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The entropy weak solution to a generalized Degasperis-Procesi equation

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Abstract

A nonlinear generalization of the Degasperis-Procesi equation is investigated. The well-posedness of entropy weak solutions for the Cauchy problem of the equation is established in the space $L^1(R) \cap L^\infty(R)$.

MSC: 35G25; 35L05

Keywords: L^1 stability; entropy weak solutions; L^∞ bounded solution

1 Introduction

The objective of this work is to study the well-posedness in the space $L^1(R) \cap L^\infty(R)$ for the generalized Degasperis-Procesi equation

$$u_t - u_{txx} + muu_x = 3u_x u_{xx} + uu_{xxx}, \quad (t,x) \in R_+ \times R, \tag{1}$$

where $m > 0$ is a constant and $R_+ = (0,\infty)$. Letting $u_0 = u(0,x)$ be an initial condition for Eq. (1), we derive the inequality

$$c_1\|u_0\|_{L^2(R)} \leq \|u\|_{L^2(R)} \leq c_2\|u_0\|_{L^2(R)}, \tag{2}$$

where c_1 and c_2 are positive constants. In our further investigation, we only assume that

$$u_0 \in L^1(R) \cap L^\infty(R). \tag{3}$$

For $m = 4$, Eq. (1) becomes the Degasperis-Procesi equation [1]

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad (t,x) \in R_+ \times R. \tag{4}$$

The formal integrability of Eq. (4) was found in [2]. It was shown in [2] that Eq. (4) possesses a bi-Hamiltonian structure with an infinite sequence of conserved quantities and has exact peakon solutions. Dullin *et al.* [3] proved that the Degasperis-Procesi equation can be obtained from the shallow water elevation equation by an appropriate Kodama transformation. The traveling wave solutions of Eq. (4) were found in Lundmark and Szmigielski [4] and Vakhnenko and Parkes [5]. Lin and Liu [6] established the L^2-stability of peakons for Degasperis-Procesi Eq. (4) under certain assumptions imposing on the initial value. The local well-posedness of Eq. (4) with initial data $u_0 \in H^s(R)$, $s > \frac{3}{2}$ and the precise blow-up scenario were analyzed in [7]. Lenells [8] classified all weak traveling wave solutions. Matsuno [9] studied multisoliton solutions and their peakon limits. The properties of infinite speed of propagation of Eq. (4) were established in Henry [10] and Mustafa [11]. For other methods to handle the problems relating to various dynamic properties of the Degasperis-Procesi equation and other shallow water equations, the reader is referred to [12–20] and the references therein.

Recently, Coclite and Karlsen [21] established the existence, uniqueness and $L^1(R)$ stability of entropy weak solutions belonging to the class $L^1(R) \cap BV(R)$ for Eq. (4). They obtained the existence of at least one weak solution satisfying a restricted set of entropy inequalities in the space $L^2(R) \cap L^4(R)$. In Coclite and Karlsen [22], the well-posedness of entropy weak solution is investigated in the space $L^1(R) \cap L^\infty(R)$.

Motivated by the desire to extend the weak solution results presented in Coclite and Karlsen [22], we consider Eq. (1) with its Cauchy problem in the form

$$\begin{cases} u_t - u_{txx} = -\partial_x(\frac{m}{2}u^2) + 3u_x u_{xx} + uu_{xxx} = -(\frac{m}{2}u^2)_x + \frac{1}{2}\partial_{xxx}^3 u^2, \\ u(0,x) = u_0(x), \end{cases} \tag{5}$$

which is equivalent to

$$\begin{cases} u_t + uu_x = -\frac{m-1}{2}\Lambda^{-2}(u^2)_x, \\ u(0,x) = u_0(x), \end{cases} \tag{6}$$

where $m > 0$ is a constant and $\Lambda = (1 - \partial_x^2)^{\frac{1}{2}}$.

The objective of this paper is to study problem (5). We establish the existence, uniqueness and L^1 stability of entropy weak solutions belonging to the space $L^1(R) \cap L^\infty(R)$ under condition (3). One of our contributions in this work is that we derive inequality (2), which leads us to establishing our main results. Here we state that we will adopt the well-known and celebrated Kruzkov technique (see [23]), which was originally introduced to analyze hyperbolic conservation laws.

The rest of this paper is organized as follows. Section 2 establishes several estimates for the viscous approximations of problem (5). The existence, uniqueness and L^1 stability of entropy weak solutions for problem (6) are presented in Section 3.

2 Viscous approximations and estimates

Defining

$$\phi(x) = \begin{cases} e^{\frac{1}{x^2-1}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

and letting $\phi_\varepsilon(x) = \varepsilon^{-\frac{1}{4}}\phi(\varepsilon^{-\frac{1}{4}}x)$ with $0 < \varepsilon < \frac{1}{4}$ and $u_{0,\varepsilon} = \phi_\varepsilon \star u_0$, we know that $u_{0,\varepsilon} \in C^\infty$ for any $u_0 \in H^s$ with $s \geq 0$. We let $L^p = L^p(R)$ $(1 \leq p < +\infty)$ be the space of all measurable functions h such that $\|h(t,\cdot)\|_{L^p}^p = \int_R |h(t,x)|^p\,dx < \infty$. We define $L^\infty = L^\infty(R)$ with the standard norm $\|h(t,\cdot)\|_{L^\infty} = \inf_{m(e)=0} \sup_{x \in R \setminus e} |h(t,x)|$.

For simplicity, throughout this article, we let c denote any positive constants which are independent of parameter ε.

Several properties for the smooth functions $u_{0,\varepsilon}$ are stated in the following lemma.

Lemma 2.1 *The following estimates hold for any ε with $0 < \varepsilon < \frac{1}{4}$ and $s \geq 0$:*

$$\|u_{0,\varepsilon}\|_{L^p(R)} \leq c\|u_0\|_{L^p(R)} \quad \text{for } 1 \leq p \leq \infty,$$
$$u_{0,\varepsilon} \to u_0 \quad (\varepsilon \to 0) \text{ in } L^p(R) \text{ for } 1 \leq p \leq \infty,$$
$$\|u_{0,\varepsilon}\|_{H^q} \leq c\|u_0\|_{H^s} \quad \text{if } q \leq s,$$

where c is a constant independent of ε.

The proof of the above lemma can be found in [18].

To establish the existence of solutions to Cauchy problem (5), we will analyze the limiting behavior of a sequence of smooth functions $\{u_\varepsilon\}_{\varepsilon>0}$, where each function u_ε satisfies the viscous problem

$$\begin{cases} \partial_t u_\varepsilon - \partial^3_{txx} u_\varepsilon + m u_\varepsilon \partial_x u_\varepsilon \\ \quad = 3\partial_x u_\varepsilon \partial^2_{xx} u_\varepsilon + u_\varepsilon \partial^3_{xxx} u_\varepsilon + \varepsilon \partial^2_{xx} u_\varepsilon - \varepsilon \partial^4_{xxxx} u_\varepsilon, \quad (t,x) \in R_+ \times R, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \quad x \in R, \end{cases} \tag{7}$$

which is equivalent to the parabolic-elliptic system

$$\begin{cases} \partial_t u_\varepsilon + \partial_x(\frac{u_\varepsilon^2}{2}) + \partial_x P_\varepsilon = \varepsilon \partial^2_{xx} u_\varepsilon, \\ P_\varepsilon - \partial^2_{xx} P_\varepsilon = \frac{m-1}{2} u_\varepsilon^2, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x). \end{cases} \tag{8}$$

From the second identity of (8), we get

$$P_\varepsilon(t,x) = \frac{m-1}{4} \int_R e^{-|x-y|} \big(u_\varepsilon(t,y)\big)^2 \, dy. \tag{9}$$

Lemma 2.2 *Provided that $u_0 \in L^2(R)$, for any fixed $\varepsilon > 0$, there exists a unique global smooth solution $u_\varepsilon = u_\varepsilon(t,x)$ to Cauchy problem* (7) *belonging to $C([0,\infty); H^s(R))$ with $s \geq 0$.*

Proof We omit the proof since it is similar to the one found in [21] or [24] by using $u_{0,\varepsilon} \in C^\infty(R)$. □

Here we state that the following lemma takes an important role in our further study of Eq. (1).

Lemma 2.3 *Assume that $u_0 \in L^2(R)$ holds and u_ε is a solution of problem* (7). *Then the following bounds hold for any $t \geq 0$:*

$$c_1\|u_0\|_{L^2(R)} \leq \|u_\varepsilon\|_{L^2(R)} \leq c_2\|u_0\|_{L^2(R)}, \tag{10}$$
$$\sqrt{\varepsilon}\|\partial_x u_\varepsilon\|_{L^2(R)} \leq c\|u_0\|_{L^2(R)}, \tag{11}$$

where c_1, c_2 and c are positive constants independent of ε and t.

We give some bounds on the nonlocal term P_ε, which all are consequences of the L^2 bound in Lemma 2.3.

Lemma 2.4 *Assume that* $u_0 \in L^2(R)$ *holds. Then*

$$\|P_\varepsilon(t,\cdot)\|_{L^1(R)}, \|\partial_x P_\varepsilon(t,\cdot)\|_{L^1(R)} \le c\|u_0\|_{L^2}^2, \tag{12}$$

$$\|P_\varepsilon\|_{L^\infty(R_+\times R)}, \|\partial_x P_\varepsilon\|_{L^\infty(R_+\times R)} \le c\|u_0\|_{L^2}^2, \tag{13}$$

$$\|\partial_{xx}^2 P_\varepsilon(t,\cdot)\|_{L^1(R)} \le c\|u_0\|_{L^2}^2, \tag{14}$$

where c is a constant independent of ε *and* t.

The proofs of Lemmas 2.3 and 2.4 are similar to those of Lemmas 2.2, 2.3 and 2.4 in Coclite and Karlsen [21]. Here we omit them.

Lemma 2.5 *If* $u_0 \in L^1(R) \cap L^\infty(R)$, *it holds that*

$$\|u_\varepsilon(t,\cdot)\|_{L^\infty} \le \|u_0\|_{L^\infty} + ct\|u_0\|_{L^2}^2. \tag{15}$$

Proof Since

$$\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon - \varepsilon \partial_{xx} u_\varepsilon = -\partial_x P_\varepsilon, \tag{16}$$

using Lemma 2.4, we have

$$\|\partial_x P_\varepsilon\|_{L^\infty(R_+\times R)} \le c\|u_0\|_{L^2}^2. \tag{17}$$

Setting $g(t) = \|u_0\|_{L^\infty(R)} + ct\|u_0\|_{L^2}^2$, we get

$$\frac{dg}{dt} = c\|u_0\|_{L^2}^2. \tag{18}$$

Using $\|u_\varepsilon(0,x)\|_{L^\infty(R)} \le g(0)$ and the comparison principle for the parabolic equations, we obtain the desired result (15). □

Applying Lemma 2.4 and the methods presented in Coclite and Karlsen [21] or [22], we obtain the following result.

Lemma 2.6 (Oleinik-type estimate) *Assume that* (3) *holds and* $T > 0$. *Then*

$$\partial_x u_\varepsilon(t,x) \le \frac{1}{t} + C_T, \quad x \in R, 0 < t \le T, \tag{19}$$

where the constant C_T *depends on* T.

We omit the proof of this lemma since it is similar to the proof of Lemma 6 in [22].

We state the concepts of weak solutions (see [21] or [22]).

Definition 2.1 (Weak solution) We call a function $u : R_+ \times R \to R$ a weak solution of Cauchy problem (8) provided

(i) $u \in L^\infty(R_+; L^2(R))$, and
(ii) $\partial_t u + \partial_x(\frac{u^2}{2}) + \partial_x P^u(t,x) = 0$ in $D'([0,\infty) \times R)$, that is, $\forall \phi \in C_c^\infty([0,\infty) \times R)$, the following identity holds:

$$\int_{R^+} \int_R \left(u\partial_t \phi + \frac{u^2}{2} \partial_x \phi - \partial_x P^u \phi \right) dx\, dt + \int_R u_0(x)\phi(0,x)\, dx = 0, \tag{20}$$

where

$$P^u(t,x) = G_1 * \left(\frac{m-1}{2} u^2 \right)(t,x) = \frac{m-1}{4} \int_R e^{-|x-y|} \big(u(t,y)\big)^2\, dy. \tag{21}$$

Definition 2.2 (Entropy weak solution) We call a function $u : R_+ \times R \to R$ an entropy weak solution of Cauchy problem (8) if

(i) u is a weak solution in the sense of Definition 2.1,
(ii) $u \in L^\infty([0,T] \times R)$ for any $T > 0$, and
(iii) for any convex C^2 entropy $\eta : R \to R$ with corresponding entropy flux $q : R \to R$ defined by $q'(u) = \eta'(u)u$, the following holds:

$$\partial_t \eta(u) + \partial_x q(u) + \eta'(u)\partial_x P^u \le 0 \quad \text{in } D'\big([0,\infty) \times R\big), \tag{22}$$

that is, $\forall \phi \in C_c^\infty([0,\infty) \times R)$, $\phi \ge 0$

$$\int_{R_+} \int_R \big(\eta(u)\partial_t \phi + q(u)\partial_x \phi - \eta'(u)\partial_x P^u \phi \big)\, dx\, dt + \int_R \eta\big(u_0(x)\big)\phi(0,x)\, dx \ge 0. \tag{23}$$

As pointed out in Coclite and Karsen [21] or [22], it takes a standard argument to know that the Kruzkov entropies/entropy fluxes

$$\eta(u) = |u-k|, \qquad q(u) := \operatorname{sign}(u-k)\left(\frac{u^2}{2} - \frac{k^2}{2} \right) \tag{24}$$

satisfy (23). Using the Kruzkov entropy fluxes, we see that the weak formulation (20) is a consequence of the entropy formulation (23).

3 Main result

Now we give the following $L^1(R)$ stability result of entropy weak solutions for Eq. (1).

Theorem 3.1 (L^1-stability) *Assume that u and v are two entropy weak solutions of Eq.* (1) *with initial data u_0 and v_0 satisfying* (3). *For an arbitrary $T > 0$, it holds that*

$$\big\| u(t,\cdot) - v(t,\cdot) \big\|_{L^1(R)} \le ce^{ct} \int_{-\infty}^{\infty} \big| u_0(x) - v_0(x) \big|\, dx, \quad t \in [0,T], \tag{25}$$

where c depends on $\|u_0\|_{L^\infty(R)}$, $\|v_0\|_{L^\infty(R)}$, $\|u_0\|_{L^2(R)}$, $\|v_0\|_{L^2(R)}$, *and* T.

For the proof of Theorem 3.1, the reader is referred to [21] or [23].

Letting $v(t,x) = 0$ in Theorem 3.1 and assuming $u_0 \in L^1(R) \cap L^\infty(R)$, we know $u(t,\cdot) \in L^1(R)$ for any $t \in [0,T]$.

We will apply the compensated compactness method presented in [25, 26] to obtain strong convergence of a subsequence of viscosity approximations.

Lemma 3.1 *Let $\{v_\gamma\}_{\gamma>0}$ be a family of functions defined on $(0,\infty)\times R$ such that*

$$\|v_\gamma\|_{L^\infty} \le M_T$$

and the family

$$\{\partial_t\eta(v_\gamma) + \partial_x q(v_\gamma)\}_{\gamma>0}$$

is compact in $H^{-1}_{\mathrm{loc}}((0,\infty)\times R)$ for any convex $\eta \in C^2(R)$, where $q(u) = u\eta'(u)$. Then there exist a sequence $\{\gamma_n\}_{n\in N}$, $\gamma_n \to 0$, and a map $v \in L^\infty((0,T)\times R)$, $T > 0$, such that

$$v_{\gamma_n} \to v \quad \text{a.e. and in } L^p_{\mathrm{loc}}\big((0,\infty)\times R\big), 1 \le p < \infty.$$

Lemma 3.1 can be found in [25] or [26]. Now, we cite a result presented in Murat [27].

Lemma 3.2 *Let Ω be a bounded open subset of R^H, $H \ge 2$. Suppose that the sequence $\{L_n\}_{n=1}^\infty$ of distributions is bounded in $W^{-1,\infty}(\Omega)$ and*

$$L_n = L_n^{(1)} + L_n^{(2)},$$

where $\{L_n^{(1)}\}_{n=1}^\infty$ lies in a compact subset of $H^{-1}_{\mathrm{loc}}(\Omega)$ and $\{L_n^{(2)}\}_{n=1}^\infty$ lies in a bounded subset of $M_{\mathrm{loc}}(\Omega)$. Then $\{L_n\}_{n=1}^\infty$ lies in a compact subset of $H^{-1}_{\mathrm{loc}}(\Omega)$.

Lemma 3.3 *Suppose that $u_0 \in L^1(R)\cap L^\infty(R)$. Then there exist a subsequence $\{u_{\varepsilon_k}\}_{k=1}^\infty$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and a limit function*

$$u \in L^\infty\big(R_+; L^2(R)\big) \cap L^\infty\big((0,T); L^\infty \cap L^1(R)\big) \quad \forall T > 0 \tag{26}$$

such that

$$u_{\varepsilon_k} \to u \quad \text{in } L^p\big((0,T)\times R\big)\ \forall T > 0, \forall p \in [1,\infty). \tag{27}$$

Proof Let $\eta : R \to R$ be any convex C^2 entropy function which is compactly supported, and let $q : R \to R$ be the corresponding entropy flux defined by $q'(u) = \eta'(u)u$. We write

$$\partial_t\eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = L_\varepsilon^{(1)} + L_\varepsilon^{(2)}, \tag{28}$$

where

$$\begin{cases} L_\varepsilon^{(1)} = \varepsilon\partial_{xx}^2\eta(u_\varepsilon), \\ L_\varepsilon^{(2)} = -\varepsilon\eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2 - \eta'(u_\varepsilon)\partial_x P_\varepsilon \end{cases} \tag{29}$$

are distributions. We claim that

$$\begin{cases} L_\varepsilon^{(1)} \to 0 \quad \text{in } H^{-1}((0,T)\times R), T > 0, \\ L_\varepsilon^{(2)} \quad \text{is uniformly bounded in } L^1((0,T)\times R). \end{cases} \tag{30}$$

Applying Lemmas 2.3, 2.4 and 2.5, we have

$$\|\varepsilon\partial_{xx}^2\eta(u_\varepsilon)\|_{H^{-1}(R_+\times R)} \le \sqrt{\varepsilon}c\|\eta'\|_{L^\infty}\|u_0\|_{L^2(R)} \to 0, \tag{31}$$

$$\|\varepsilon\eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2\| \le c\|\eta''\|_{L^\infty(R)}\|u_0\|_{L^2(R)}, \tag{32}$$

$$\|\eta'(u_\varepsilon)\|_{L^1((0,T)\times R)} \le c\|\eta'\|_{L^\infty(R)}\|u_0\|_{L^2(R)}. \tag{33}$$

Hence, (30) follows. Therefore, from Lemmas 3.1 and 3.2, we know that there exist a subsequence $\{u_{\varepsilon_k}\}_{k=1}^\infty$ and a limit function u satisfying (26) such that as $k \to \infty$

$$u_{\varepsilon_k} \to u \quad \text{in } L^p_{\text{loc}}(R_+ \times R) \text{ for any } p \in [1,\infty) \quad \text{and} \tag{34}$$

$$u_{\varepsilon_k} \to u \quad \text{a.e in } R_+ \times R. \tag{35}$$

Using Lemma 2.5, from (34) and (35), we get (27). □

Lemma 3.4 *Suppose that $u_0 \in L^1(R) \cap L^\infty(R)$ holds. Then*

$$P_{\varepsilon_k} \to P^u \quad \text{in } L^p\big((0,T); W^{1,p}(R)\big)\ \forall T > 0, \forall p \in [1,2), \tag{36}$$

where the sequence $\{\varepsilon_k\}_{k=1}^\infty$ and the function u are constructed in Lemma 3.3.

The proof is similar to that of Lemma 9 in [22]. Here we omit it.

Theorem 3.2 (Existence) *Assume that* (3) *holds. Then there exists at least one entropy weak solution to problem* (7).

Proof Let $\varphi \in C_c^\infty(R_+ \times R)$. It follows from (8) that

$$\int_{R_+}\int_R \left(u_\varepsilon\partial_t\varphi + \frac{u_\varepsilon^2}{2}\partial_x\varphi - \partial_x P_\varepsilon\varphi + \varepsilon u_\varepsilon\partial_{xx}^2\varphi\right)dx\,dt + \int_R u_{0,\varepsilon}\varphi(0,x)\,dx = 0. \tag{37}$$

From Lemmas 2.1 and 3.3, we derive that the function u presented in Lemma 3.3 is a weak solution of problem (8) in the sense of Definition 2.1. We have to verify that u satisfies the entropy inequalities in Definition 2.2. Let $\eta \in C^2(R)$ be a convex entropy with flux q defined by $q'(u) = u\eta'(u)$. The convexity of η and (8) yield

$$\partial_t\eta(u_\varepsilon) + \partial_x q(u_\varepsilon) + \eta'(u_\varepsilon)\partial_x P_\varepsilon = \varepsilon\partial_{xx}^2\eta(u_\varepsilon) - \varepsilon\eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2 \le \varepsilon\partial_{xx}^2\eta(u_\varepsilon). \tag{38}$$

Therefore, the entropy inequalities follow from Lemmas 3.3 and 3.4. □

From Theorems 3.1 and 3.2, we have the following theorem.

Theorem 3.3 *Assume that* (3) *holds. Then Cauchy problem* (7) *has a unique entropy weak solution in the sense of Definition* 2.2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The article is a joint work of three authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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