Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Boundedness of sublinear operators and their commutators on generalized central Morrey spaces

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Abstract

In this paper, we introduce the generalized central Morrey spaces $\dot{B}^{p,\varphi}(\mathbb{R}^n)$ and get the boundedness of a large class of rough operators on them. We also consider the CBMO estimates of their commutators on generalized central Morrey spaces. As applications, we obtain the boundedness characterizations of rough Hardy-Littlewood maximal function, rough Calderón-Zygmund singular integral, rough fractional integral, *etc.* on generalized central Morrey spaces.

MSC: 42B20; 42B25

Keywords: sublinear operators; commutator; central Morrey spaces; rough kernel

1 Introduction

Let $\Omega \in L^s(S^{n-1})$ be homogeneous of degree zero on $\mathbb{R}^n$, where S^{n-1} denotes the unit sphere of $\mathbb{R}^n$ and $s > 1$. We define $s' = s/(s-1)$ for any $s > 1$. Suppose that T_Ω represents a sublinear operator, which satisfies that for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$,

$$\left|T_\Omega f(x)\right| \le C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} \left|f(y)\right| dy, \tag{1.1}$$

where $C > 0$ is an absolute constant. Similarly, for any $0 < \alpha < n$, we assume that $T_{\Omega,\alpha}$ represents a sublinear operator, which satisfies that

$$\left|T_{\Omega,\alpha} f(x)\right| \le C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} \left|f(y)\right| dy \tag{1.2}$$

for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$.

Let T be a linear operator. For a locally integrable function b on $\mathbb{R}^n$, we define the commutator $[T, b]$ by

$$[T, b]f(x) = b(x)Tf(x) - T(bf)(x) \tag{1.3}$$

for any suitable function f.

To study the local behavior of solutions to second-order elliptic partial differential equations, Morrey [1] introduced the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$. The readers can find more details in [2].

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Let $1 \leq p < \infty$ and $\lambda \geq 0$. $B(x_0, t)$ denotes a ball centered at x_0 of radius t. Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ are defined by

$$L^{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x_0 \in \mathbb{R}^n, t>0} \left(\frac{1}{t^\lambda} \int_{B(x_0,t)} |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

When $1 \leq p < \infty$, $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$; when $\lambda > n$, $L^{p,\lambda}(\mathbb{R}^n) = \{0\}$.

Many authors have studied the mapping properties of many operators on Morrey spaces; see [3–5] and [6]. Alvarez *et al.* [7], in order to study the relationship between central *BMO* spaces and Morrey spaces, introduced λ-central bounded mean oscillation spaces and central Morrey spaces.

Let $\lambda < \frac{1}{n}$ and $1 < p < \infty$. A function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to the λ-central bounded mean oscillation spaces $CBMO^{p,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{CBMO^{p,\lambda}} = \sup_{r>0} \left(\frac{1}{|B(0,r)|^{1+\lambda p}} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^p\right)^{\frac{1}{p}} < \infty,$$

where $f_{B(0,r)} = \frac{1}{|B(0,r)|} \int_{B(0,r)} f(y)\, dy$. If two functions which differ by a constant are regarded as functions in the spaces $CBMO^{p,\lambda}(\mathbb{R}^n)$, then $CBMO^{p,\lambda}(\mathbb{R}^n)$ spaces become Banach spaces. $CBMO^{p,\lambda}(\mathbb{R}^n)$ spaces become the spaces of constants when $\lambda < -\frac{1}{p}$ and they coincide with $L^p(\mathbb{R}^n)$ modulo constants when $\lambda = -\frac{1}{p}$.

Let $\lambda \in \mathbb{R}$ and $1 < p < \infty$. The central Morrey spaces $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ are defined by

$$\|f\|_{\dot{B}^{p,\lambda}} = \sup_{r>0} \left(\frac{1}{|B(0,r)|^{1+\lambda p}} \int_{B(0,r)} |f(x)|^p\right)^{\frac{1}{p}} < \infty.$$

It follows that $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ spaces are Banach spaces continuously included in $CBMO^{p,\lambda}(\mathbb{R}^n)$ spaces. $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ spaces reduce to $\{0\}$ when $\lambda < -\frac{1}{p}$, and it is true that $\dot{B}^{p,-\frac{1}{p}}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, $\dot{B}^{p,0}(\mathbb{R}^n) = \dot{B}^p(\mathbb{R}^n)$.

Recently, Guliyev [8] introduced the generalized Morrey spaces $L^{p,\varphi}(\mathbb{R}^n)$, where $\varphi(x,r)$ is a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \leq p < \infty$. For all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, the generalized Morrey spaces $L^{p,\varphi}(\mathbb{R}^n)$ are defined by

$$\|f\|_{L^{p,\varphi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L^p(B(x,r))} < \infty.$$

Obviously, if $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$, $L^{p,\lambda}(\mathbb{R}^n) = L^{p,\varphi}(\mathbb{R}^n)$.

When $\Omega \equiv 1$, Guliyev obtained the sufficient condition on φ_1 and φ_2

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\, dt \leq C\varphi_2(r)$$

for the boundedness of T_Ω satisfying (1.1) from $L^{p,\varphi_1}(\mathbb{R}^n)$ to $L^{p,\varphi_2}(\mathbb{R}^n)$ in [9] and gave the condition on the pair of (φ_1, φ_2)

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}+1}}\,dt \le C\varphi_2(r)$$

for the boundedness of $T_{\Omega,\alpha}$ satisfying (1.2) from $L^{p,\varphi_1}(\mathbb{R}^n)$ to $L^{q,\varphi_2}(\mathbb{R}^n)$ in [10], where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

Inspired by the above, we consider the boundedness of sublinear operators on the following generalized central Morrey spaces and give the λ-central bounded mean oscillation estimates for linear operator commutators.

Definition 1.1 Let $\varphi(r)$ be a positive measurable function on $\mathbb{R}_+$ and $1 < p < \infty$. We denote by $\dot{B}^{p,\varphi}(\mathbb{R}^n)$ the generalized central Morrey spaces, the spaces of all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{\dot{B}^{p,\varphi}(\mathbb{R}^n)} = \sup_{r>0} \varphi(r)^{-1}\bigl|B(0,r)\bigr|^{-\frac{1}{p}}\|f\|_{L^p(B(0,r))}.$$

We can recover the spaces $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ under the choice $\varphi(r) = r^{n\lambda}$.

Recall that in 1994 the doctoral thesis [11] by Guliyev (see also [12–15]) introduced the local Morrey-type space $LM_{p\theta,\omega}$ given by

$$\|f\|_{LM_{p\theta,\omega}} = \bigl\|\omega(r)\|f\|_{L^p(B(0,r))}\bigr\|_{L^\theta(0,\infty)} < \infty,$$

where ω is a positive measurable function defined on $(0,\infty)$. The main purpose of [11] (also of [12–15]) is to give some sufficient conditions for the boundedness of fractional integral operators and singular integral operators defined on homogeneous Lie groups in the local Morrey-type space $LM_{p\theta,\omega}$. In a series of papers by Burenkov, H Guliyev and V Guliyev, *etc.* (see [16–21]), some necessary and sufficient conditions for the boundedness of fractional maximal operators, fractional integral operators and singular integral operators in local Morrey-type spaces $LM_{p\theta,\omega}$ were given.

Particularly, if $\theta = \infty$, $LM_{p\theta,\omega} = LM_{p,\omega}$, then the generalized central Morrey spaces $\dot{B}^{p,\varphi}(\mathbb{R}^n)$ are the same spaces as the local Morrey spaces $LM_{p,\omega}$ with $\omega(r) = \varphi(r)^{-1}r^{-\frac{n}{p}}$.

The following statements were proved in [11] (see also [14]).

Theorem A *Let $1 < p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \varphi_1(t)\frac{dt}{t} \le C\varphi_2(r),$$

where C does not depend on r. Then the Calderón-Zygmund operator T is bounded from $\dot{B}^{p,\varphi_1}$ to $\dot{B}^{p,\varphi_2}$.

Theorem B *Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty t^\alpha\varphi_1(t)\frac{dt}{t} \le C\varphi_2(r),$$

where C does not depend on r. Then the Riesz potential I_α is bounded from $\dot{B}^{p,\varphi_1}$ to $\dot{B}^{q,\varphi_2}$.

From Lemmas 4.4 and 5.3 in [9] we get the following for the generalized central (local) Morrey spaces $\dot{B}^{p,\varphi}$.

Theorem C *Let* $1 < p < \infty$, T *be a sublinear operator satisfying that for any* $f \in L^1(\mathbb{R}^n)$ *with compact support and* $x \notin \operatorname{supp} f$,

$$\left|Tf(x)\right| \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\,dy,$$

and bounded on $f \in L^p(\mathbb{R}^n)$. *Let also the pair* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\,dt \le C\varphi_2(r),$$

where C *does not depend on* r. *Then the operator* T *is bounded from* $\dot{B}^{p,\varphi_1}$ *to* $\dot{B}^{p,\varphi_2}$.

Theorem D *Let* $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, T_α *be a sublinear operator satisfying that for any* $f \in L^1(\mathbb{R}^n)$ *with compact support and* $x \notin \operatorname{supp} f$,

$$\left|T_\alpha f(x)\right| \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy,$$

and bounded from $f \in L^p(\mathbb{R}^n)$ *to* $f \in L^q(\mathbb{R}^n)$. *Let also the pair* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}+1}}\,dt \le C\varphi_2(r),$$

where C *does not depend on* r. *Then the operator* T_α *is bounded from* $\dot{B}^{p,\varphi_1}$ *to* $\dot{B}^{q,\varphi_2}$.

2 Sublinear operator with rough kernel

Theorem E *Let* ω *be a positive weight function on* $(0,\infty)$. *The inequality*

$$\operatorname*{ess\,sup}_{t>0} \nu_2(t) H_\omega g(t) \le c \operatorname*{ess\,sup}_{t>0} \nu_1(t) g(t)$$

holds for all non-negative and non-increasing g *on* $(0,\infty)$ *if and only if*

$$A := \sup_{t>0} \frac{\nu_2(t)}{t} \int_0^t \frac{\omega(r)\,dr}{\operatorname{ess\,sup}_{0<\tau<r} \nu_1(\tau)} < \infty,$$

and $c \approx A$, *where the* H_ω *is the weighted Hardy operator*

$$H_\omega g(t) := \frac{1}{t}\int_0^t g(r)\omega(r)\,dr, \quad 0 < t < \infty.$$

Note that Theorem E can be proved analogously to Theorem 1 in [22]; particularly, when $\omega \equiv 1$, it was proved in [23].

In this section we are going to discuss the boundedness of T_Ω and $T_{\Omega,\alpha}$ on generalized central Morrey spaces.

Lemma 2.1 *Let* $1 < p < \infty$, T_Ω *be a sublinear operator and satisfy* (1.1) *with* $\Omega \in L^s(S^{n-1})$.

When $s' \le p$ and T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, then the inequality

$$\|T_\Omega f\|_{L^p(B(0,r))} \le Cr^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L^p(B(0,t))}\,dt$$

holds for any ball $B(0,r)$ and for all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$; or $p < s$ and T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, then the inequality

$$\|T_\Omega f\|_{L^p(B(0,r))} \le Cr^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty} t^{-\frac{n}{p}+\frac{n}{s}-1} \|f\|_{L^p(B(0,t))}\,dt$$

holds for any ball $B(0,r)$ and for all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof Let $1 < p < \infty$. For any $r > 0$, set $B = B(0,r)$ and $2B = B(0,2r)$. We write

$$f(x) = f(x)\chi_{2B}(x) + f(x)\chi_{(2B)^c}(x) := f_1(x) + f_2(x)$$

and have

$$\|T_\Omega f\|_{L^p(B)} \le \|T_\Omega f_1\|_{L^p(B)} + \|T_\Omega f_2\|_{L^p(B)}.$$

Since $T_\Omega f_1$ is bounded on $L^p(\mathbb{R}^n)$, it follows that

$$\|T_\Omega f_1\|_{L^p(B)} \le \|T_\Omega f_1\|_{L^p(\mathbb{R}^n)} \le C\|f_1\|_{L^p(\mathbb{R}^n)} = C\|f\|_{L^p(2B)},$$

where the constant $C > 0$ is independent of f.

It is known that $x \in B$, $y \in (2B)^c$, which implies $\frac{1}{2}|y| \le |x-y| \le \frac{3}{2}|y|$. Thus

$$\left|T_\Omega f_2(x)\right| \le C \int_{(2B)^c} \left|f(y)\right|\left|\Omega(x-y)\right| \frac{dy}{|y|^n}.$$

(i) When $s' \le p$ and by Fubini's theorem, we have

$$\begin{aligned}
&\int_{(2B)^c} \left|f(y)\right|\left|\Omega(x-y)\right| \frac{dy}{|y|^n} \\
&\quad = C \int_{(2B)^c} \left|f(y)\right|\left|\Omega(x-y)\right| \int_{|y|}^{\infty} \frac{dt}{t^{n+1}}\,dy \\
&\quad \le C \int_{2r}^{\infty} \int_{2r\le|y|<t} \left|f(y)\right|\left|\Omega(x-y)\right| dy \frac{dt}{t^{n+1}} \\
&\quad \le C \int_{2r}^{\infty} \|f\|_{L^p(B(0,t))} \left(\int_{B(0,t)} \left|\Omega(x-y)\right|^s dy\right)^{\frac{1}{s}} \left|B(0,t)\right|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{n+1}} \\
&\quad \le C \int_{2r}^{\infty} \|f\|_{L^p(B(0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.
\end{aligned}$$

Hence, for all $p \in (0,\infty)$, the inequality

$$\|T_\Omega f_2\|_{L^p(B)} \le Cr^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(0,t))} \frac{dt}{t^{\frac{n}{p}+1}}$$

holds.

(ii) When $p < s$, by Fubini's theorem and the Minkowski inequality, we get

$$\begin{aligned}\|T_\Omega f_2\|_{L^p(B)} &\le \left(\int_B \left|\int_{2r}^\infty \int_{B(0,t)} |f(y)||\Omega(x-y)|\,dy \frac{dt}{t^{n+1}}\right|^p dx\right)^{\frac{1}{p}} \\ &\le \int_{2r}^\infty \int_{B(0,t)} |f(y)|\left(\int_B |\Omega(x-y)|^p\,dx\right)^{\frac{1}{p}} dy \frac{dt}{t^{n+1}} \\ &\le C\int_{2r}^\infty \int_{B(0,t)} |f(y)|\left(\int_{B(0,t)} |\Omega(x-y)|^s\,dx\right)^{\frac{1}{s}} |B|^{\frac{1}{p}-\frac{1}{s}}\,dy \frac{dt}{t^{n+1}} \\ &\le Cr^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^\infty \int_{B(0,t)} |f(y)|\,dy \frac{dt}{t^{n-\frac{n}{s}+1}} \\ &\le Cr^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^\infty \|f\|_{L^p(B(0,t))} \frac{dt}{t^{\frac{n}{p}-\frac{n}{s}+1}}.\end{aligned}$$

On the other hand, for any $q > 0$, we have

$$\|f\|_{L^p(2B)} = Cr^{\frac{n}{q}} \|f\|_{L^p(2B)} \int_{2r}^\infty \frac{dt}{t^{\frac{n}{q}+1}} \le Cr^{\frac{n}{q}} \int_{2r}^\infty \|f\|_{L^p(B(0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

Combining the above estimates, we complete the proof of Lemma 2.1. □

Theorem 2.2 *Let $1 < p < \infty$ and $\Omega \in L^s(S^{n-1})$. Let T_Ω be a sublinear operator satisfying* (1.1) *and bounded on $L^p(\mathbb{R}^n)$ for $p > 1$. If either of the two conditions*

(i) *when $s' \le p$, (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\,dt \le C\varphi_2(r),$$

(ii) *when $p < s$, (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}-\frac{n}{s}+1}}\,dt \le C\varphi_2(r)r^{\frac{n}{s}}$$

is satisfied, then the operator T_Ω is bounded from $\dot{B}^{p,\varphi_1}$ to $\dot{B}^{p,\varphi_2}$.

Proof When $s' \le p$, by Lemma 2.1 and Theorem E, for $\omega \equiv 1$, $p > 1$, we have

$$\begin{aligned}\|T_\Omega f\|_{\dot{B}^{p,\varphi_2}} &\le C\sup_{r>0} \varphi_2(r)^{-1} \int_r^\infty \|f\|_{L^p(B(0,t))} \frac{dt}{t^{\frac{n}{p}+1}} \\ &= C\sup_{r>0} \varphi_2(r)^{-1} \int_0^{r^{-\frac{n}{p}}} \|f\|_{L^p(B(0,t^{-\frac{p}{n}}))}\,dt \\ &= C\sup_{r>0} \varphi_2\big(r^{-\frac{p}{n}}\big)^{-1} \int_0^r \|f\|_{L^p(B(0,t^{-\frac{p}{n}}))}\,dt \\ &\le C\sup_{r>0} \varphi_1\big(r^{-\frac{p}{n}}\big)^{-1} r\|f\|_{L^p(B(0,r^{-\frac{p}{n}}))} = C\|f\|_{\dot{B}^{p,\varphi_1}}.\end{aligned}$$

For the case of $p < s$, we can use the same method to prove the desirable conclusion. □

The Calderón-Zygmund operator with rough kernel $\tilde{T}_\Omega$ has the following integral expression:

$$\tilde{T}_\Omega f(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\Omega(y)\,dy$$

for any test function f and $x \notin \operatorname{supp} f$. The kernel is a locally integral function defined away from the diagonal satisfying the size condition

$$K(x,y) \le C\frac{1}{|x-y|^n}$$

for all $x, y \in \mathbb{R}^n$ and $x \neq y$.

$f \in L^1_{\mathrm{loc}}$, the rough Hardy-Littlewood maximal function M_Ω is defined by

$$M_\Omega f(x) = \sup_{t>0} \frac{1}{|B(x,t)|} \int_{B(x,t)} \bigl|\Omega(y)\bigr|\bigl|f(y)\bigr|\,dy.$$

Then we can get the following corollary.

Corollary 2.3 *Let $1 < p < \infty$ and $\Omega \in L^s(S^{n-1})$. If either of the two conditions*

(i) *when $s' \le p$, (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} \le C\varphi_2(r),$$

(ii) *when $p < s$, (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}-\frac{n}{s}+1}} \le C\varphi_2(r)r^{\frac{n}{s}}$$

is satisfied, then M_Ω and $\tilde{T}_\Omega$ are both bounded from $\dot{B}^{p,\varphi_1}$ to $\dot{B}^{p,\varphi_2}$.

In the following statements, the boundedness of $T_{\Omega,\alpha}$ satisfying (1.2) in generalized central Morrey spaces is proved.

Lemma 2.4 *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $T_{\Omega,\alpha}$ be a sublinear operator and satisfy (1.2) with $\Omega \in L^s(S^{n-1})$.*

When $s' \le p$ and $T_{\Omega,\alpha}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then the inequality

$$\|T_{\Omega,\alpha} f\|_{L^q(B(0,r))} \le Cr^{\frac{n}{q}} \int_{2r}^\infty t^{-\frac{n}{q}-1} \|f\|_{L^p(B(0,t))}\,dt$$

holds for any ball $B(0,r)$ and for all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$; or $q < s$ and $T_{\Omega,\alpha}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then the inequality

$$\|T_{\Omega,\alpha} f\|_{L^q(B(0,r))} \le Cr^{\frac{n}{q}-\frac{n}{s}} \int_{2r}^\infty t^{-\frac{n}{q}+\frac{n}{s}-1} \|f\|_{L^p(B(0,t))}\,dt$$

holds for any ball $B(0,r)$ and for all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. For any $r > 0$, set $B = B(0,r)$ and $2B = B(0,2r)$. We write

$$f(x) = f(x)\chi_{2B}(x) + f(x)\chi_{(2B)^c}(x) := f_1(x) + f_2(x)$$

and have

$$\|T_{\Omega,\alpha}f\|_{L^q(B)} \le \|T_{\Omega,\alpha}f_1\|_{L^q(B)} + \|T_{\Omega,\alpha}f_2\|_{L^q(B)}.$$

Since $T_{\Omega,\alpha}f_1$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, it follows that

$$\|T_{\Omega,\alpha}f_1\|_{L^q(B)} \le \|T_{\Omega,\alpha}f_1\|_{L^q(\mathbb{R}^n)} \le C\|f_1\|_{L^p(\mathbb{R}^n)} = C\|f\|_{L^p(2B)},$$

where the constant $C > 0$ is independent of f.

Since $x \in B$, $y \in (2B)^c$, thus

$$\left|T_{\Omega,\alpha}f_2(x)\right| \le C\int_{(2B)^c} \left|f(y)\right|\left|\Omega(x-y)\right|\frac{dy}{|y|^{n-\alpha}}.$$

(i) When $s' \le p$ and by Fubini's theorem, we have

$$\begin{aligned}
&\int_{(2B)^c} \left|f(y)\right|\left|\Omega(x-y)\right|\frac{dy}{|y|^{n-\alpha}} \\
&\quad = C\int_{(2B)^c} \left|f(y)\right|\left|\Omega(x-y)\right|\int_{|y|}^{\infty}\frac{dt}{t^{n+1-\alpha}}\,dy \\
&\quad \le C\int_{2r}^{\infty}\int_{2r\le|y|<t}\left|f(y)\right|\left|\Omega(x-y)\right|\,dy\frac{dt}{t^{n+1-\alpha}} \\
&\quad \le C\int_{2r}^{\infty}\|f\|_{L^p(B(0,t))}\left(\int_{B(0,t)}\left|\Omega(x-y)\right|^s\,dy\right)^{\frac{1}{s}}\left|B(0,t)\right|^{1-\frac{1}{p}-\frac{1}{s}}\frac{dt}{t^{n+1-\alpha}} \\
&\quad \le C\int_{2r}^{\infty}\|f\|_{L^p(B(0,t))}\frac{dt}{t^{\frac{n}{q}+1}}.
\end{aligned}$$

Hence, for all $p \in (0,\infty)$, the inequality

$$\|T_{\Omega,\alpha}f_2\|_{L^q(B)} \le Cr^{\frac{n}{q}}\int_{2r}^{\infty}\|f\|_{L^p(B(0,t))}\frac{dt}{t^{\frac{n}{q}+1}}$$

holds.

(ii) When $q < s$, by Fubini's theorem and the Minkowski inequality, we get

$$\begin{aligned}
\|T_{\Omega,\alpha}f_2\|_{L^p(B)} &\le \left(\int_B\left|\int_{2r}^{\infty}\int_{B(0,t)}\left|f(y)\right|\left|\Omega(x-y)\right|\,dy\frac{dt}{t^{n+1-\alpha}}\right|^q dx\right)^{\frac{1}{q}} \\
&\le \int_{2r}^{\infty}\int_{B(0,t)}\left|f(y)\right|\left(\int_B\left|\Omega(x-y)\right|^q\,dx\right)^{\frac{1}{q}}dy\frac{dt}{t^{n+1-\alpha}} \\
&\le C\int_{2r}^{\infty}\int_{B(0,t)}\left|f(y)\right|\left(\int_{B(0,t)}\left|\Omega(x-y)\right|^s\,dx\right)^{\frac{1}{s}}|B|^{\frac{1}{q}-\frac{1}{s}}\,dy\frac{dt}{t^{n+1-\alpha}}
\end{aligned}$$

$$\leq Cr^{\frac{n}{q}-\frac{n}{s}} \int_{2r}^{\infty} \int_{B(0,t)} |f(y)| \, dy \frac{dt}{t^{n-\frac{n}{s}+1-\alpha}}$$

$$\leq Cr^{\frac{n}{q}-\frac{n}{s}} \int_{2r}^{\infty} \|f\|_{L^p(B(0,t))} \frac{dt}{t^{\frac{n}{q}-\frac{n}{s}+1}}.$$

Similarly, combining the above estimates, we finish this proof. □

Theorem 2.5 *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $\Omega \in L^s(S^{n-1})$. *Let* $T_{\Omega,\alpha}$ *be a sublinear operator* T_Ω *satisfying* (1.2) *and bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$. *If either of the two conditions*

(i) *when* $s' \leq p$, (φ_1, φ_2) *satisfies the condition*

$$\int_r^{\infty} \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} \, dt \leq C\varphi_2(r),$$

(ii) *when* $q < s$, (φ_1, φ_2) *satisfies the condition*

$$\int_r^{\infty} \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}} \, dt \leq C\varphi_2(r) r^{\frac{n}{s}}$$

is satisfied, then the operator $T_{\Omega,\alpha}$ *is bounded from* $\dot{B}^{p,\varphi_1}$ *to* $\dot{B}^{q,\varphi_2}$.

Proof When $s' \leq p$, by Lemma 2.1 and Theorem E, for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, we have

$$\begin{aligned}
\|T_{\Omega,\alpha} f\|_{\dot{B}^{q,\varphi_2}} &\leq C \sup_{r>0} \varphi_2(r)^{-1} \int_r^{\infty} \|f\|_{L^p(B(0,t))} \frac{dt}{t^{\frac{n}{q}+1}} \\
&= C \sup_{r>0} \varphi_2(r)^{-1} \int_0^{r^{-\frac{n}{q}}} \|f\|_{L^p(B(0,t^{-\frac{q}{n}}))} \, dt \\
&= C \sup_{r>0} \varphi_2\bigl(r^{-\frac{q}{n}}\bigr)^{-1} \int_0^{r} \|f\|_{L^p(B(0,t^{-\frac{q}{n}}))} \, dt \\
&\leq C \sup_{r>0} \varphi_1\bigl(r^{-\frac{q}{n}}\bigr)^{-1} r^{\frac{p}{q}} \|f\|_{L^p(B(0,r^{-\frac{q}{n}}))} = C\|f\|_{\dot{B}^{p,\varphi_1}}.
\end{aligned}$$

For the case of $q < s$, we can also use the same method to prove the desirable conclusion. □

$f \in L^1_{\mathrm{loc}}$, the rough fractional maximal function $M_{\Omega,\alpha}$ and the rough fractional integral $I_{\Omega,\alpha}$ are defined by

$$M_{\Omega,\alpha} f(x) = \sup_{t>0} \frac{1}{|B(x,t)|^{1+\alpha}} \int_{B(x,t)} |\Omega(y)| |f(y)| \, dy,$$

$$I_{\Omega,\alpha} f(x) = \sup_{t>0} \int_{\mathbb{R}^n} \frac{\Omega(y) f(y)}{|x-y|^{n-\alpha}} \, dy$$

for $0 < \alpha < n$.

Corollary 2.6 *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* $\Omega \in L^s(S^{n-1})$. *If either of the two conditions*

(i) *when* $s' \leq p$, (φ_1, φ_2) *satisfies the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}+1}}\,dt \leq C\varphi_2(r),$$

(ii) *when* $q < s$, (φ_1, φ_2) *satisfies the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}-\frac{n}{s}+1}}\,dt \leq C\varphi_2(r)r^{\frac{n}{s}}$$

is satisfied, then $M_{\Omega,\alpha}$ *and* $I_{\Omega,\alpha}$ *are both bounded from* $\dot{B}^{p,\varphi_1}$ *to* $\dot{B}^{q,\varphi_2}$.

Remark 1 When $s = \infty$, the comments in Theorem 2.2 and in Theorem 2.5 can be obtained from Lemmas 4.4 and 5.3 in [9].

3 The commutators of a linear operator with rough kernel

Let $\tilde{T}$ be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. The commutator operator $[\tilde{T}, b]$ is defined by

$$[\tilde{T}, b]f(x) = b(x)\tilde{T}f(x) - \tilde{T}(bf)(x).$$

A well-known result of Coifman *et al.* [24] states that the commutator $[\tilde{T}, b]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in BMO(\mathbb{R}^n)$.

Since $BMO \subset \bigcap_{p>1} CBMO^{p,\lambda}$ when $\lambda = 0$, if we only assume $b \in CBMO^{p,\lambda}$, then $[\tilde{T}, b]$ may not be a bounded operator on $L^p(\mathbb{R}^n)$. However, it has some boundedness properties on other spaces. As a matter of fact, in [25] and [26], they considered the commutators with $b \in CBMO^{p,\lambda}$. Here we also obtain some boundedness of the commutators with $b \in CBMO^{p,\lambda}$ on generalized central Morrey spaces.

We need the following statement on the boundedness of the Hardy-type operator

$$H_1 g(t) := \frac{1}{t}\int_0^t \ln\left(1 + \frac{t}{r}\right) g(r)\,dr, \quad 0 < t < \infty.$$

Theorem F [27] *The inequality*

$$\operatorname*{ess\,sup}_{t>0} \nu_2(t) H_1 g(t) \leq c \operatorname*{ess\,sup}_{t>0} \nu_1(t) g(t)$$

holds for all non-negative and non-increasing g *on* $(0,\infty)$ *if and only if*

$$A := \sup_{t>0} \frac{\nu_2(t)}{t} \int_0^t \ln\left(1 + \frac{t}{r}\right) \frac{dr}{\operatorname{ess\,sup}_{0<\tau<r} \nu_1(\tau)} < \infty,$$

and $c \approx A$.

Lemma 3.1 *Let* $1 < p < \infty$, $b \in CBMO^{p_2,\lambda}$, $0 < \lambda < \frac{1}{n}$ *and* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, T_Ω *is a sublinear operator and satisfies* (1.1) *with* $\Omega \in L^s(S^{n-1})$.

When $s' \leq p$ and T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, then the inequality

$$\big\|[T_\Omega, b]f\big\|_{L^p(B(0,r))} \leq Cr^{\frac{n}{p}} \int_{2r}^{\infty} t^{n\lambda}\left(1 + \ln\frac{t}{r}\right)\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}+1}}$$

holds for any ball $B(0,r)$ and for all $f \in L^{p_1}_{\mathrm{loc}}(\mathbb{R}^n)$; or $p_1 < s$ and T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, then the inequality

$$\big\|[T_\Omega, b]f\big\|_{L^p(B(0,r))} \leq Cr^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty} t^{n\lambda}\left(1 + \ln\frac{t}{r}\right)\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}-\frac{n}{s}+1}}$$

holds for any ball $B(0,r)$ and for all $f \in L^{p_1}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof Let $1 < p < \infty$, $b \in CBMO^{p_2,\lambda}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. For any $r > 0$, set $B = B(0,r)$ and $2B = B(0,2r)$. We can write

$$f(x) = f(x)\chi_{2B}(x) + f(x)\chi_{(2B)^c}(x) := f_1(x) + f_2(x)$$

and

$$\begin{aligned}[T_\Omega, b]f(x) &= \big(b(x) - b_B\big)T_\Omega f_1(x) - T_\Omega\big(\big(b(\cdot) - b_B\big)f_1\big)(x)\\ &\quad + \big(b(x) - b_B\big)T_\Omega f_2(x) - T_\Omega\big(\big(b(\cdot) - b_B\big)f_2\big)(x)\\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Hence, we have

$$\big\|[T_\Omega, b]f\big\|_{L^p(B)} \leq \|I_1\|_{L^p(B)} + \|I_2\|_{L^p(B)} + \|I_3\|_{L^p(B)} + \|I_4\|_{L^p(B)}.$$

Since T_Ω is bounded on $L^p(\mathbb{R}^n)$, it follows that

$$\begin{aligned}\|I_1\|_{L^p(B)} &= \left(\int_B \big|b(x) - b_B\big|^p\big|T_\Omega f_1(x)\big|^p\,dx\right)^{\frac{1}{p}}\\ &\leq \left(\int_B \big|b(x) - b_B\big|^{p_2}\,dx\right)^{\frac{1}{p_2}}\left(\int_B \big|T_\Omega f_1(x)\big|^{p_1}\,dx\right)^{\frac{1}{p_1}}\\ &\leq Cr^{\frac{n}{p_2}+n\lambda}\|b\|_{CBMO^{p_2,\lambda}}\|f\|_{L^{p_1}(2B)},\end{aligned}$$

where the constant $C > 0$ is independent of f.

For I_2, we have

$$\begin{aligned}\|I_2\|_{L^p(B)} &= \left(\int_B \big|T_\Omega\big(\big(b(\cdot) - b_B\big)f_1\big)(x)\big|^p\,dx\right)^{\frac{1}{p}}\\ &\leq C\left(\int_{\mathbb{R}^n} \big|b(x) - b_B\big|^p\big|f_1(x)\big|^p\,dx\right)^{\frac{1}{p}}\\ &\leq C\left(\int_B \big|b(x) - b_B\big|^{p_2}\,dx\right)^{\frac{1}{p_2}}\left(\int_B \big|f(x)\big|^{p_1}\,dx\right)^{\frac{1}{p_1}}\\ &\leq Cr^{\frac{n}{p_2}+n\lambda}\|b\|_{CBMO^{p_2,\lambda}}\|f\|_{L^{p_1}(2B)}.\end{aligned}$$

For I_3, it is known that $x \in B$, $y \in (2B)^c$, which implies $\frac{1}{2}|y| \leq |x-y| \leq \frac{3}{2}|y|$.
(i) When $s' \leq p$ and by Fubini's theorem, we have

$$\begin{aligned}
\left|T_\Omega f_2(x)\right| &\leq C\int_{(2B)^c}\left|f(y)\right|\left|\Omega(x-y)\right|\frac{dy}{|y|^n} \\
&= C\int_{(2B)^c}\left|f(y)\right|\left|\Omega(x-y)\right|\int_{|y|}^{\infty}\frac{dt}{t^{n+1}}\,dy \\
&= C\int_{2r}^{\infty}\int_{2r\leq|y|<t}\left|f(y)\right|\left|\Omega(x-y)\right|dy\frac{dt}{t^{n+1}} \\
&\leq C\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}\left(\int_{B(0,t)}\left|\Omega(x-y)\right|^s dy\right)^{\frac{1}{s}}\left|B(0,t)\right|^{1-\frac{1}{p_1}-\frac{1}{s}}\frac{dt}{t^{n+1}} \\
&\leq C\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}+1}},
\end{aligned}$$

thus

$$\begin{aligned}
\|I_3\|_{L^p(B)} &\leq C\left(\int_B\left|b(x)-b_B\right|^p dx\right)^{\frac{1}{p}}\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}+1}} \\
&\leq Cr^{\frac{n}{p}+n\lambda}\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}+1}} \\
&\leq Cr^{\frac{n}{p}}\int_{2r}^{\infty}t^{n\lambda}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}+1}}.
\end{aligned}$$

(ii) When $p_1 < s$, by Fubini's theorem and the Minkowski inequality, we get

$$\begin{aligned}
\|I_3\|_{L^p(B)} &\leq \left(\int_B\left|\int_{2r}^{\infty}\int_{B(0,t)}\left|b(x)-b_B\right|\left|f(y)\right|\left|\Omega(x-y)\right|dy\frac{dt}{t^{n+1}}\right|^p dx\right)^{\frac{1}{p}} \\
&\leq \int_{2r}^{\infty}\int_{B(0,t)}\left|f(y)\right|\left(\int_B\left|b(x)-b_B\right|^p\left|\Omega(x-y)\right|^p dx\right)^{\frac{1}{p}}dy\frac{dt}{t^{n+1}} \\
&\leq \int_{2r}^{\infty}\int_{B(0,t)}\left|f(y)\right|\left(\int_B\left|b(x)-b_B\right|^{p_2}dx\right)^{\frac{1}{p_2}}\left(\int_B\left|\Omega(x-y)\right|^{p_1}dx\right)^{\frac{1}{p_1}}dy\frac{dt}{t^{n+1}} \\
&\leq Cr^{\frac{n}{p_2}+n\lambda}\int_{2r}^{\infty}\int_{B(0,t)}\left|f(y)\right|\left(\int_{B(0,t)}\left|\Omega(x-y)\right|^s dx\right)^{\frac{1}{s}}|B|^{\frac{1}{p_1}-\frac{1}{s}}dy\frac{dt}{t^{n+1}} \\
&\leq Cr^{\frac{n}{p}-\frac{n}{s}+n\lambda}\int_{2r}^{\infty}\int_{B(0,t)}\left|f(y)\right|dy\frac{dt}{t^{n-\frac{n}{s}+1}} \\
&\leq Cr^{\frac{n}{p}-\frac{n}{s}}\int_{2r}^{\infty}t^{n\lambda}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}-\frac{n}{s}+1}}.
\end{aligned}$$

On the other hand, for I_4, by Fubini's theorem, we have

$$\begin{aligned}
\left|[T_\Omega,b]f_2(x)\right| &\leq C\int_{(2B)^c}\left|b(y)-b_B\right|\left|f(y)\right|\left|\Omega(x-y)\right|\frac{dy}{|y|^n} \\
&\leq C\int_{2r}^{\infty}\int_{B(0,t)}\left|b(y)-b_B\right|\left|f(y)\right|\left|\Omega(x-y)\right|dy\frac{dt}{t^{n+1}} \\
&\leq C\int_{2r}^{\infty}\int_{B(0,t)}\left|b(y)-b_{B(0,t)}\right|\left|f(y)\right|\left|\Omega(x-y)\right|dy\frac{dt}{t^{n+1}}
\end{aligned}$$

$$+ C\int_{2r}^{\infty}\int_{B(0,t)}\left|b_{B(0,r)} - b_{B(0,t)}\right|\left|f(y)\right|\left|\Omega(x-y)\right|dy\frac{dt}{t^{n+1}}$$
$$=: I_{41} + I_{42}.$$

(i) When $s' \le p$, we obtain

$$\begin{aligned}
I_{41} &\le C\int_{2r}^{\infty}\left(\int_{B(0,t)}\left|b(y) - b_{B(0,t)}\right|^{p}\left|f(y)\right|^{p}dy\right)^{\frac{1}{p}} \\
&\quad\cdot\left(\int_{B(0,t)}\left|\Omega(x-y)\right|^{s}dy\right)^{\frac{1}{s}}\left|B(0,t)\right|^{1-\frac{1}{p}-\frac{1}{s}}\frac{dt}{t^{n+1}} \\
&\le C\int_{2r}^{\infty}\left(\int_{B(0,t)}\left|b(y) - b_{B(0,t)}\right|^{p_2}dy\right)^{\frac{1}{p_2}}\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p}+1}} \\
&\le C\int_{2r}^{\infty}t^{n\lambda}\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}+1}},
\end{aligned}$$

then

$$\|I_{41}\|_{L^{p}(B)} \le Cr^{\frac{n}{p}}\int_{2r}^{\infty}t^{n\lambda}\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}+1}}.$$

Moreover,

$$\begin{aligned}
I_{42} &\le C\int_{2r}^{\infty}|b_{B(0,r)} - b_{B(0,t)}|\int_{B(0,t)}\left|f(y)\right|\left|\Omega(x-y)\right|dy\frac{dt}{t^{n+1}} \\
&\le C\int_{2r}^{\infty}|b_{B(0,r)} - b_{B(0,t)}|\left(\int_{B(0,t)}\left|f(y)\right|^{p_1}dy\right)^{\frac{1}{p_1}} \\
&\quad\cdot\left(\int_{B(0,t)}\left|\Omega(x-y)\right|^{s}dy\right)^{\frac{1}{s}}\left|B(0,t)\right|^{1-\frac{1}{p_1}-\frac{1}{s}}\frac{dt}{t^{n+1}} \\
&\le C\int_{2r}^{\infty}\left(1 + \ln\frac{t}{r}\right)t^{n\lambda}\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}+1}},
\end{aligned}$$

then

$$\|I_{42}\|_{L^{p}(B)} \le Cr^{\frac{n}{p}}\int_{2r}^{\infty}t^{n\lambda}\left(1 + \ln\frac{t}{r}\right)\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}+1}}.$$

By estimating I_{41} and I_{42}, we obtain

$$\|I_{4}\|_{L^{p}(B)} \le Cr^{\frac{n}{p}}\int_{2r}^{\infty}t^{n\lambda}\left(1 + \ln\frac{t}{r}\right)\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}+1}}.$$

(ii) When $p_1 < s$, by the Minkowski inequality, we get

$$\begin{aligned}
\|I_{41}\|_{L^{p}(B)} &\le C\int_{2r}^{\infty}\int_{B(0,t)}\left|b(y) - b_{B(0,t)}\right|\left|f(y)\right|\left(\int_{B}\left|\Omega(x-y)\right|^{p}dx\right)^{\frac{1}{p}}dy\frac{dt}{t^{n+1}} \\
&\le Cr^{\frac{n}{p}-\frac{n}{s}}\int_{2r}^{\infty}\int_{B(0,t)}\left|b(y) - b_{B(0,t)}\right|\left|f(y)\right|dy\frac{dt}{t^{n-\frac{n}{s}+1}}
\end{aligned}$$

$$\le Cr^{\frac{n}{p}-\frac{n}{s}}\int_{2r}^{\infty}\left(\int_{B(0,t)}\left|b(y)-b_{B(0,t)}\right|^{p_2}dy\right)^{\frac{1}{p_2}}\|f\|_{L^{p_1}(B(0,t))}t^{n-\frac{n}{p}}\frac{dt}{t^{n-\frac{n}{s}+1}}$$

$$\le Cr^{\frac{n}{p}-\frac{n}{s}}\int_{2r}^{\infty}t^{n\lambda}\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}-\frac{n}{s}+1}}$$

and

$$\begin{aligned}\|I_{42}\|_{L^p(B)} &\le C\int_{2r}^{\infty}|b_{B(0,r)}-b_{B(0,t)}|\int_{B(0,t)}\left|f(y)\right|\left(\int_B\left|\Omega(x-y)\right|^p dx\right)^{\frac{1}{p}}dy\frac{dt}{t^{n+1}}\\ &\le Cr^{\frac{n}{p}-\frac{n}{s}}\int_{2r}^{\infty}|b_{B(0,r)}-b_{B(0,t)}|\int_{B(0,t)}\left|f(y)\right|dy\frac{dt}{t^{n-\frac{n}{s}+1}}\\ &\le Cr^{\frac{n}{p}-\frac{n}{s}}\int_{2r}^{\infty}|b_{B(0,r)}-b_{B(0,t)}|\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}-\frac{n}{s}+1}}\\ &\le Cr^{\frac{n}{p}-\frac{n}{s}}\int_{2r}^{\infty}t^{n\lambda}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}-\frac{n}{s}+1}}.\end{aligned}$$

Hence, we have

$$\|I_4\|_{L^p(B)}\le Cr^{\frac{n}{p}-\frac{n}{s}}\int_{2r}^{\infty}t^{n\lambda}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{p_1}-\frac{n}{s}+1}}.$$

Moreover, for any $q>0$, we have

$$\begin{aligned}r^{\frac{n}{p_2}+n\lambda}\|f\|_{L^{p_1}(2B)} &= Cr^{\frac{n}{p_2}+n\lambda+\frac{n}{q}}\|f\|_{L^{p_1}(2B)}\int_{2r}^{\infty}\frac{dt}{t^{\frac{n}{q}+1}}\\ &\le Cr^{\frac{n}{p_2}+\frac{n}{q}}\int_{2r}^{\infty}t^{n\lambda}\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{q}+1}}\\ &\le Cr^{\frac{n}{p_2}+\frac{n}{q}}\int_{2r}^{\infty}t^{n\lambda}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{q}+1}}.\end{aligned}$$

Now combining all the above estimates, we end the proof. □

Then we have the following conclusions.

Theorem 3.2 *Let* $1<p<\infty$, $b\in CBMO^{p_2,\lambda}$, $0<\lambda<\frac{1}{n}$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ *and* $\Omega\in L^s(S^{n-1})$. *Let* T_Ω *be a linear operator satisfying* (1.1) *and bounded on* $L^p(\mathbb{R}^n)$ *for* $p>1$. *If either of the two conditions*

(i) *when* $s'\le p$, (φ_1,φ_2) *satisfies the condition*

$$\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)t^{n\lambda}\frac{\operatorname{ess\,inf}_{t<\tau<\infty}\varphi_1(\tau)\tau^{\frac{n}{p_1}}}{t^{\frac{n}{p_1}+1}}dt\le C\varphi_2(r),$$

(ii) *when* $p_1<s$, (φ_1,φ_2) *satisfies the condition*

$$\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)t^{n\lambda}\frac{\operatorname{ess\,inf}_{t<\tau<\infty}\varphi_1(\tau)\tau^{\frac{n}{p_1}}}{t^{\frac{n}{p_1}-\frac{n}{s}+1}}dt\le C\varphi_2(r)r^{\frac{n}{s}}$$

is satisfied, then the operator $[T_\Omega,b]$ *is bounded from* $\dot{B}^{p_1,\varphi_1}$ *to* $\dot{B}^{p,\varphi_2}$.

Corollary 3.3 *Let* $1 < p < \infty$, $b \in CBMO^{p_2,\lambda}$, $0 < \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\Omega \in L^s(S^{n-1})$. *If either of the two conditions*

(i) *when* $s' \le p$, (φ_1, φ_2) *satisfies the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) t^{n\lambda} \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(\tau)\tau^{\frac{n}{p_1}}}{t^{\frac{n}{p_1}+1}}\, dt \le C\varphi_2(r),$$

(ii) *when* $p_1 < s$, (φ_1, φ_2) *satisfies the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) t^{n\lambda} \frac{\operatorname{ess\,inf}_{t<\tau<\infty} \varphi_1(\tau)\tau^{\frac{n}{p_1}}}{t^{\frac{n}{p_1}-\frac{n}{s}+1}}\, dt \le C\varphi_2(r) r^{\frac{n}{s}}$$

is satisfied, then the operator $[\tilde{T}_\Omega, b]$ *is bounded from* $\dot{B}^{p_1,\varphi_1}$ *to* $\dot{B}^{p,\varphi_2}$.

About the commutator of linear operator $T_{\Omega,\alpha}$ satisfying (1.2), we get the following corresponding results.

Lemma 3.4 *Let* $0 < \alpha < n$, $1 < p_1 < \frac{n}{\alpha}$, $b \in CBMO^{p_2,\lambda}$, $0 < \lambda < \frac{1}{n}$, $p_1' < p_2 < \infty$ *and let*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}, \qquad \frac{1}{q} = \frac{1}{p_1} - \frac{\alpha}{n}, \qquad \frac{1}{h} = \frac{1}{p_1} + \frac{1}{p_2}.$$

$T_{\Omega-\alpha}$ *is a sublinear operator and satisfies* (1.2) *with* $\Omega \in L^s(S^{n-1})$.

When $s' \le h$, $T_{\Omega,\alpha}$ *is bounded from* $L^t(\mathbb{R}^n)$ *to* $L^m(\mathbb{R}^n)$ *for any* $1 < t < \frac{n}{\alpha}$ *and* $\frac{1}{m} = \frac{1}{t} - \frac{\alpha}{n}$, *then the inequality*

$$\big\| [T_{\Omega,\alpha}, b]f \big\|_{L^p(B(0,r))} \le C r^{\frac{n}{p}} \int_{2r}^\infty t^{n\lambda} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L^{p_1}(B(0,t))} \frac{dt}{t^{\frac{n}{q}+1}}$$

holds for any ball $B(0,r)$ *and all* $f \in L^{p_1}_{\mathrm{loc}}(\mathbb{R}^n)$; *or* $q < s$, $T_{\Omega,\alpha}$ *is bounded from* $L^t(\mathbb{R}^n)$ *to* $L^m(\mathbb{R}^n)$ *for any* $1 < t < \frac{n}{\alpha}$ *and* $\frac{1}{m} = \frac{1}{t} - \frac{\alpha}{n}$, *then the inequality*

$$\big\| [T_{\Omega,\alpha}, b]f \big\|_{L^p(B(0,r))} \le C r^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^\infty t^{n\lambda} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L^{p_1}(B(0,t))} \frac{dt}{t^{\frac{n}{q}-\frac{n}{s}+1}}$$

holds for any ball $B(0,r)$ *and all* $f \in L^{p_1}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof Let $0 < \alpha < n$, $1 < p_1 < \frac{n}{\alpha}$, $b \in CBMO^{p_2,\lambda}$, $0 < \lambda < \frac{1}{n}$. For any $r > 0$, set $B = B(0,r)$ and $2B = B(0,2r)$. We also write

$$f(x) = f(x)\chi_{2B}(x) + f(x)\chi_{(2B)^c}(x) := f_1(x) + f_2(x)$$

and

$$\begin{aligned}
[T_{\Omega,\alpha}, b]f(x) &= \big(b(x) - b_B\big) T_{\Omega,\alpha} f_1(x) - T_{\Omega,\alpha}\big(\big(b(\cdot) - b_B\big) f_1\big)(x) \\
&\quad + \big(b(x) - b_B\big) T_{\Omega,\alpha} f_2(x) - T_{\Omega,\alpha}\big(\big(b(\cdot) - b_B\big) f_2\big)(x) \\
&=: II_1 + II_2 + II_3 + II_4.
\end{aligned}$$

Hence, we have

$$\big\|[T_{\Omega,\alpha},b]f\big\|_{L^p(B)}\le\|II_1\|_{L^p(B)}+\|II_2\|_{L^p(B)}+\|II_3\|_{L^p(B)}+\|II_4\|_{L^p(B)}.$$

Since $T_{\Omega,\alpha}$ is bounded from $L^{p_1}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ and $\frac{1}{q}=\frac{1}{p_1}-\frac{\alpha}{n}$, it follows that

$$\begin{aligned}\|II_1\|_{L^p(B)}&=\left(\int_B\big|b(x)-b_B\big|^p\big|T_{\Omega,\alpha}f_1(x)\big|^p\,dx\right)^{\frac{1}{p}}\\&\le\left(\int_B\big|b(x)-b_B\big|^{p_2}\,dx\right)^{\frac{1}{p_2}}\left(\int_B\big|T_{\Omega,\alpha}f_1(x)\big|^q\,dx\right)^{\frac{1}{q}}\\&\le Cr^{\frac{n}{p_2}+n\lambda}\|b\|_{CBMO^{p_2,\lambda}}\|f\|_{L^{p_1}(2B)},\end{aligned}$$

where the constant $C>0$ is independent of f.

For II_2, $\frac{1}{p}=\frac{1}{h}-\frac{\alpha}{n}$ and $\frac{1}{h}=\frac{1}{p_1}+\frac{1}{p_2}$,

$$\begin{aligned}\|II_2\|_{L^p(B)}&=\left(\int_B\big|T_{\Omega,\alpha}\big(\big(b(\cdot)-b_B\big)f_1\big)(x)\big|^p\,dx\right)^{\frac{1}{p}}\\&\le\left(\int_{\mathbb{R}^n}\big|b(x)-b_B\big|^h\big|f_1(x)\big|^h\,dx\right)^{\frac{1}{h}}\\&\le\left(\int_B\big|b(x)-b_B\big|^{p_2}\,dx\right)^{\frac{1}{p_2}}\left(\int_B\big|f(x)\big|^{p_1}\,dx\right)^{\frac{1}{p_1}}\\&\le Cr^{\frac{n}{p_2}+n\lambda}\|b\|_{CBMO^{p_2,\lambda}}\|f\|_{L^{p_1}(2B)}.\end{aligned}$$

For I_3,

(i) when $s'\le h$, by Fubini's theorem, since $x\in B$, we have

$$\begin{aligned}\big|T_{\Omega,\alpha}f_2(x)\big|&\le C\int_{(2B)^c}\big|f(y)\big|\big|\Omega(x-y)\big|\frac{dy}{|y|^{n-\alpha}}\\&=C\int_{(2B)^c}\big|f(y)\big|\big|\Omega(x-y)\big|\int_{|y|}^{\infty}\frac{dt}{t^{n-\alpha+1}}\,dy\\&=C\int_{2r}^{\infty}\int_{2r\le|y|<t}\big|f(y)\big|\big|\Omega(x-y)\big|\,dy\frac{dt}{t^{n-\alpha+1}}\\&\le C\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}\left(\int_{B(0,t)}\big|\Omega(x-y)\big|^s\,dy\right)^{\frac{1}{s}}\big|B(0,t)\big|^{1-\frac{1}{p_1}-\frac{1}{s}}\frac{dt}{t^{n-\alpha+1}}\\&\le C\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{q}+1}},\end{aligned}$$

thus

$$\begin{aligned}\|I_3\|_{L^p(B)}&\le C\left(\int_B\big|b(x)-b_B\big|^p\,dx\right)^{\frac{1}{p}}\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{q}+1}}\\&\le C\left(\int_B\big|b(x)-b_B\big|^{p_2}\,dx\right)^{\frac{1}{p_2}}r^{\frac{n}{q}}\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{q}+1}}\end{aligned}$$

$$\begin{aligned}
&\le Cr^{\frac{n}{p}+n\lambda}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{q}+1}}\\
&\le Cr^{\frac{n}{p}}\int_{2r}^{\infty}t^{n\lambda}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{q}+1}};
\end{aligned}$$

(ii) when $q < s$, by Fubini's theorem and the Minkowski inequality, we get

$$\begin{aligned}
&\|I_3\|_{L^p(B)}\\
&\quad\le\left(\int_B\left|\int_{2r}^{\infty}\int_{B(0,t)}\left|b(x)-b_B\right|\left|f(y)\right|\left|\Omega(x-y)\right|dy\frac{dt}{t^{n-\alpha+1}}\right|^p dx\right)^{\frac{1}{p}}\\
&\quad\le\int_{2r}^{\infty}\int_{B(0,t)}\left|f(y)\right|\left(\int_B\left|b(x)-b_B\right|^p\left|\Omega(x-y)\right|^p dx\right)^{\frac{1}{p}}dy\frac{dt}{t^{n-\alpha+1}}\\
&\quad\le\int_{2r}^{\infty}\int_{B(0,t)}\left|f(y)\right|\left(\int_B\left|b(x)-b_B\right|^{p_2}dx\right)^{\frac{1}{p_2}}\left(\int_B\left|\Omega(x-y)\right|^q dx\right)^{\frac{1}{q}}dy\frac{dt}{t^{n-\alpha+1}}\\
&\quad\le Cr^{\frac{n}{p_2}+n\lambda}\int_{2r}^{\infty}\int_{B(0,t)}\left|f(y)\right|\left(\int_{B(0,t)}\left|\Omega(x-y)\right|^s dx\right)^{\frac{1}{s}}|B|^{\frac{1}{q}-\frac{1}{s}}dy\frac{dt}{t^{n-\alpha+1}}\\
&\quad\le Cr^{\frac{n}{p}-\frac{n}{s}+n\lambda}\int_{2r}^{\infty}\int_{B(0,t)}\left|f(y)\right|dy\frac{dt}{t^{n-\frac{n}{s}-\alpha+1}}\\
&\quad\le Cr^{\frac{n}{p}-\frac{n}{s}}\int_{2r}^{\infty}t^{n\lambda}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{q}-\frac{n}{s}+1}}.
\end{aligned}$$

On the other hand, for I_4, by Fubini's theorem, we have

$$\begin{aligned}
\left|[T_{\Omega,\alpha},b]f_2(x)\right|&\le C\int_{(2B)^c}\left|b(y)-b_B\right|\left|f(y)\right|\left|\Omega(x-y)\right|\frac{dy}{|y|^{n-\alpha}}\\
&\le C\int_{2r}^{\infty}\int_{B(0,t)}\left|b(y)-b_B\right|\left|f(y)\right|\left|\Omega(x-y)\right|dy\frac{dt}{t^{n-\alpha+1}}\\
&\le C\int_{2r}^{\infty}\int_{B(0,t)}\left|b(y)-b_{B(0,t)}\right|\left|f(y)\right|\left|\Omega(x-y)\right|dy\frac{dt}{t^{n-\alpha+1}}\\
&\quad+C\int_{2r}^{\infty}\int_{B(0,t)}|b_{B(0,r)}-b_{B(0,t)}|\left|f(y)\right|\left|\Omega(x-y)\right|dy\frac{dt}{t^{n-\alpha+1}}\\
&=:I_{41}+I_{42}.
\end{aligned}$$

(i) When $s' \le h$, we obtain

$$\begin{aligned}
I_{41}&\le C\int_{2r}^{\infty}\left(\int_{B(0,t)}\left|b(y)-b_{B(0,t)}\right|^h\left|f(y)\right|^h dy\right)^{\frac{1}{h}}\\
&\quad\cdot\left(\int_{B(0,t)}\left|\Omega(x-y)\right|^s dy\right)^{\frac{1}{s}}\left|B(0,t)\right|^{1-\frac{1}{h}-\frac{1}{s}}\frac{dt}{t^{n-\alpha+1}}\\
&\le C\int_{2r}^{\infty}\left(\int_{B(0,t)}\left|b(y)-b_{B(0,t)}\right|^{p_2}dy\right)^{\frac{1}{p_2}}\|f\|_{L^{p_1}(B(0,t))}t^{n-\frac{n}{h}}\frac{dt}{t^{n-\alpha+1}}\\
&\le C\int_{2r}^{\infty}t^{n\lambda}\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{q}+1}},
\end{aligned}$$

then

$$\|I_{41}\|_{L^p(B)} \le Cr^{\frac{n}{p}} \int_{2r}^{\infty} t^{n\lambda} \|f\|_{L^{p_1}(B(0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

For I_{42}, we have

$$\begin{aligned} I_{42} &\le C \int_{2r}^{\infty} |b_{B(0,r)} - b_{B(0,t)}| \int_{B(0,t)} |f(y)||\Omega(x-y)|\,dy \frac{dt}{t^{n-\alpha+1}} \\ &\le C \int_{2r}^{\infty} |b_{B(0,r)} - b_{B(0,t)}| \left(\int_{B(0,t)} |f(y)|^{p_1}\,dy \right)^{\frac{1}{p_1}} \\ &\quad \cdot \left(\int_{B(0,t)} |\Omega(x-y)|^s\,dy \right)^{\frac{1}{s}} |B(0,t)|^{1-\frac{1}{p_1}-\frac{1}{s}} \frac{dt}{t^{n-\alpha+1}} \\ &\le C \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{n\lambda} \|f\|_{L^{p_1}(B(0,t))} \frac{dt}{t^{\frac{n}{q}+1}}, \end{aligned}$$

then

$$\|I_{42}\|_{L^p(B)} \le Cr^{\frac{n}{p}} \int_{2r}^{\infty} t^{n\lambda} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L^{p_1}(B(0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

Then, by estimating I_{41} and I_{42}, we obtain

$$\|I_4\|_{L^p(B)} \le Cr^{\frac{n}{p}} \int_{2r}^{\infty} t^{n\lambda} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L^{p_1}(B(0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

(ii) When $q < s$, by the Minkowski inequality, we get

$$\begin{aligned} \|I_{41}\|_{L^p(B)} &\le C \int_{2r}^{\infty} \int_{B(0,t)} |b(y) - b_{B(0,t)}||f(y)| \left(\int_B |\Omega(x-y)|^p\,dx \right)^{\frac{1}{p}} dy \frac{dt}{t^{n-\alpha+1}} \\ &\le Cr^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty} \int_{B(0,t)} |b(y) - b_{B(0,t)}||f(y)|\,dy \frac{dt}{t^{n-\frac{n}{s}-\alpha+1}} \\ &\le Cr^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty} \left(\int_{B(0,t)} |b(y) - b_{B(0,t)}|^{p_2}\,dy \right)^{\frac{1}{p_2}} \|f\|_{L^{p_1}(B(0,t))} t^{n-\frac{n}{h}} \frac{dt}{t^{n-\frac{n}{s}-\alpha+1}} \\ &\le Cr^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty} t^{n\lambda} \|f\|_{L^{p_1}(B(0,t))} \frac{dt}{t^{\frac{n}{q}-\frac{n}{s}+1}} \end{aligned}$$

and

$$\begin{aligned} \|I_{42}\|_{L^p(B)} &\le C \int_{2r}^{\infty} |b_{B(0,r)} - b_{B(0,t)}| \int_{B(0,t)} |f(y)| \left(\int_B |\Omega(x-y)|^p\,dx \right)^{\frac{1}{p}} dy \frac{dt}{t^{n-\alpha+1}} \\ &\le Cr^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty} |b_{B(0,r)} - b_{B(0,t)}| \int_{B(0,t)} |f(y)|\,dy \frac{dt}{t^{n-\frac{n}{s}-\alpha+1}} \\ &\le Cr^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty} |b_{B(0,r)} - b_{B(0,t)}| \|f\|_{L^{p_1}(B(0,t))} \frac{dt}{t^{\frac{n}{q}-\frac{n}{s}+1}} \\ &\le Cr^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty} t^{n\lambda} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L^{p_1}(B(0,t))} \frac{dt}{t^{\frac{n}{q}-\frac{n}{s}+1}}. \end{aligned}$$

Hence, we have

$$\|I_4\|_{L^p(B)} \le Cr^{\frac{n}{p}-\frac{n}{s}} \int_{2r}^{\infty} t^{n\lambda}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p_1}(B(0,t))}\frac{dt}{t^{\frac{n}{q}-\frac{n}{s}+1}}.$$

Then we end this proof. □

Theorem 3.5 *Let* $0<\alpha<n$, $1<p_1<\frac{n}{\alpha}$, $b\in CBMO^{p_2,\lambda}$, $0<\lambda<\frac{1}{n}$, $p_1'<p_2<\infty$ *and*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n},\qquad \frac{1}{q}=\frac{1}{p_1}-\frac{\alpha}{n},\qquad \frac{1}{h}=\frac{1}{p_1}+\frac{1}{p_2}.$$

Let $T_{\Omega,\alpha}$ *be a linear operator satisfying* (1.2) *with* $\Omega\in L^s(S^{n-1})$, *which is bounded from* $L^t(\mathbb{R}^n)$ *to* $L^m(\mathbb{R}^n)$ *for any* $1<t<\frac{n}{\alpha}$, $\frac{1}{m}=\frac{1}{t}-\frac{\alpha}{n}$. *If either of the two conditions*

(i) *when* $s'\le h$, (φ_1,φ_2) *satisfies the condition*

$$\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)t^{n\lambda}\frac{\operatorname{ess\,inf}_{t<\tau<\infty}\varphi_1(\tau)\tau^{\frac{n}{p_1}}}{t^{\frac{n}{q}+1}}\,dt\le C\varphi_2(r),$$

(ii) *when* $q<s$, (φ_1,φ_2) *satisfies the condition*

$$\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)t^{n\lambda}\frac{\operatorname{ess\,inf}_{t<\tau<\infty}\varphi_1(\tau)\tau^{\frac{n}{p_1}}}{t^{\frac{n}{q}-\frac{n}{s}+1}}\,dt\le C\varphi_2(r)r^{\frac{n}{s}}$$

is satisfied, then the operator $[T_{\Omega,\alpha},b]$ *is bounded from* $\dot{B}^{p_1,\varphi_1}$ *to* $\dot{B}^{p,\varphi_2}$.

Corollary 3.6 *Let* $0<\alpha<n$, $1<p_1<\frac{n}{\alpha}$, $b\in CBMO^{p_2,\lambda}$, $0<\lambda<\frac{1}{n}$, $p_1'<p_2<\infty$, $\Omega\in L^s(S^{n-1})$ *and*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n},\qquad \frac{1}{q}=\frac{1}{p_1}-\frac{\alpha}{n},\qquad \frac{1}{h}=\frac{1}{p_1}+\frac{1}{p_2}.$$

If either of the two conditions

(i) *when* $s'\le h$, (φ_1,φ_2) *satisfies the condition*

$$\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)t^{n\lambda}\frac{\operatorname{ess\,inf}_{t<\tau<\infty}\varphi_1(\tau)\tau^{\frac{n}{p_1}}}{t^{\frac{n}{q}+1}}\,dt\le C\varphi_2(r),$$

(ii) *when* $q<s$, (φ_1,φ_2) *satisfies the condition*

$$\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)t^{n\lambda}\frac{\operatorname{ess\,inf}_{t<\tau<\infty}\varphi_1(\tau)\tau^{\frac{n}{p_1}}}{t^{\frac{n}{q}-\frac{n}{s}+1}}\,dt\le C\varphi_2(r)r^{\frac{n}{s}}$$

is satisfied, then the operator $[I_{\Omega,\alpha},b]$ *is bounded from* $\dot{B}^{p_1,\varphi_1}$ *to* $\dot{B}^{p,\varphi_2}$.

Remark 2 In our main results, if we let $\varphi_1=r^{n\lambda_1}$ and $\varphi_2=r^{n\lambda_2}$, then by calculating we can recover some known results in [7] and [25].

Competing interests

The author declares that they have no competing interests.

Author's contributions

The author contributed to all the main results in this paper.

Acknowledgements

This work was supported by NSF of Zhejiang Province (No. LQ13A010005). The author thanks the referees for their valuable suggestions.

Received: 24 July 2012 Accepted: 8 August 2013 Published: 27 August 2013

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doi:10.1186/1029-242X-2013-411

Cite this article as: Fan: **Boundedness of sublinear operators and their commutators on generalized central Morrey spaces.** *Journal of Inequalities and Applications* 2013 **2013**:411.