R,\lambda}(I-\lambda A)S_\mu(I-\mu B)x$. We note that $I - tF$ is positive and $\|I - tF\| \leq 1 - t\rho$. Hence we have

$$\begin{aligned}\|W_t x - W_t y\| &= \big\|\big[I - t(F - \gamma f)\big]J_{R,\lambda}(I - \lambda A)S_\mu(I - \mu B)x \\ &\quad - \big[I - t(F - \gamma f)\big]J_{R,\lambda}(I - \lambda A)S_\mu(I - \mu B)y\big\| \\ &\leq \|I - tF\|\big\|J_{R,\lambda}(I - \lambda A)S_\mu(I - \mu B)x - J_{R,\lambda}(I - \lambda A)S_\mu(I - \mu B)y\big\| \\ &\quad + t\gamma\big\|f\big(J_{R,\lambda}(I - \lambda A)S_\mu(I - \mu B)x\big) - f\big(J_{R,\lambda}(I - \lambda A)S_\mu(I - \mu B)y\big)\big\| \\ &\leq (1 - \rho t)\|x - y\| + t\gamma\tau\|x - y\| = \big[1 - (\rho - \gamma\tau)t\big]\|x - y\|.\end{aligned}$$

This shows that W is a contraction. Therefore, W has a unique fixed point, which is denoted by x_t.

Theorem 3.3 *The net* $\{x_t\}$ *defined by* (3.1) *converges strongly to the unique solution* $\tilde{x} \in \Omega$ *of the following variational inequality*:

$$\big\langle (F - \gamma f)\tilde{x}, y - \tilde{x}\big\rangle \geq 0, \quad \forall y \in \Omega. \tag{3.2}$$

Remark 3.4 First, we can check easily that $F - \gamma f$ is strongly monotone with coefficient $\rho - \gamma\tau$. Now, we show the uniqueness of the solution of the variational inequality (3.2). Suppose that $x^* \in \Omega$ and $\tilde{x} \in \Omega$ both are solutions to (3.2). Then we have

$$\big\langle (F - \gamma f)x^*, \tilde{x} - x^*\big\rangle \geq 0, \qquad \big\langle (F - \gamma f)\tilde{x}, x^* - \tilde{x}\big\rangle \geq 0.$$

Adding up the last two inequalities gives

$$\big\langle (F - \gamma f)\tilde{x} - (F - \gamma f)x^*, \tilde{x} - x^*\big\rangle \leq 0.$$

The strong monotonicity of $F - \gamma f$ implies that $\tilde{x} = x^*$, and so, the uniqueness is proved.

Next, we give the detail proofs of Theorem 3.3.

Proof Pick up $x^* \in \Omega$. It is clear that $S_\mu(x^* - \mu Bx^*) = J_{R,\lambda}(x^* - \lambda Ax^*) = x^*$. Set $z_t = S_\mu(x_t - \mu Bx_t)$ and $y_t = J_{R,\lambda}(z_t - \lambda Az_t)$ for all $t \in [0,1]$. It follows from Lemma 2.3 that

$$\begin{aligned}\big\|y_t - x^*\big\| &= \big\|J_{R,\lambda}(z_t - \lambda Az_t) - J_{R,\lambda}\big(x^* - \lambda Ax^*\big)\big\| \\ &\leq \big\|(z_t - \lambda Az_t) - \big(x^* - \lambda Ax^*\big)\big\| \\ &\leq \big\|z_t - x^*\big\|\end{aligned}$$

and

$$\begin{aligned}\big\|z_t - x^*\big\|^2 &= \big\|S_\mu(x_t - \mu Bx_t) - S_\mu\big(x^* - \mu Bx^*\big)\big\|^2 \\ &\leq \big\|(x_t - \mu Bx_t) - \big(x^* - \mu Bx^*\big)\big\|^2 \\ &\leq \big\|x_t - x^*\big\|^2 + \mu(\mu - 2\beta)\big\|Bx_t - Bx^*\big\|^2 \\ &\leq \big\|x_t - x^*\big\|^2.\end{aligned} \tag{3.3}$$

Therefore, we have

$$\|y_t - x^*\| \leq \|x_t - x^*\|.$$

From (3.1), we get

$$\begin{aligned}
\|x_t - x^*\| &= \|[I - t(F - \gamma f)]y_t - x^*\| \\
&\leq \|(I - tF)(y_t - x^*)\| + t\gamma\|f(y_t) - f(x^*)\| + t\|(F - \gamma f)x^*\| \\
&\leq (1 - \rho t)\|y_t - x^*\| + t\gamma\tau\|y_t - x^*\| + t\|(F - \gamma f)x^*\| \\
&\leq [1 - (\rho - \gamma\tau)t]\|x_t - x^*\| + (\rho - \gamma\tau)t\frac{\|(F - \gamma f)x^*\|}{\rho - \gamma\tau},
\end{aligned}$$

and so,

$$\|x_t - x^*\| \leq \frac{\|(F - \gamma f)x^*\|}{\rho - \gamma\tau}.$$

Therefore, the net (x_t) is bounded, and so $\{y_t\}$, $\{z_t\}$, $\{Fy_t\}$ and $\{fy_t\}$ are all bounded. It follows from (3.3) and Lemma 2.3 that

$$\begin{aligned}
\|y_t - x^*\|^2 &= \|J_{R,\lambda}(z_t - \lambda Az_t) - J_{R,\lambda}(x^* - \lambda Ax^*)\|^2 \\
&\leq \|(z_t - \lambda Az_t) - (x^* - \lambda Ax^*)\|^2 \\
&\leq \|z_t - x^*\|^2 + \lambda(\lambda - 2\alpha)\|Az_t - Ax^*\|^2 \\
&\leq \|x_t - x^*\|^2 + \mu(\mu - 2\beta)\|Bx_t - Bx^*\|^2 + \lambda(\lambda - 2\alpha)\|Az_t - Ax^*\|^2. \qquad (3.4)
\end{aligned}$$

By (3.1), we obtain

$$\begin{aligned}
\|x_t - x^*\|^2 &= \|[I - t(F - \gamma f)]y_t - x^*\|^2 \\
&\leq [\|y_t - x^*\| + t\|(F - \gamma f)y_t\|]^2 \\
&= \|y_t - x^*\|^2 + t(2\|y_t - x^*\|\|(F - \gamma f)y_t\| + t\|(F - \gamma f)y_t\|^2) \\
&\leq \|y_t - x^*\|^2 + tM, \qquad (3.5)
\end{aligned}$$

where $M > 0$ is some constant satisfying

$$\sup\left\{2\|y_t - x^*\|\|(F - \gamma f)y_t\| + t\|(F - \gamma f)y_t\|^2 : t \in \left(0, \frac{1}{\rho - \gamma\tau}\right)\right\} \leq M.$$

By (3.4) and (3.5), we have

$$\|y_t - x^*\|^2 \leq \|y_t - x^*\|^2 + tM + \mu(\mu - 2\beta)\|Bx_t - Bx^*\|^2 + \lambda(\lambda - 2\alpha)\|Az_t - Ax^*\|^2,$$

and so,

$$\mu(2\beta - \mu)\|Bx_t - Bx^*\|^2 + \lambda(2\alpha - \lambda)\|Az_t - Ax^*\|^2 \leq tM,$$

which implies that

$$\lim_{t\to 0}\|Az_t - Ax^*\| = 0, \qquad \lim_{t\to 0}\|Bx_t - Bx^*\| = 0.$$

Since S_μ is firmly nonexpansive, we have

$$\begin{aligned}
\|z_t - x^*\|^2 &= \|S_\mu(x_t - \mu Bx_t) - S_\mu(x^* - \mu Bx^*)\|^2 \\
&\le \langle x_t - \mu Bx_t - (x^* - \mu Bx^*), z_t - x^*\rangle \\
&= \frac{1}{2}\big(\|x_t - \mu Bx_t - (x^* - \mu Bx^*)\|^2 + \|z_t - x^*\|^2 \\
&\quad - \|x_t - \mu Bx_t - (x^* - \mu Bx^*) - (z_t - x^*)\|^2\big) \\
&\le \frac{1}{2}\big(\|x_t - x^*\|^2 + \|z_t - x^*\|^2 - \|x_t - z_t - \mu(Bx_t - Bx^*)\|^2\big) \\
&= \frac{1}{2}\big(\|x_t - x^*\|^2 + \|z_t - x^*\|^2 - \|x_t - z_t\|^2 \\
&\quad + 2\mu\langle Bx_t - Bx^*, x_t - z_t\rangle - \mu^2\|Bx_t - Bx^*\|^2\big),
\end{aligned}$$

and so,

$$\|z_t - x^*\|^2 \le \|x_t - x^*\|^2 - \|x_t - z_t\|^2 + 2\mu\|Bx_t - Bx^*\|\|x_t - z_t\|. \tag{3.6}$$

Since $J_{R,\lambda}$ is 1-inverse strongly monotone, we have

$$\begin{aligned}
\|y_t - x^*\|^2 &= \|J_{R,\lambda}(z_t - \lambda Az_t) - J_{R,\lambda}(x^* - \lambda Ax^*)\|^2 \\
&\le \langle z_t - \lambda Az_t - (x^* - \lambda Ax^*), y_t - x^*\rangle \\
&= \frac{1}{2}\big(\|z_t - \lambda Az_t - (x^* - \lambda Ax^*)\|^2 + \|y_t - x^*\|^2 \\
&\quad - \|z_t - \lambda Az_t - (x^* - \lambda Ax^*) - (y_t - x^*)\|^2\big) \\
&\le \frac{1}{2}\big(\|z_t - x^*\|^2 + \|y_t - x^*\|^2 - \|z_t - y_t - \lambda(Az_t - Ax^*)\|^2\big) \\
&= \frac{1}{2}\big(\|z_t - x^*\|^2 + \|y_t - x^*\|^2 - \|z_t - y_t\|^2 \\
&\quad + 2\lambda\langle Az_t - Ax^*, z_t - y_t\rangle - \lambda^2\|Az_t - Ax^*\|^2\big),
\end{aligned}$$

which implies that

$$\|y_t - x^*\|^2 \le \|z_t - x^*\|^2 - \|z_t - y_t\|^2 + 2\lambda\|Az_t - Ax^*\|\|z_t - y_t\|. \tag{3.7}$$

Thus, by (3.6) and (3.7), we obtain

$$\begin{aligned}
\|y_t - x^*\|^2 &\le \|x_t - x^*\|^2 - \|x_t - z_t\|^2 + 2\mu\|Bx_t - Bx^*\|\|x_t - z_t\| \\
&\quad - \|z_t - y_t\|^2 + 2\lambda\|Az_t - Ax^*\|\|z_t - y_t\|.
\end{aligned} \tag{3.8}$$

Substituting (3.5) into (3.8), we get

$$\begin{aligned}\|y_t - x^*\|^2 &\le \|y_t - x^*\|^2 + tM - \|x_t - z_t\|^2 + 2\mu\|Bx_t - Bx^*\|\|x_t - z_t\| \\ &\quad - \|z_t - y_t\|^2 + 2\lambda\|Az_t - Ax^*\|\|z_t - y_t\|.\end{aligned}$$

Thus, we derive

$$\|x_t - z_t\|^2 + \|z_t - y_t\|^2 \le tM + 2\mu\|Bx_t - Bx^*\|\|x_t - z_t\| + 2\lambda\|Az_t - Ax^*\|\|z_t - y_t\|,$$

and so,

$$\lim_{t\to 0}\|x_t - z_t\| = 0, \qquad \lim_{t\to 0}\|z_t - y_t\| = 0.$$

By (3.1), we obtain

$$\begin{aligned}&\|x_t - x^*\|^2 \\ &\quad = \langle [I - t(F - \gamma f)]y_t - x^*, x_t - x^*\rangle \\ &\quad = \langle [I - t(F - \gamma f)]y_t - [I - t(F - \gamma f)]x^*, x_t - x^*\rangle - t\langle (F - \gamma f)x^*, x_t - x^*\rangle \\ &\quad \le (1 - \rho t)\|y_t - x^*\|\|x_t - x^*\| + t\gamma\|f(y_t) - f(x^*)\|\|x_t - x^*\| - t\langle (F - \gamma f)x^*, x_t - x^*\rangle \\ &\quad \le [1 - (\rho - \gamma\tau)t]\|x_t - x^*\|^2 - t\langle (F - \gamma f)x^*, x_t - x^*\rangle.\end{aligned}$$

It follows that

$$\|x_t - x^*\|^2 \le -\frac{1}{\rho - \gamma\tau}\langle (F - \gamma f)x^*, x_t - x^*\rangle. \tag{3.9}$$

Next, we show that the net $\{x_t\}$ is relatively norm-compact as $t \to 0^+$. In fact, assume that $\{t_n\} \subset (0,1)$ is such that $t_n \to 0^+$ as $n \to \infty$. Put $x_n := x_{t_n}$, $y_n := y_{t_n}$ and $z_n := z_{t_n}$. From (3.9), we have

$$\|x_n - x^*\|^2 \le -\frac{1}{\rho - \gamma\tau}\langle (F - \gamma f)x^*, x_n - x^*\rangle, \quad \forall x^* \in \Omega. \tag{3.10}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $\tilde{x} \in H$.

Next, we prove that $\tilde{x} \in \Omega$. We first show that $\tilde{x} \in EP$. By $z_n = S_\mu(x_n - \mu Bx_n)$, we know that

$$\Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \frac{1}{\mu}\langle y - z_n, z_n - (x_n - \mu Bx_n)\rangle \ge 0, \quad \forall y \in H.$$

It follows from (H2) that

$$\varphi(y) - \varphi(z_n) + \frac{1}{\mu}\langle y - z_n, z_n - (x_n - \mu Bx_n)\rangle \ge \Theta(y, z_n), \quad \forall y \in H,$$

and so,

$$\varphi(y) - \varphi(z_{n_i}) + \left\langle y - z_{n_i}, \frac{z_{n_i} - (x_{n_i} - \mu B x_{n_i})}{\mu} \right\rangle \geq \Theta(y, z_{n_i}), \quad \forall y \in H. \tag{3.11}$$

For any $t \in (0,1]$ and $y \in H$, let $u_t = ty + (1-t)\tilde{x}$. It follows from (3.11) that

$$\begin{aligned} \langle u_t - z_{n_i}, Bu_t \rangle &\geq \langle u_t - z_{n_i}, Bu_t \rangle - \varphi(u_t) + \varphi(z_{n_i}) \\ &\quad - \left\langle u_t - z_{n_i}, \frac{z_{n_i} - (x_{n_i} - \mu B x_{n_i})}{\mu} \right\rangle + \Theta(u_t, z_{n_i}) \\ &= \langle u_t - z_{n_i}, Bu_t - Bz_{n_i} \rangle + \langle u_t - z_{n_i}, Bz_{n_i} - Bx_{n_i} \rangle - \varphi(u_t) \\ &\quad + \varphi(z_{n_i}) - \left\langle u_t - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\mu} \right\rangle + \Theta(u_t, z_{n_i}). \end{aligned}$$

Since $\|z_{n_i} - x_{n_i}\| \to 0$, we have $\|Bz_{n_i} - Bx_{n_i}\| \to 0$. Further, by the monotonicity of B, we have $\langle u_t - z_{n_i}, Bu_t - Bz_{n_i} \rangle \geq 0$. Thus, from (H4) and the weakly lower semi-continuity of φ, $\frac{z_{n_i} - x_{n_i}}{\mu} \to 0$ and $z_{n_i} \to \tilde{x}$ weakly, it follows that

$$\langle u_t - \tilde{x}, Bu_t \rangle \geq -\varphi(u_t) + \varphi(\tilde{x}) + \Theta(u_t, \tilde{x}). \tag{3.12}$$

From conditions (H1), (H4) and (3.12), we also have

$$\begin{aligned} 0 &= \Theta(u_t, u_t) + \varphi(u_t) - \varphi(u_t) \\ &\leq t\Theta(u_t, y) + (1-t)\Theta(u_t, \tilde{x}) + t\varphi(y) + (1-t)\varphi(\tilde{x}) - \varphi(u_t) \\ &= t\big[\Theta(u_t, y) + \varphi(y) - \varphi(u_t)\big] + (1-t)\big[\Theta(u_t, \tilde{x}) + \varphi(\tilde{x}) - \varphi(u_t)\big] \\ &\leq t\big[\Theta(u_t, y) + \varphi(y) - \varphi(u_t)\big] + (1-t)\langle u_t - \tilde{x}, Bu_t \rangle \\ &= t\big[\Theta(u_t, y) + \varphi(y) - \varphi(u_t)\big] + (1-t)t\langle y - \tilde{x}, Bu_t \rangle, \end{aligned}$$

and hence

$$0 \leq \Theta(u_t, y) + \varphi(y) - \varphi(u_t) + (1-t)\langle y - \tilde{x}, Bu_t \rangle.$$

Letting $t \to 0$, we have

$$\Theta(\tilde{x}, y) + \varphi(y) - \varphi(\tilde{x}) + \langle y - \tilde{x}, B\tilde{x} \rangle \geq 0, \quad \forall y \in H.$$

This implies that $\tilde{x} \in EP$.

Next, we show that $\tilde{x} \in I(A, R)$. In fact, since A is α-inverse strongly monotone, A is a Lipschitz continuous monotone mapping. It follows from Lemma 2.2 that $R + A$ is maximal monotone. Let $(v, g) \in G(R + A)$, *i.e.*, $g - Av \in R(v)$. Again, since $y_{n_i} = J_{R,\lambda}(z_{n_i} - \lambda A z_{n_i})$, we have $z_{n_i} - \lambda A z_{n_i} \in (I + \lambda R)(y_{n_i})$, *i.e.*, $\frac{1}{\lambda}(z_{n_i} - y_{n_i} - \lambda A z_{n_i}) \in R(y_{n_i})$. By virtue of the maximal monotonicity of R, we have

$$\left\langle v - y_{n_i}, g - Av - \frac{1}{\lambda}(z_{n_i} - y_{n_i} - \lambda A z_{n_i}) \right\rangle \geq 0,$$

and so,

$$\begin{aligned}\langle v - y_{n_i}, g\rangle &\geq \left\langle v - y_{n_i}, Av + \frac{1}{\lambda}(z_{n_i} - y_{n_i} - \lambda Az_{n_i})\right\rangle \\ &= \left\langle v - y_{n_i}, Av - Ay_{n_i} + Ay_{n_i} - Az_{n_i} + \frac{1}{\lambda}(z_{n_i} - y_{n_i})\right\rangle \\ &\geq \langle v - y_{n_i}, Ay_{n_i} - Az_{n_i}\rangle + \left\langle v - y_{n_i}, \frac{1}{\lambda}(z_{n_i} - y_{n_i})\right\rangle.\end{aligned}$$

It follows from $\|z_n - y_n\| \to 0$, $\|Az_n - Ay_n\| \to 0$ and $y_{n_i} \to \tilde{x}$ weakly that

$$\lim_{n_i\to\infty} \langle v - y_{n_i}, g\rangle = \langle v - \tilde{x}, g\rangle \geq 0.$$

It follows from the maximal monotonicity of $A + R$ that $\theta \in (R + A)(\tilde{x})$, *i.e.*, $\tilde{x} \in I(A,R)$. Hence $\tilde{x} \in \Omega$. Therefore, if we can substitute $\tilde{x}$ for x^* in (3.10), then we get

$$\|x_n - \tilde{x}\|^2 \leq -\frac{1}{\rho - \gamma\tau}\big\langle (F - \gamma f)\tilde{x}, x_n - \tilde{x}\big\rangle. \tag{3.13}$$

Consequently, the weak convergence of $\{x_n\}$ to $\tilde{x}$ actually implies that $x_n \to \tilde{x}$ strongly. This shows the relative norm-compactness of the net $\{x_t\}$ as $t \to 0^+$.

Now, we return to (3.10). If we take the limit as $n \to \infty$ in (3.10), then we get

$$\big\|\tilde{x} - x^*\big\|^2 \leq -\frac{1}{\rho - \gamma\tau}\big\langle (F - \gamma f)x^*, \tilde{x} - x^*\big\rangle, \quad \forall x^* \in \Omega.$$

In particular, $\tilde{x}$ solves the following variational inequality

$$\tilde{x} \in \Omega, \quad \big\langle (F - \gamma f)x^*, x^* - \tilde{x}\big\rangle \geq 0, \quad \forall x^* \in \Omega. \tag{3.14}$$

We know that the variational inequality (3.14) is equivalent to its dual variational inequality

$$\tilde{x} \in \Omega, \quad \big\langle (F - \gamma f)\tilde{x}, x^* - \tilde{x}\big\rangle \geq 0, \quad \forall x^* \in \Omega.$$

Thus, by the uniqueness of the variational inequality, we deduce that the entire net $\{x_t\}$ converges in norm to $\tilde{x}$ as $t \to 0^+$. This completes the proof. □

Next, we introduce an explicit scheme and prove its strong convergence to the unique solution of the variational inequality (3.2).

Algorithm 3.5 For any $x_0 \in H$, define the sequence $\{x_n\}$ generated iteratively by

$$x_{n+1} = \big[I - \alpha_n(F - \gamma f)\big]J_{R,\lambda}(I - \lambda A)S_\mu(I - \mu B)x_n, \quad \forall n \geq 0, \tag{3.15}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$.

Theorem 3.6 *Assume the following conditions are also satisfied*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C2) $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Then the sequence $\{x_n\}$ *generated by* (3.15) *converges strongly to the unique solution* $\tilde{x} \in \Omega$ *of the variational inequality* (3.2).

Proof We write $z_n = S_\mu(I - \mu B)x_n$ and $y_n = J_{R,\lambda}(z_n - \lambda A z_n)$ for all $n \geq 0$. Then it follows from Lemma 2.3 that, for any $x^* \in \Omega$,

$$\begin{aligned}\|y_n - x^*\| &= \|J_{R,\lambda}(z_n - \lambda A z_n) - J_{R,\lambda}(x^* - \lambda A x^*)\| \\ &\leq \|(z_n - \lambda A z_n) - (x^* - \lambda A x^*)\| \\ &\leq \|z_n - x^*\|\end{aligned}$$

and

$$\begin{aligned}\|z_n - x^*\|^2 &= \|S_\mu(x_n - \mu B x_n) - S_\mu(x^* - \mu B x^*)\|^2 \\ &\leq \|x_n - \mu B x_n - (x^* - \mu B x^*)\|^2 \\ &\leq \|x_n - x^*\|^2 + \mu(\mu - 2\beta)\|Bx_n - Bx^*\|^2 \\ &\leq \|x_n - x^*\|^2. \end{aligned} \tag{3.16}$$

Hence we have

$$\|y_n - x^*\| \leq \|x_n - x^*\|.$$

By induction, it follows from (3.15) that

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|[I - \alpha_n(F - \gamma f)]y_n - x^*\| \\ &\leq \|[I - \alpha_n(F - \gamma f)]y_n - [I - \alpha_n(F - \gamma f)]x^*\| + \alpha_n\|(F - \gamma f)x^*\| \\ &\leq \|(I - \alpha_n F)(y_n - x^*)\| + t\gamma\|f(y_n) - f(x^*)\| + \alpha_n\|(F - \gamma f)x^*\| \\ &\leq (1 - \rho\alpha_n)\|y_n - x^*\| + \alpha_n\gamma\tau\|y_n - x^*\| + \alpha_n\|(F - \gamma f)x^*\| \\ &\leq [1 - (\rho - \gamma\tau)\alpha_n]\|x_n - x^*\| + (\rho - \gamma\tau)\alpha_n\frac{\|(F - \gamma f)x^*\|}{\rho - \gamma\tau} \\ &\leq \max\left\{\|x_n - x^*\|, \frac{\|(F - \gamma f)x^*\|}{\rho - \gamma\tau}\right\} \\ &\leq \max\left\{\|x_0 - x^*\|, \frac{\|(F - \gamma f)x^*\|}{\rho - \gamma\tau}\right\}.\end{aligned}$$

Therefore, $\{x_n\}$ is bounded, and so, $\{z_n\}$, $\{y_n\}$, $\{Fy_n\}$ and $\{fy_n\}$ are all bounded. It follows from (3.15) that

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| &= \|[I - \alpha_{n+1}(F - \gamma f)]y_{n+1} - [I - \alpha_n(F - \gamma f)]y_n\| \\ &\leq \|[I - \alpha_{n+1}(F - \gamma f)]y_{n+1} - [I - \alpha_{n+1}(F - \gamma f)]y_n\| \\ &\quad + |\alpha_{n+1} - \alpha_n|\|(F - \gamma f)y_n\|\end{aligned}$$

$$\begin{aligned}
&\le \|I-\alpha_{n+1}F\|\|y_{n+1}-y_n\| + \alpha_{n+1}\gamma\|f(y_{n+1})-f(y_n)\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\|(F-\gamma f)y_n\| \\
&\le (1-\rho\alpha_{n+1})\|y_{n+1}-y_n\| + \alpha_{n+1}\gamma\tau\|y_{n+1}-y_n\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\|(F-\gamma f)y_n\| \\
&= \bigl[1-(\rho-\gamma\tau)\alpha_{n+1}\bigr]\|y_{n+1}-y_n\| + |\alpha_{n+1}-\alpha_n|\|(F-\gamma f)y_n\|. \qquad (3.17)
\end{aligned}$$

Note that

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|J_{R,\lambda}(z_{n+1}-\lambda Az_{n+1}) - J_{R,\lambda}(z_n-\lambda Az_n)\| \\
&\le \|z_{n+1}-\lambda Az_{n+1}-(z_n-\lambda Az_n)\| \\
&\le \|z_{n+1}-z_n\| \\
&= \|S_\mu(x_{n+1}-\mu Bx_{n+1}) - S_\mu(x_n-\mu Bx_n)\| \\
&\le \|(x_{n+1}-\mu Bx_{n+1})-(x_n-\mu Bx_n)\| \\
&\le \|x_{n+1}-x_n\|. \qquad (3.18)
\end{aligned}$$

Substituting (3.18) into (3.17), we get

$$\begin{aligned}
&\|x_{n+2}-x_{n+1}\| \\
&\quad\le \bigl[1-(\rho-\gamma\tau)\alpha_{n+1}\bigr]\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\|(F-\gamma f)y_n\| \\
&\quad= \bigl[1-(\rho-\gamma\tau)\alpha_{n+1}\bigr]\|y_{n+1}-y_n\| + (\rho-\gamma\tau)\alpha_{n+1}\frac{|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}}\frac{\|(F-\gamma f)y_n\|}{\rho-\gamma\tau}.
\end{aligned}$$

Notice that $\lim_{n\to\infty}|1-\frac{\alpha_n}{\alpha_{n+1}}| = 0$. This, together with the last inequality and Lemma 2.4, implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

Again, using Lemma 2.3 and (3.16), we get

$$\begin{aligned}
&\|y_n-x^*\|^2 \\
&\quad= \|J_{R,\lambda}(z_n-\lambda Az_n) - J_{R,\lambda}(x^*-\lambda Ax^*)\|^2 \\
&\quad\le \|(z_n-\lambda Az_n)-(x^*-\lambda Ax^*)\|^2 \\
&\quad\le \|z_n-x^*\|^2 + \lambda(\lambda-2\alpha)\|Az_n-Ax^*\|^2 \\
&\quad\le \|x_n-x^*\|^2 + \mu(\mu-2\beta)\|Bx_n-Bx^*\|^2 + \lambda(\lambda-2\alpha)\|Az_n-Ax^*\|^2. \qquad (3.19)
\end{aligned}$$

By (3.15), we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|[I-\alpha_n(F-\gamma f)]y_n-x^*\|^2 \\
&\le \bigl[\|y_n-x^*\| + \alpha_n\|(F-\gamma f)y_n\|\bigr]^2
\end{aligned}$$

$$= \|y_n - x^*\|^2 + \alpha_n\big[2\|y_n - x^*\|\|(F-\gamma f)y_n\| + \alpha_n\|(F-\gamma f)y_n\|^2\big]$$
$$\le \|y_n - x^*\|^2 + \alpha_n M_1, \tag{3.20}$$

where $M_1 > 0$ is a constant satisfying

$$\sup\big\{2\|y_n - x^*\|\|(F-\gamma f)y_n\| + \alpha_n\|(F-\gamma f)y_n\|^2 : n \ge 1\big\} \le M_1.$$

From (3.19) and (3.20), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \le{}& \|x_n - x^*\|^2 + \mu(\mu - 2\beta)\|Bx_n - Bx^*\|^2 \\ &+ \lambda(\lambda - 2\alpha)\|Az_n - Ax^*\|^2 + \alpha_n M_1,\end{aligned}$$

and so,

$$\begin{aligned}&\mu(2\beta - \mu)\|Bx_n - Bx^*\|^2 + \lambda(2\alpha - \lambda)\|Az_n - Ax^*\|^2 \\ &\quad\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n M_1 \\ &\quad\le \big(\|x_n - x^*\| + \|x_{n+1} - x^*\|\big)\|x_{n+1} - x_n\| + \alpha_n M_1,\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|Bx_n - Bx^*\| = 0, \qquad \lim_{n\to\infty}\|Az_n - Ax^*\| = 0.$$

Since S_r is firmly nonexpansive, we have

$$\begin{aligned}\|z_n - x^*\|^2 &= \|S_\mu(x_n - \mu Bx_n) - S_\mu(x^* - \mu Bx^*)\|^2 \\ &\le \langle x_n - \mu Bx_n - (x^* - \mu Bx^*), z_n - x^*\rangle \\ &= \frac{1}{2}\big(\|x_n - \mu Bx_n - (x^* - \mu Bx^*)\|^2 + \|z_n - x^*\|^2 \\ &\quad - \|x_n - \mu Bx_n - (x^* - \mu Bx^*) - (z_n - x^*)\|^2\big) \\ &\le \frac{1}{2}\big(\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|x_n - z_n - \mu(Bx_n - Bx^*)\|^2\big) \\ &= \frac{1}{2}\big(\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|x_n - z_n\|^2 \\ &\quad + 2\mu\langle Bx_n - Bx^*, x_n - z_n\rangle - \mu^2\|Bx_n - Bx^*\|^2\big),\end{aligned}$$

and hence

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\mu\|Bx_n - Bx^*\|\|x_n - z_n\|. \tag{3.21}$$

Since $J_{R,\lambda}$ is 1-inverse strongly monotone, we have

$$\begin{aligned}\|y_n - x^*\|^2 &= \|J_{R,\lambda}(z_n - \lambda Az_n) - J_{R,\lambda}(x^* - \lambda Ax^*)\|^2 \\ &\le \langle z_n - \lambda Az_n - (x^* - \lambda Ax^*), y_n - x^*\rangle\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\Big(\big\|z_n - \lambda Az_n - \big(x^* - \lambda Ax^*\big)\big\|^2 + \big\|y_n - x^*\big\|^2 \\
&\quad - \big\|z_n - \lambda Az_n - \big(x^* - \lambda Ax^*\big) - \big(y_n - x^*\big)\big\|^2\Big) \\
&\le \frac{1}{2}\Big(\big\|z_n - x^*\big\|^2 + \big\|y_n - x^*\big\|^2 - \big\|z_n - y_n - \lambda\big(Az_n - Ax^*\big)\big\|^2\Big) \\
&= \frac{1}{2}\Big(\big\|z_n - x^*\big\|^2 + \big\|y_n - x^*\big\|^2 - \|z_n - y_n\|^2 \\
&\quad + 2\lambda\big\langle Az_n - Ax^*, z_n - y_n\big\rangle - \lambda^2\big\|Az_n - Ax^*\big\|^2\Big),
\end{aligned}$$

which implies that

$$\big\|y_n - x^*\big\|^2 \le \big\|z_n - x^*\big\|^2 - \|z_n - y_n\|^2 + 2\lambda\big\|Az_n - Ax^*\big\|\|z_n - y_n\|. \tag{3.22}$$

Thus, by (3.21) and (3.22), we obtain

$$\begin{aligned}
\big\|y_n - x^*\big\|^2 &\le \big\|x_n - x^*\big\|^2 - \|x_n - z_n\|^2 + 2\mu\big\|Bx_n - Bx^*\big\|\|x_n - z_n\| \\
&\quad - \|z_n - y_n\|^2 + 2\lambda\big\|Az_n - Ax^*\big\|\|z_n - y_n\|.
\end{aligned} \tag{3.23}$$

Substituting (3.20) into (3.23), we get

$$\begin{aligned}
\big\|y_n - x^*\big\|^2 &\le \big\|y_n - x^*\big\|^2 + \alpha_n M_1 - \|x_n - z_n\|^2 + 2\mu\big\|Bx_n - Bx^*\big\|\|x_n - z_n\| \\
&\quad - \|z_n - y_n\|^2 + 2\lambda\big\|Az_n - Ax^*\big\|\|z_n - y_n\|.
\end{aligned}$$

Thus, we derive

$$\begin{aligned}
&\|x_n - z_n\|^2 + \|z_n - y_n\|^2 \\
&\quad \le \alpha_n M_1 + 2\mu\big\|Bx_n - Bx^*\big\|\|x_n - z_n\| + 2\lambda\big\|Az_n - Ax^*\big\|\|z_n - y_n\|,
\end{aligned}$$

and so,

$$\lim_{n\to\infty}\|x_n - z_n\| = 0, \qquad \lim_{n\to\infty}\|z_n - y_n\| = 0.$$

Next, we prove that

$$\limsup_{n\to\infty} -\frac{2}{\rho - \gamma\tau}\big\langle (F - \gamma f)\tilde{x}, x_n - \tilde{x}\big\rangle \le 0, \tag{3.24}$$

where $\tilde{x}$ is the unique solution of the variational inequality (3.2). To see this, we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying

$$\limsup_{n\to\infty} -\frac{2}{\rho - \gamma\tau}\big\langle (F - \gamma f)\tilde{x}, x_n - \tilde{x}\big\rangle = \lim_{k\to\infty} -\frac{2}{\rho - \gamma\tau}\big\langle (F - \gamma f)\tilde{x}, x_{n_k} - \tilde{x}\big\rangle \tag{3.25}$$

and $\{x_{n_k}\}$ converges weakly to a point x^* as $k \to \infty$. By the similar argument as in Theorem 3.3, we can deduce $x^* \in \Omega$. Since $\tilde{x}$ solves the variational inequality (3.2), by combining

(3.24) and (3.25), we get

$$\limsup_{n\to\infty} -\frac{2}{\rho-\gamma\tau}\langle (F-\gamma f)\tilde{x}, x_n-\tilde{x}\rangle = -\frac{2}{\rho-\gamma\tau}\langle (F-\gamma f)\tilde{x}, x^*-\tilde{x}\rangle \le 0.$$

Finally, we show that $x_n \to \tilde{x}$ as $n\to\infty$. It follows from (3.15) that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \langle x_{n+1}-x^*, x_{n+1}-x^*\rangle \\
&= \langle [I-\alpha_n(F-\gamma f)]y_n - x^*, x_{n+1}-x^*\rangle \\
&= \langle [I-\alpha_n(F-\gamma f)]y_n - [I-\alpha_n(F-\gamma f)]x^*, x_{n+1}-x^*\rangle \\
&\quad - \alpha_n\langle (F-\gamma f)x^*, x_{n+1}-x^*\rangle \\
&\le (1-\alpha_n\rho)\|y_n-x^*\|\|x_{n+1}-x^*\| + \alpha_n\gamma\|f(y_n)-f(x^*)\|\|x_{n+1}-x^*\| \\
&\quad - \alpha_n\langle (F-\gamma f)x^*, x_{n+1}-x^*\rangle \\
&\le [1-(\rho-\gamma\tau)\alpha_n]\|y_n-x^*\|\|x_{n+1}-x^*\| - \alpha_n\langle (F-\gamma f)x^*, x_{n+1}-x^*\rangle \\
&\le [1-(\rho-\gamma\tau)\alpha_n]\|x_n-x^*\|\|x_{n+1}-x^*\| - \alpha_n\langle (F-\gamma f)x^*, x_{n+1}-x^*\rangle \\
&\le \frac{1-(\rho-\gamma\tau)\alpha_n}{2}\|x_n-x^*\|^2 + \frac{1}{2}\|x_{n+1}-x^*\|^2 \\
&\quad - \alpha_n\langle (F-\gamma f)x^*, x_{n+1}-x^*\rangle,
\end{aligned}$$

that is,

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le [1-(\rho-\gamma\tau)\alpha_n]\|x_n-x^*\|^2 - 2\alpha_n\langle (F-\gamma f)x^*, x_{n+1}-x^*\rangle \\
&= (1-\delta_n)\|x_n-x^*\|^2 + \delta_n\sigma_n,
\end{aligned}$$

where $\delta_n = (\rho-\gamma\tau)\alpha_n$ and $\sigma_n = -\frac{2}{\rho-\gamma\tau}\langle (F-\gamma f)x^*, x_{n+1}-x^*\rangle$. It is easy to see that $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \le 0$. Hence, by Lemma 2.4, we conclude that the sequence $\{x_n\}$ converges strongly to the point x^*. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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