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A reconsideration of Jensen's inequality and its applications

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Abstract

A finite form of Jensen's inequality for a continuous *convex* function from a topological abelian semigroup to another topological ordered abelian semigroup is obtained under some assumption. As an application, a refinement of a mean inequality is also obtained.

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1 Introduction

This is the inheritance of the idea of [1, Theorem 1] which gives a new interpretation of Jensen's inequality by φ-mean. The finite form of Jensen's inequality proved by Jensen in 1906 asserts that if $t_1, \ldots, t_n$ are positive numbers with $\sum_{i=1}^{n} t_i = 1$ and f is a continuous convex (resp. concave) function on a real interval I, then

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i f(x_i) \quad \left(\text{resp.} f\left(\sum_{i=1}^{n} t_i x_i\right) \geq \sum_{i=1}^{n} t_i f(x_i)\right)$$

holds for all $x_1, \ldots, x_n \in I$.

We first introduce a concept called '$(*, \circ)$-convex (or concave)' for a continuous function from a topological abelian semigroup $(I, *)$ to another topological ordered abelian semigroup $(J, \circ)$ and give an interesting example of such a function (see Remark 1). Our purpose of this paper is to give a finite form of Jensen's inequality for such a function under some assumption (see Theorem 1). Also, as an application, we give a refinement of a mean inequality (see Theorem 2).

2 Terminology and main theorem

Let I be a topological space, and let $*$ be a topological abelian semigroup operation on I. For any $x \in I$ and $n \in \mathbf{N}$ with $n \geq 1$, define the nth power $x^{(n)*}$ of x recursively by $x^{(1)*} = x$ and $x^{(n+1)*} = x^{(n)*} * x$ for $n \geq 1$. We assume that

($\sharp_1$) any nth-power function $x \mapsto x^{(n)*}$ is a bijection of I onto itself.

By the assumption ($\sharp_1$), for each $x \in I$ and $n \in \mathbf{N}$, there exists a unique element a of I such that $a^{(n)*} = x$. Denote by $x^{(1/n)*}$ such an element a. Moreover, we define

$$x^{(m/n)*} = \left(x^{(1/n)*}\right)^{(m)*}$$

for each $m, n \in \mathbf{N}$. Then we can easily see that this definition is well defined, that is,

$$\frac{m}{n} = \frac{m'}{n'} \quad \Rightarrow \quad \left(x^{(1/n)*}\right)^{(m)*} = \left(x^{(1/n')*}\right)^{(m')*} \quad (\forall x \in I).$$

In this case, we can easily show that the following power laws:

$$x^{(p+q)*} = x^{(p)*} * x^{(q)*}, \qquad x^{(pq)*} = \left(x^{(p)*}\right)^{(q)*} \quad \text{and} \quad (x * y)^{(p)*} = x^{(p)*} * y^{(p)*} \tag{1}$$

for all $p, q \in \mathbf{Q}_+$ and $x, y \in I$. Here $\mathbf{Q}_+$ denotes the set of all positive rational numbers. Moreover, we assume that

($\sharp_2$) for each $x \in I$, the function $p \mapsto x^{(p)*}$ is continuous on $\mathbf{Q}_+$ and it has a continuous extension to $\mathbf{R}_+$, say $t \mapsto x^{(t)*}$.

Here $\mathbf{R}_+$ denotes the set of all positive real numbers. Therefore power laws (1) hold for all $p, q \in \mathbf{R}_+$. Denote by $\mathcal{A}_+(I)$ the set of all topological abelian semigroup operations on I satisfying the assumptions ($\sharp_1$) and ($\sharp_2$). Our assumption ($\sharp_1$) leads to the following important concept called 'mean'. For each $x, y \in I$, put

$$M_*(x, y) = (x * y)^{(1/2)*}.$$

We call $M_*(x, y)$ the mean of x and y with respect to the operation $*$.

Moreover, let J be a topological ordered space with relation $\leq$, and denote by $\mathcal{A}_+^0(J, \leq)$ the set of all operations $\circ \in \mathcal{A}_+(J)$ such that

($\flat_1$) $a \leq b \Leftrightarrow a \circ c \leq b \circ c$ ($a, b, c \in J$)

and

($\flat_2$) $a \leq b \Rightarrow a^{(t)\circ} \leq b^{(t)\circ}$ ($a, b \in J, t \in \mathbf{R}_+$).

Let $C(I, J)$ be the set of all continuous functions from I to J. Take $* \in \mathcal{A}_+(I)$, $\circ \in \mathcal{A}_+^0(J, \leq)$ and $f \in C(I, J)$ arbitrarily. If f satisfies

$$f\left(M_*(x, y)\right) \leq M_\circ\left(f(x), f(y)\right) \quad \left(\text{resp.} f\left(M_*(x, y)\right) \geq M_\circ\left(f(x), f(y)\right)\right)$$

for all $x, y \in I$, then we say that f is said to be $(*, \circ)$-convex (resp. concave).

The following theorem states a finite form of Jensen's inequality for a $(*, \circ)$-convex (or concave) function.

Theorem 1 *Let $* \in \mathcal{A}_+(I)$ and $\circ \in \mathcal{A}_+^0(J, \leq)$. If $f \in C(I, J)$ is $(*, \circ)$-convex, then*

$$f\left(x_1^{(t_1)*} * \cdots * x_n^{(t_n)*}\right) \leq f(x_1)^{(t_1)\circ} \circ \cdots \circ f(x_n)^{(t_n)\circ}$$

holds for all $n \in \mathbf{N}$, $x_1, \ldots, x_n \in I$ and $t_1, \ldots, t_n \in \mathbf{R}_+$ with $t_1 + \cdots + t_n = 1$.

If f is $(*, \circ)$-concave, then the inequality above is reversed.

Remark 1 Let $\mathbf{R}_+^2$ be the product space of $\mathbf{R}_+$ with ordinary topology. Let $*$ be the operation on $\mathbf{R}_+^2$ defined by

$$(a, b) * (c, d) = (ac, ad + bc)$$

for each $(a, b), (c, d) \in \mathbf{R}_+^2$. Then $*$ is a topological abelian semigroup operation on $\mathbf{R}_+^2$ (*cf.* [2, p.157-160]). In fact, $(\mathbf{R}_+^2, *)$ is topologically isomorphic to an abelian subsemigroup of the semigroup of all 2×2 matrices with usual product under the following mapping:

$$(a, b) \mapsto \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Note that

$$(a, b)^{(n)*} = \left(a^n, na^{n-1}b\right) \quad \text{and} \quad (a, b)^{(1/n)*} = \left(a^{1/n}, \frac{b}{na^{1-1/n}}\right)$$

for all $n \in \mathbf{N}$. Then a simple calculation implies that

$$(a, b)^{(m/n)*} = \left(a^{m/n}, \frac{m}{n}a^{\frac{m}{n}-1}b\right)$$

for each $m, n \in \mathbf{N}$. Therefore we see that $* \in \mathcal{A}_+(\mathbf{R}_+^2)$. In this case, we obtain from a simple calculation that

$$M_*\big((a, b), (c, d)\big) = \left(\sqrt{ac}, \frac{ad + bc}{2\sqrt{ac}}\right)$$

for each $(a, b), (c, d) \in \mathbf{R}_+^2$.

Now let $\cdot$ be the ordinary multiplication on $\mathbf{R}_+$. Since $\mathbf{R}_+$ becomes a topological ordered space with the ordinary topology and the ordinary order $\leq$, we have that $\cdot \in \mathcal{A}_+^0(\mathbf{R}_+, \leq)$ and $M_\cdot(x, y) = \sqrt{xy}$ for each $x, y \in \mathbf{R}_+$. Let α and β be real numbers and put

$$f_{\alpha, \beta}(a, b) = a^\alpha b^\beta$$

for each $(a, b) \in \mathbf{R}_+^2$. Then $f_{\alpha, \beta}$ is a continuous function from $\mathbf{R}_+^2$ to $\mathbf{R}_+$ such that

$$f_{\alpha, \beta}\big(M_*\big((a, b), (c, d)\big)\big) = (\sqrt{ac})^{\alpha - \beta}\left(\frac{ad + bc}{2}\right)^\beta$$

and

$$M_\cdot\big(f_{\alpha, \beta}(a, b), f_{\alpha, \beta}(c, d)\big) = (\sqrt{ac})^\alpha(\sqrt{bd})^\beta$$

for each $(a, b), (c, d) \in \mathbf{R}_+^2$. Therefore we can easily see that if $\beta \leq 0$ (resp. $\beta \geq 0$), then $f_{\alpha, \beta}$ is $(*, \cdot)$-convex (resp. $(*, \cdot)$-concave).

Remark 2 Let E be a nontrivial real interval with the ordinary topology and the ordinary order $\leq$. In this case, Craigen and Pales [3] showed that if $\circ$ is a continuous cancellative

semigroup operation on E, then there exists a continuous order-preserving bijection φ of E onto another (necessarily unbounded) real interval such that

$$a \circ b = \varphi^{-1}\big(\varphi(a) + \varphi(b)\big)$$

for all $a, b \in E$ (*cf.* Aczel [4, 5]). Therefore we can easily see that if $\varphi(E) = \mathbf{R}_+$, then all continuous cancellative semigroup operations on E are in $\mathcal{A}_+^0(E, \leq)$.

Remark 3 It is clear that a direct product of $\mathbf{R}_+$ admits a semigroup operation in $\mathcal{A}_+^0$ that is given as the product of semigroup operations on each $\mathbf{R}_+$. However, the semigroup operation on $\mathbf{R}_+^2$ described in Remark 1 does not satisfy properties ($\flat_1$) and ($\flat_2$). So, it would be of interest to give an example of an ordered abelian semigroup with a semigroup operation in $\mathcal{A}_+^0$, which is not isomorphic, as a topological semigroup, to the direct product of the topological semigroup $\mathbf{R}_+$.

3 Lemmas and proof of Theorem 1

Throughout this section, let I and J be as in Section 2 and suppose that $* \in \mathcal{A}_+(I)$, $\circ \in \mathcal{A}_+^0(J, \leq)$ and that $f \in C(I, J)$ is $(*, \circ)$-convex.

Remark 4 If $f \in C(I, J)$ is $(*, \circ)$-concave, then all inequalities in this section are reversed.

Lemma 1 *The inequality*

$$f\big(x_1^{(1/2)*} * \cdots * x_n^{(1/2^n)*}\big) \leq f(x_1)^{(1/2)\circ} \circ \cdots f(x_{n-1})^{(1/2^{n-1})\circ} \circ \big[f\big(x_n^{(1/2)*}\big)\big]^{(1/2^{n-1})\circ}$$

holds for all $n \in \mathbf{N}$ and $x_1, \ldots, x_n \in I$.

Proof Let $n \in \mathbf{N}$ and $x_1, \ldots, x_n \in I$. Since f is $(*, \circ)$-convex, it follows that

$$f\big(x_1^{(1/2)*} * x_2^{(1/2^2)*} * \cdots * x_n^{(1/2^n)*}\big)$$

$$= f\big(\big(x_1 * x_2^{(1/2)*} * \cdots * x_n^{(1/2^{n-1})*}\big)^{(1/2)*}\big)$$

$$\leq \big(f(x_1) \circ f\big(x_2^{(1/2)*} * \cdots * x_n^{(1/2^{n-1})*}\big)\big)^{(1/2)\circ}$$

$$= f(x_1)^{(1/2)\circ} \circ \big[f\big(x_2^{(1/2)*} * \cdots * x_n^{(1/2^{n-1})*}\big)\big]^{(1/2)\circ}$$

$$= f(x_1)^{(1/2)\circ} \circ \big[f\big(\big(x_2 * x_3^{(1/2)*} * \cdots * x_n^{(1/2^{n-2})*}\big)^{(1/2)*}\big)\big]^{(1/2)\circ}$$

$$\leq f(x_1)^{(1/2)\circ} \circ \big[\big(f(x_2) \circ f\big(x_3^{(1/2)*} * \cdots * x_n^{(1/2^{n-2})*}\big)\big)^{(1/2)\circ}\big]^{(1/2)\circ}$$

$$= f(x_1)^{(1/2)\circ} \circ f(x_2)^{(1/2^2)\circ} \circ \big[f\big(x_3^{(1/2)*} * \cdots * x_n^{(1/2^{n-2})*}\big)\big]^{(1/2^2)\circ}$$

$$\cdots$$

$$\leq f(x_1)^{(1/2)\circ} \circ f(x_2)^{(1/2^2)\circ} \circ \cdots \circ f(x_{n-1})^{(1/2^{n-1})\circ} \circ \big[f\big(x_n^{(1/2)*}\big)\big]^{(1/2^{n-1})\circ}.$$

Therefore we obtain the desired inequality. □

Let p and q be two nonnegative rational numbers. We define $x^{(p)*} * y^{(q)*} = y^{(q)*}$ if $p = 0$, $q > 0$ and $x^{(p)*} * y^{(q)*} = x^{(p)*}$ if $q = 0$, $p > 0$. We also define in the same way as for $\circ$. Applying these notations, we show the following lemma.

Lemma 2 *The inequality $f(x^{(p)*} * y^{(1-p)*}) \leq f(x)^{(p)\circ} \circ f(y)^{(1-p)\circ}$ holds for all $x, y \in I$ and $p \in \mathbf{R}_+$ with $0 < p < 1$.*

Proof Let $x, y \in I$ and $p \in \mathbf{R}_+$ with $0 < p < 1$. By the binary system, we have the expansion $p = \sum_{i=1}^{\infty} p_i/2^i$, where $p_i \in \{0, 1\}$ $(i = 1, 2, \ldots)$. Since $p_i \in \{0, 1\}$ $(i = 1, 2, \ldots)$, it follows that

$$\lim_{n \to \infty} \left[f\left(x^{(p_1)*} * y^{(1-p_1)*}\right) \right]^{(1/2)\circ} \circ \cdots \circ \left[f\left(x^{(p_{n-1})*} * y^{(1-p_{n-1})*}\right) \right]^{(1/2^{n-1})\circ}$$

$$= \lim_{n \to \infty} \left[f(x)^{(p_1/2)\circ} \circ f(y)^{((1-p_1)/2)\circ} \right] \circ \cdots$$

$$\circ \left[f(x)^{(p_n/2^{n-1})\circ} \circ f(y)^{((1-p_{n-1})/2^{n-1})\circ} \right]$$

$$= \lim_{n \to \infty} f(x)^{(\sum_{i=1}^{n-1} p_i/2^i)\circ} \circ f(y)^{(\sum_{i=1}^{n-1}(1-p_i)/2^i)\circ}$$

$$= f(x)^{(p)\circ} \circ f(y)^{(1-p)\circ}.$$

So, putting

$$a_n = \left[f\left(x^{(p_1)*} * y^{(1-p_1)*}\right) \right]^{(1/2)\circ} \circ \cdots \circ \left[f\left(x^{(p_{n-1})*} * y^{(1-p_{n-1})*}\right) \right]^{(1/2^{n-1})\circ}$$

for each $n \in \mathbf{N}$, we have

$$\lim_{n \to \infty} a_n = f(x)^{(p)\circ} \circ f(y)^{(1-p)\circ}. \tag{2}$$

Put

$$\mathbf{N}_1 = \left\{ n \in \mathbf{N} : f\left(\left(x^{(p_n)*} * y^{(1-p_n)*}\right)^{(1/2)*}\right) = f\left(x^{(1/2)*}\right) \right\}$$

and

$$\mathbf{N}_0 = \left\{ n \in \mathbf{N} : f\left(\left(x^{(p_n)*} * y^{(1-p_n)*}\right)^{(1/2)*}\right) = f\left(y^{(1/2)*}\right) \right\}.$$

Since

$$f\left(\left(x^{(p_n)*} * y^{(1-p_n)*}\right)^{(1/2)*}\right) = \begin{cases} f(x^{(1/2)*}) & (p_n = 1), \\ f(y^{(1/2)*}) & (p_n = 0) \end{cases}$$

for each $n \in \mathbf{N}$, it follows that $\mathbf{N} = \mathbf{N}_1 \cup \mathbf{N}_0$. Then either $\mathbf{N}_1$ or $\mathbf{N}_0$ are infinite. Put

$$e_n = \left[f\left(\left(x^{(p_n)*} * y^{(1-p_n)*}\right)^{(1/2)*}\right) \right]^{(1/2^{n-1})\circ}$$

for each $n \in \mathbf{N}$. If $\mathbf{N}_1$ is infinite, then we have

$$\lim_{n \in \mathbf{N}_1} e_n \circ f\left(x^{(1/2)*}\right) = \lim_{n \in \mathbf{N}_1} \left(f\left(x^{(1/2)*}\right)\right)^{(1+1/2^{n-1})\circ} = f\left(x^{(1/2)*}\right). \tag{3}$$

Therefore it follows from (2) and (3) that

$$f(x)^{(p)\circ} \circ f(y)^{(1-p)\circ} \circ f\left(x^{(1/2)*}\right) = \lim_{n \in \mathbf{N}_1} a_n \circ e_n \circ f\left(x^{(1/2)*}\right). \tag{4}$$

Also we have from Lemma 1 that

$$f\big(x^{(\sum_{i=1}^n p_i/2^i)*} * y^{(\sum_{i=1}^n (1-p_i)/2^i)*}\big)$$

$$= f\big(\big(x^{(p_1)*} * y^{(1-p_1)*}\big)^{(1/2)*} * \cdots * \big(x^{(p_n)*} * y^{(1-p_n)*}\big)^{(1/2^n)*}\big)$$

$$\leq \big[f\big(x^{(p_1)*} * y^{(1-p_1)*}\big)\big]^{(1/2)\circ} \circ \cdots \circ \big[f\big(x^{(p_{n-1})*} * y^{(1-p_{n-1})*}\big)\big]^{(1/2^{n-1})\circ}$$

$$\circ \big[f\big(\big(x^{(p_n)*} * y^{(1-p_n)*}\big)^{(1/2)*}\big)\big]^{(1/2^{n-1})\circ}$$

$$= a_n \circ e_n$$

for all $n \in \mathbf{N}_1$. Then it follows from $(\flat_1)$ that

$$f\big(x^{(\sum_{i=1}^n p_i/2^i)*} * y^{(\sum_{i=1}^n (1-p_i)/2^i)*}\big) \circ f\big(x^{(1/2)*}\big) \leq a_n \circ e_n \circ f\big(x^{(1/2)*}\big) \tag{5}$$

for all $n \in \mathbf{N}_1$. Letting $n \in \mathbf{N}_1 \to \infty$ in (5), we obtain from (4) that

$$f\big(x^{(p)*} * y^{(1-p)*}\big) \circ f\big(x^{(1/2)*}\big) \leq f(x)^{(p)\circ} \circ f(y)^{(1-p)\circ} \circ f\big(x^{(1/2)*}\big). \tag{6}$$

Canceling $f(x^{(1/2)*})$ in (6) by $(\flat_1)$, we obtain the desired inequality. Similarly, the desired inequality is obtained in case that $\mathbf{N}_0$ is infinite. □

Lemma 3 *The inequality*

$$f\big((x_1 * \cdots * x_n)^{(1/n)*}\big) \leq \big(f(x_1) \circ \cdots \circ f(x_n)\big)^{(1/n)\circ}$$

holds for all $n \in \mathbf{N}$ and $x_1, \ldots, x_n \in I$.

Proof It is clear that the lemma holds for $n = 1$. Suppose the lemma holds for $n = k$. Let $x_1, \ldots, x_k, x_{k+1} \in I$ and put

$$a = \big(f(x_1) \circ \cdots \circ f(x_k)\big)^{(1/k)\circ} \quad \text{and} \quad x = (x_1 * \cdots * x_k)^{(1/k)*}.$$

Then $a \geq f(x)$ by hypothesis, and hence $a^{(k/k+1)\circ} \geq f(x)^{(k/k+1)\circ}$ by $(\flat_2)$. It follows from $(\flat_1)$ that

$$a^{(k/k+1)\circ} \circ f(x_{k+1})^{(1/k+1)\circ} \geq f(x)^{(k/k+1)\circ} \circ f(x_{k+1})^{(1/k+1)\circ}.$$

Therefore we have from Lemma 2 that

$$\begin{aligned}
\big(f(x_1) \circ \cdots \circ f(x_{k+1})\big)^{(1/k+1)\circ} &= \big(a^{(k)\circ} \circ f(x_{k+1})\big)^{(1/k+1)\circ} \\
&= a^{(k/k+1)\circ} \circ f(x_{k+1})^{(1/k+1)\circ} \\
&\geq f(x)^{(k/k+1)\circ} \circ f(x_{k+1})^{(1/k+1)\circ} \\
&\geq f\big(x^{(k/k+1)*} * x_{k+1}^{(1/k+1)*}\big) \\
&= f\big((x_1 * \cdots * x_k * x_{k+1})^{(1/k+1)*}\big).
\end{aligned}$$

In other words, the lemma holds for $n = k + 1$. Then, by mathematical induction, the lemma holds for all $n \in \mathbf{N}$. □

Lemma 4 *The inequality*

$$f\big(x_1^{(p_1)*} * \cdots * x_n^{(p_n)*}\big) \leq f(x_1)^{(p_1)\circ} \circ \cdots \circ f(x_n)^{(p_n)\circ}$$

holds for all $x_1, \ldots, x_n \in I$, $n \in \mathbf{N}$ and $p_1, \ldots, p_n \in \mathbf{Q}_+$ with $p_1 + \cdots + p_n = 1$.

Proof This result follows directly from Lemma 3. □

We are now in a position to prove Theorem 1.

Proof Let $n \in \mathbf{N}$, $x_1, \ldots, x_n \in I$ and $t_1, \ldots, t_n \in \mathbf{R}_+$ with $t_1 + \cdots + t_n = 1$. For each $1 \leq i \leq n$, choose a sequence $\{p_{ik}\}_{k=1}^\infty$ in $\mathbf{Q}_+$ which converges to t_i. Put $q_k = p_{1k} + \cdots + p_{nk}$ for each $k \in \mathbf{N}$. Then we have from Lemma 4 that

$$f\big(x_1^{(p_{1k}/q_k)*} * \cdots * x_n^{(p_{nk}/q_k)*}\big) \leq f(x_1)^{(p_{1k}/q_k)\circ} \circ \cdots \circ f(x_n)^{(p_{nk}/q_k)\circ}. \tag{7}$$

Hence, after taking the limit with respect to k in (7), we obtain the desired inequality:

$$f\big(x_1^{(t_1)*} * \cdots * x_n^{(t_n)*}\big) \leq f(x_1)^{(t_1)\circ} \circ \cdots \circ f(x_n)^{(t_n)\circ}.$$

Of course, if f is $(*, \circ)$-concave, the above inequality is reversed, as stated in Remark 4. This completes the proof of Theorem 1. □

4 Applications

Let K be a topological ordered space with order $\leq$, and let $\circ$, $*$ and $\diamond$ be three operations in $\mathcal{A}_+^0(K, \leq)$ which have the following properties:

$$(a \circ b) * c = (a * c) \circ (b * c) \quad (\forall a, b, c \in K), \tag{8}$$

$$\exists e \in K : e * x = x \quad (\forall x \in K), \tag{9}$$

$$a \diamond b = a \circ b \circ (a * b) \quad (\forall a, b \in K). \tag{10}$$

In this case, it is clear that an element e in (9) is unique.

Lemma 5 *The equality $x^{(t)\diamond} \circ e = (x \circ e)^{(t)*}$ holds for each $x \in K$ and $t \in \mathbf{R}_+$.*

Proof Take $x \in K$ arbitrarily. By (8) and (9), we have $(a \diamond b) \circ e = (a \circ e) * (b \circ e)$ for all $a, b \in K$. By mathematical induction, we have

$$(x_1 \diamond \cdots \diamond x_k) \circ e = (x_1 \circ e) * \cdots * (x_k \circ e) \tag{11}$$

for all $k \in \mathbf{N}$ and $x_1, \ldots, x_k \in K$. In particular, we have

$$x^{(k)\diamond} \circ e = (x \circ e)^{(k)*} \tag{12}$$

for all $k \in \mathbf{N}$. Then we have

$$
\begin{aligned}
(x \circ e)^{(n/m)*} &= \left((x \circ e)^{(n)*}\right)^{(1/m)*} \\
&= \left(x^{(n)\diamond} \circ e\right)^{(1/m)*} \quad \text{by (12)} \\
&= \left(\left(x^{(n/m)\diamond}\right)^{(m)\diamond} \circ e\right)^{(1/m)*} \\
&= \left(\left(x^{(n/m)\diamond} \circ e\right)^{(m)*}\right)^{(1/m)*} \quad \text{by (12)} \\
&= x^{(n/m)\diamond} \circ e
\end{aligned}
$$

for all $n, m \in \mathbf{N}$. Therefore $x^{(p)\diamond} \circ e = (x \circ e)^{(p)*}$ holds for all $p \in \mathbf{Q}_+$. Take $t \in \mathbf{R}_+$ arbitrarily, and choose a sequence $\{p_n\}$ in $\mathbf{Q}_+$ which converges to t. Then

$$
x^{(t)\diamond} \circ e = \lim_{n \to \infty} x^{(p_n)\diamond} \circ e = \lim_{n \to \infty} (x \circ e)^{(p_n)*} = (x \circ e)^{(t)*}
$$

holds and so the proof is complete. □

Lemma 6 *Suppose that* $M_*(a, b) \leq M_\circ(a, b)$ *holds for all* $a, b \in K$. *Then* $M_*(a, b) \leq M_\diamond(a, b) \leq M_\circ(a, b)$ *holds for each* $a, b \in K$.

Proof Let $a, b \in K$. We first show that $M_*(a, b) \leq M_\diamond(a, b)$. Since

$$
\left((a * b)^{(1/2)*}\right)^{(2)\diamond} \leq a \circ b,
$$

it follows from Lemma 5 that

$$
\begin{aligned}
\left((a * b)^{(1/2)*} \circ e\right)^{(2)*} &= (a * b) \circ \left((a * b)^{(1/2)*}\right)^{(2)\diamond} \circ e \\
&\leq (a * b) \circ (a \circ b) \circ e \\
&= (a \diamond b) \circ e \\
&= \left((a \diamond b)^{(1/2)\diamond}\right)^{(2)\diamond} \circ e \\
&= \left((a \diamond b)^{(1/2)\diamond} \circ e\right)^{(2)*}.
\end{aligned}
$$

Therefore we obtain from ($\flat_2$) for $*$ that

$$
(a * b)^{(1/2)*} \circ e \leq (a \diamond b)^{(1/2)\diamond} \circ e.
$$

Canceling e in the above inequality, we obtain the desired inequality.

We next show that $M_\diamond(a, b) \leq M_\circ(a, b)$. Since

$$
a * b \leq \left((a \circ b)^{(1/2)\circ}\right)^{(2)*},
$$

it follows from Lemma 5 that

$$
\begin{aligned}
\left((a \diamond b)^{(1/2)\diamond} \circ e\right)^{(2)*} &= \left((a \diamond b)^{(1/2)\diamond}\right)^{(2)\diamond} \circ e \\
&= (a \diamond b) \circ e
\end{aligned}
$$

$$= (a * b) \circ a \circ b \circ e$$

$$\leq \left((a \circ b)^{(1/2)\circ} \right)^{(2)*} \circ a \circ b \circ e$$

$$= \left((a \circ b)^{(1/2)\circ} \circ e \right)^{(2)*}.$$

Therefore we obtain from ($\flat_2$) for $*$ that

$$(a \diamond b)^{(1/2)\diamond} \circ e \leq (a \circ b)^{(1/2)\circ} \circ e.$$

Canceling e in the above inequality, we obtain the desired inequality and so the proof is complete. □

The following result is a refinement of the mean inequality.

Theorem 2 *Let K be a topological ordered space with order $\leq$ and $*, \circ, \diamond \in \mathcal{A}_+^0(K, \leq)$ satisfying* (8), (9) *and* (10). *If $M_*(x, y) \leq M_\circ(x, y)$ holds for all $x, y \in K$, then*

$$x_1^{(t_1)*} * \cdots * x_n^{(t_n)*} \leq x_1^{(t_1)\diamond} \diamond \cdots \diamond x_n^{(t_n)\diamond} \leq x_1^{(t_1)\circ} \circ \cdots \circ x_n^{(t_n)\circ}$$

holds for all $n \in \mathbf{N}$, $x_1, \ldots, x_n \in K$ and $t_1, \ldots, t_n \in \mathbf{R}_+$ with $t_1 + \cdots + t_n = 1$.

Proof This follows immediately from Theorem 1 and Lemma 6. □

5 Examples

Throughout this section, let $\mathbf{R}_+$ be an ordinary topological ordered space.

Example 1 Put $x \circ_t y = (x^t + y^t)^{1/t}$ for each $x, y \in \mathbf{R}_+$ and $t \in \mathbf{R} \backslash \{0\}$. Then each $\circ_t$ is a topological abelian semigroup operation on $\mathbf{R}_+$ such that $x^{(n)\circ_t} = n^{1/t}x$ and $x^{(1/n)\circ_t} = (1/n)^{1/t}x$ for all $n \in \mathbf{N}$ and $x \in \mathbf{R}_+$. Also, since

$$x^{(n/m)\circ_t} = \left(x^{(1/m)\circ_t} \right)^{(n)\circ_t} = n^{1/t}(1/m)^{1/t}x = (n/m)^{1/t}x$$

for each $m, n \in \mathbf{N}$, $t \in \mathbf{R} \backslash \{0\}$ and $x \in \mathbf{R}_+$, it follows that $\circ_t \in \mathcal{A}_+^0(\mathbf{R}_+, \leq)$ for each $t \in \mathbf{R} \backslash \{0\}$. Let $x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_n \in \mathbf{R}_+$ with $\alpha_1 + \cdots + \alpha_n = 1$. Since $x^{(\alpha)\circ_t} = \alpha^{1/t}x$ for each $x, \alpha \in \mathbf{R}_+$ and $t \in \mathbf{R} \backslash \{0\}$, we have that

$$x_1^{(\alpha_1)\circ_t} \circ_t \cdots \circ_t x_n^{(\alpha_n)\circ_t} = \left(\alpha_1 x_1^t + \cdots + \alpha_n x_n^t \right)^{1/t} \tag{13}$$

for all $t \in \mathbf{R} \backslash \{0\}$. Note that

$$M_{\circ_s}(x, y) = \left(\frac{x^s + y^s}{2} \right)^{1/s} \leq \left(\frac{x^t + y^t}{2} \right)^{1/t} = M_{\circ_t}(x, y)$$

for all $x, y \in \mathbf{R}_+$ and $s, t \in \mathbf{R} \backslash \{0\}$ with $s \leq t$. Therefore Theorem 1 implies the following well-known inequality:

$$\left(\alpha_1 x_1^s + \cdots + \alpha_n x_n^s \right)^{1/s} \leq \left(\alpha_1 x_1^t + \cdots + \alpha_n x_n^t \right)^{1/t},$$

where $s, t \in \mathbf{R} \backslash \{0\}$ with $s \leq t$.

Example 2 Let t be a positive real number, and let $\circ_t$ be the operation on $\mathbf{R}_+$ defined in Example 1. Then we have $\circ_t \in \mathcal{A}_+^0(\mathbf{R}_+, \leq)$. Also, if $x * y = xy$ $(x, y \in \mathbf{R}_+)$, then $* \in \mathcal{A}_+^0(\mathbf{R}_+, \leq)$ and $x^{(\alpha)*} = x^\alpha$ for all $x, \alpha \in \mathbf{R}_+$. Hence we have

$$M_*(x, y) = \sqrt{xy} \leq \left(\frac{x^t + y^t}{2} \right)^{1/t} = M_{\circ_t}(x, y)$$

for each $x, y \in \mathbf{R}_+$.

Let $n \in \mathbf{N}$ and $x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_n \in \mathbf{R}_+$ with $\alpha_1 + \cdots + \alpha_n = 1$. Then it is clear that

$$x_1^{(\alpha_1)*} * \cdots * x_n^{(\alpha_n)*} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}. \tag{14}$$

If $x \diamond y = x \circ_t y \circ_t (x * y)$ $(x, y \in \mathbf{R}_+)$, then $\diamond \in \mathcal{A}_+^0(\mathbf{R}_+, \leq)$ and three operations $*$, $\circ$ and $\diamond$ on $\mathbf{R}_+$ satisfy (8), (9) and (10). If $x, \alpha \in \mathbf{R}_+$, we have from Lemma 5 that

$$x^{(\alpha)\diamond} \circ_t 1 = (x \circ_t 1)^{(\alpha)*} = (x \circ_t 1)^\alpha = \left(x^t + 1 \right)^{\alpha/t}. \tag{15}$$

Then we have from (11) and (15) that

$$
\begin{aligned}
x_1^{(\alpha_1)\diamond} \diamond \cdots \diamond x_n^{(\alpha_n)\diamond} \circ_t 1 &= \left(x_1^{(\alpha_1)\diamond} \circ_t 1 \right) * \cdots * \left(x_n^{(\alpha_n)\diamond} \circ_t 1 \right) \\
&= \left(x_1^{(\alpha_1)\diamond} \circ_t 1 \right) \cdots \left(x_n^{(\alpha_n)\diamond} \circ_t 1 \right) \\
&= \left(x_1^t + 1 \right)^{\alpha_1/t} \cdots \left(x_n^t + 1 \right)^{\alpha_n/t},
\end{aligned}
$$

and hence

$$x_1^{(\alpha_1)\diamond} \diamond \cdots \diamond x_n^{(\alpha_n)\diamond} = \left(\left(x_1^t + 1 \right)^{\alpha_1} \cdots \left(x_n^t + 1 \right)^{\alpha_n} - 1 \right)^{1/t}. \tag{16}$$

Then we obtain from (13), (14), (16) and Theorem 2 that

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \leq \left(\left(x_1^t + 1 \right)^{\alpha_1} \cdots \left(x_n^t + 1 \right)^{\alpha_n} - 1 \right)^{1/t} \leq \left(\alpha_1 x_1^t + \cdots + \alpha_n x_n^t \right)^{1/t}.$$

This is a refinement of the geometric-arithmetic mean inequality.

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