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The existence of solutions of periodic BVP for second order impulsive differential equations

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Abstract

In this paper, we study the existence of solutions of periodic boundary value problems for impulsive differential equations depending on a parameter λ. By employing an existing critical point theorem, we find the range of the control parameter in which the boundary value problem admits at least one non-zero weak solution. An example illustrates our results.

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1 Introduction

The well-known impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states, which are often investigated in various fields of science and technology [1–3]. For example, in the motion of spacecraft, one has to consider instantaneous impulses at a position with jump discontinuities in velocity, but no change in the position [4–7]. This motivates us to consider the following particular periodic boundary value problems:

$$\begin{cases} -u''(t) + u(t) = \lambda f(t, u(t)), & t \neq t_j, t \in [0,1], \\ \Delta u'(t_j) = I_j(u(t_j)), & j = 1, 2, \ldots, m, \\ u(0) - u(1) = u'(0) - u'(1) = 0, \end{cases} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $I_j \in C(\mathbb{R}, \mathbb{R})$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$, λ is a positive real parameter and the operator Δ is defined as $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$, where $u'(t_j^+)(u'(t_j^-))$ denotes the right-hand (left-hand) limit of u' at t_j.

In the literature, some classical tools have been used to study impulsive differential equations. These classical techniques include some fixed point theorems, the lower and upper solutions and the coincidence degree theory [8–10]. Moreover, in the last few years, some researchers have gradually paid more attention to applying variational methods to deal with the existence of solutions for impulsive differential equation boundary value problems [11–16]. The same tool has also already been used for a Neumann nonlinear differential problem in [17] (see also [18, 19] and [20] for two-point and mixed problems). In this paper, we use critical point theory and variational methods to establish the existence of at least one weak solution for problem (1.1).

The rest of this paper is organized as follows. In Section 2 we present several definitions and main tools. In Section 3, under suitable hypotheses, we prove that problem (1.1) possesses at least one non-zero weak solution when λ lies in an exactly determined open interval. Finally, an example is provided to verify our results.

2 Preliminaries

In the following, we first introduce some notations. Take $X := \{u(t) | u(t) \in W^{1,2}([0,1]), u(0) = u(1)\}$, in which we consider the inner product

$$(u, v) = \int_0^1 u'(t)v'(t)\,dt + \int_0^1 u(t)v(t)\,dt,$$

and the norm

$$\|u\|_X = \left(\int_0^1 |u'(t)|^2\,dt + \int_0^1 |u(t)|^2\,dt \right)^{\frac{1}{2}}.$$

Note that this norm is equivalent to the usual norm

$$\|u\| = \left(\int_0^1 |u'(t)|^2\,dt \right)^{\frac{1}{2}}.$$

Definition 2.1 $f : [0,1] \times R \to R$ is an L^1-Carathéodory function if:

(i) $t \mapsto f(t,u)$ is measurable for every $u \in \mathbb{R}$;

(ii) $u \mapsto f(t,u)$ is continuous for almost every $t \in [0,1]$;

(iii) for every $s > 0$, there exists a function $l_s \in L^1([0,1])$ such that

$$\sup_{|u| \leq s} |f(t,u)| \leq l_s(t) \quad \text{for a.e. } t \in [0,1].$$

Definition 2.2 The function $u : [0,1] \to R$ is called a weak solution of problem (1.1) if $u \in X$ and

$$\int_0^1 u'(t)v'(t)\,dt + \int_0^1 u(t)v(t)\,dt + \sum_{j=1}^m I_j\big(u(t_j)\big)v(t_j) = \lambda \int_0^1 f\big(t, u(t)\big)v(t)\,dt$$

for all $v \in X$.

Note that if f is continuous, each weak solution is a classical solution of problem (1.1), *i.e.*, $u \in C^2(t_{j-1}, t_j)$, satisfies the equation of (1.1) a.e. on $t \in [0,1]$, the limits $u'(t_j^+)$, $u'(t_j^-)$, $j = 1, 2, \ldots, m$, exist and $\Delta u'(t_j) = I_j(u(t_j))$ holds.

We recall the following inequality which will be used later.

Lemma 2.3 *If $u \in X$, then*

$$\|u\|_\infty \leq \sqrt{2}\|u\|_X, \tag{2.1}$$

where $\|u\|_\infty = \max_{t \in [0,1]} |u(t)|$.

Proof The proof follows easily from the mean value theorem and the Hölder inequality, so we omit it here. □

Next we define a functional φ_λ as

$$\varphi_\lambda(u) = \Phi(u) - \lambda\Psi(u), \quad u \in X, \tag{2.2}$$

where

$$\Phi(u) = \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^{m} \int_0^{u(t_j)} I_j(s)\,ds \tag{2.3}$$

and

$$\Psi(u) = \int_0^1 F\big(t, u(t)\big)\,dt, \tag{2.4}$$

with

$$F(t, u) = \int_0^{u(t)} f(t, s)\,ds.$$

Note that φ_λ is Fréchet differentiable at any $u \in X$ and for any $v \in X$, we have

$$\varphi_\lambda'(u)(v) = \int_0^1 \big(u'(t)v'(t) + u(t)v(t)\big)\,dt + \sum_{j=1}^{m} I_j\big(u(t_j)\big)v(t_j)$$

$$- \lambda \int_0^1 f\big(t, u(t)\big)v(t)\,dt. \tag{2.5}$$

Obviously, φ_λ' is continuous and a critical point of φ_λ, by (2.5), gives a weak solution of problem (1.1).

For all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, we define

$$\beta(r_1, r_2) = \inf_{v \in \Phi^{-1}((r_1, r_2))} \frac{\sup_{u \in \Phi^{-1}((r_1, r_2))} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}, \tag{2.6}$$

$$\alpha(r_1, r_2) = \sup_{v \in \Phi^{-1}((r_1, r_2))} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}((-\infty, r_1))} \Psi(u)}{\Phi(v) - r_1}. \tag{2.7}$$

Note that for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, we have $\beta(r_1, r_2) \geq 0$, $\alpha(r_1, r_2) \geq 0$.

To prove our main results, we need the following critical point theorem.

Theorem 2.4 [21, Theorem 5.1] *Let X be a reflexive real Banach space. Let $\Phi : X \to R$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*; let $\Psi : X \to R$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put $\varphi_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \alpha(r_1, r_2), \tag{2.8}$$

where β and α are given by (2.6) *and* (2.7). *Then, for each* $\lambda \in (1/\alpha(r_1, r_2), 1/\beta(r_1, r_2))$, *there is* $u_{0,\lambda} \in \Phi^{-1}((r_1, r_2))$ *such that* $\varphi_\lambda(u_{0,\lambda}) \le \varphi_\lambda(u)$ *for all* $u \in \Phi^{-1}((r_1, r_2))$ *and* $\varphi'_\lambda(u_{0,\lambda}) = 0$.

For the sake of convenience, we list the following conditions.

(H1) f is an L^1-Carathéodory function.

(H2) $0 \le \int_0^u I_j(s)\,ds < \frac{|u|^2}{2m}$, $u \in \mathbb{R}$, $j = 1, 2, \ldots, m$.

3 Main results

In this section, we establish existence results for the periodic boundary value problem (1.1).

Given three nonnegative constants c_1, c_2, d, with $c_1 < \sqrt{2}d < 2d < c_2$, put

$$a(c_2, d) = 4\frac{\int_0^1 \max_{|u(t)| \le c_2} F(t, u(t))\,dt - \int_0^1 F(t, d)\,dt}{c_2^2 - 4d^2} \tag{3.1}$$

and

$$b(c_1, d) = 4\frac{\int_0^1 F(t, d)\,dt - \int_0^1 \max_{|u(t)| \le c_1} F(t, u(t))\,dt}{4d^2 - c_1^2}. \tag{3.2}$$

Theorem 3.1 *Assume that* (H1), (H2) *are satisfied and there exist three nonnegative constants* c_1, c_2, d, *with* $c_1 < \sqrt{2}d < 2d < c_2$, *such that*

$$a(c_2, d) < b(c_1, d). \tag{3.3}$$

Then, for each $\lambda \in (1/b(c_1, d), 1/a(c_2, d))$, *problem* (1.1) *admits at least one weak solution* $u(t)$, $t \in [0, 1]$, *such that* $\sqrt{6}c_1/6 < \|u\|_X < \sqrt{2}c_2/2$.

Proof By (2.3) and (2.4), we have that Φ is a nonnegative Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and Ψ is a continuously Gâteaux differentiable functional whose Gâteaux is compact. Let

$$r_1 = \frac{c_1^2}{4}, \qquad r_2 = \frac{c_2^2}{4}, \qquad u_0(t) = d, \quad \text{for } t \in [0, 1]. \tag{3.4}$$

By condition (H2), we have

$$\begin{aligned}
\Phi(u_0) &= \frac{1}{2}\|u_0\|_X^2 + \sum_{j=1}^m \int_0^{u_0(t_j)} I_j(s)\,ds \\
&= \frac{1}{2}d^2 + \sum_{j=1}^m \int_0^d I_j(s)\,ds \\
&\le d^2,
\end{aligned} \tag{3.5}$$

and

$$\Phi(u_0) \ge \frac{d^2}{2}. \tag{3.6}$$

Combining $c_1 < \sqrt{2}d < 2d < c_2$, (3.4), (3.5) and (3.6), we have

$$r_1 < \Phi(u_0) < r_2.$$

Clearly, we have $\Psi(u_0) = \int_0^1 F(t, u_0(t)) \, dt = \int_0^1 F(t, d) \, dt$. From Lemma 2.3, the estimate $\Phi(u) < r_2$, $u \in X$, implies that

$$\left| u(t) \right|^2 \leq 2\|u\|_X^2 \leq 2 \times 2\Phi(u) < 4r_2 = c_2^2, \quad t \in [0,1],$$

and

$$\int_0^1 F\big(t, u(t)\big) \, dt \leq \int_0^1 \max_{|u(t)| \leq c_2} F(t, u) \, dt.$$

Therefore

$$\sup_{u \in \Phi^{-1}((r_1, r_2))} \Psi(u) \leq \sup_{u \in \Phi^{-1}((-\infty, r_2))} \Psi(u) \leq \int_0^1 \max_{|u(t)| \leq c_2} F\big(t, u(t)\big) \, dt.$$

For $u \in X$ with $\Phi(u) < r_1$, one can similarly obtain

$$\sup_{u \in \Phi^{-1}((-\infty, r_1))} \Psi(u) \leq \int_0^1 \max_{|u(t)| \leq c_1} F\big(t, u(t)\big) \, dt.$$

Therefore, we have

$$
\begin{aligned}
\beta(r_1, r_2) &\leq \frac{\sup_{u \in \Phi^{-1}((-\infty, r_2))} \Psi(u) - \Psi(u_0)}{r_2 - \Phi(u_0)} \\
&\leq 4 \frac{\int_0^1 \max_{|u(t)| \leq c_2} F(t, u(t)) \, dt - \int_0^1 F(t, d) \, dt}{c_2^2 - 4d^2} \\
&= a(c_2, d).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\alpha(r_1, r_2) &\geq \frac{\Psi(u_0) - \sup_{u \in \Phi^{-1}((-\infty, r_1))} \Psi(u)}{\Phi(u_0) - r_1} \\
&\geq 4 \frac{\int_0^1 F(t, d) \, dt - \int_0^1 \max_{|u(t)| \leq c_1} F(t, u(t)) \, dt}{4d^2 - c_1^2} \\
&= b(c_1, d).
\end{aligned}
$$

So, by (3.3), we induce

$$\beta(r_1, r_2) < \alpha(r_1, r_2).$$

Therefore, by Theorem 2.4, for each $\lambda \in (1/b(c_1, d), 1/a(c_2, d))$, we have that $\Phi - \lambda\Psi$ admits at least one critical point u such that $r_1 < \Phi(u) < r_2$. Combining (2.3), we get

$$\frac{c_1^2}{4} < \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^m \int_0^{u(t_j)} I_j(s) \, ds \leq \frac{3}{2}\|u\|_X^2,$$

and

$$\frac{1}{2}\|u\|_X^2 \leq \frac{1}{2}\|u\|_X^2 + \sum_{j=1}^{m} \int_0^{u(t_j)} I_j(s)\, ds < \frac{c_2^2}{4}.$$

So, problem (1.1) admits at least one weak solution $u(t)$, $t \in [0,1]$, such that $\frac{\sqrt{6}}{6}c_1 < \|u\|_X < \frac{\sqrt{2}}{2}c_2$. $\qquad\square$

Theorem 3.2 *Assume that* (H1), (H2) *hold and there exist two positive constants* c, d, *with* $2d < c$, *such that*

$$\left(\int_0^1 \max_{|u(t)|\leq c} F(t,u(t))\, dt\right) \bigg/ c^2 < \left(\int_0^1 F(t,d)\, dt\right) \bigg/ 4d^2. \tag{3.7}$$

Then, for each $\lambda \in (d^2 / \int_0^1 F(t,d)\, dt, c^2/4 \int_0^1 \max_{|u(t)|\leq c} F(t,u(t))\, dt)$, *problem* (1.1) *admits at least one nontrivial weak solution* $u(t)$, $t \in [0,1]$, *such that* $\|u\|_X < \frac{\sqrt{2}}{2}c$.

Proof Let $c_1 = 0$ and $c_2 = c$, then by (3.1) and (3.2) we get

$$\begin{aligned}
a(c_2, d) &= 4 \frac{\int_0^1 \max_{|u(t)|\leq c_2} F(t,u(t))\, dt - \int_0^1 F(t,d)}{c_2^2 - 4d^2} \\
&\leq 4 \frac{\int_0^1 \max_{|u(t)|\leq c} F(t,u(t))\, dt - \frac{4d^2}{c^2} \int_0^1 \max_{|u(t)|\leq c} F(t,u(t))}{c^2 - 4d^2} \\
&= 4 \frac{\int_0^1 \max_{|u(t)|\leq c} F(t,u(t))\, dt}{c^2},
\end{aligned}$$

and

$$b(c_1, d) = 4 \frac{\int_0^1 F(t,d)\, dt}{4d^2}.$$

Therefore, owing to (3.6) we have $a(c,d) < b(0,d)$. Moreover, by Theorem 3.1, we have that for each $\lambda \in (d^2 / \int_0^1 F(t,d)\, dt, c^2/4 \int_0^1 \max_{|u(t)|\leq c} F(t,u(t))\, dt)$, problem (1.1) admits at least one nontrivial weak solution u such that $\|u\|_X < \frac{\sqrt{2}}{2}c$. $\qquad\square$

Now, when the nonlinear term of problem (1.1) is with separable variables, we have the following results. To be precise, let $\alpha \in L^1([0,1])$ be such that $\alpha(t) \geq 0$ a.e. $t \in [0,1]$, $\alpha \not\equiv 0$, and let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Consider the boundary value problem

$$\begin{cases}
-u''(t) + u(t) = \lambda \alpha(t) g(u(t)), & t \neq t_j, t \in [0,1], \\
\Delta u'(t_j) = I_j(u(t_j)), & j = 1, 2, \ldots, m, \\
u(0) - u(1) = u'(0) - u'(1) = 0.
\end{cases} \tag{3.8}$$

Put

$$G(u) = \int_0^{u(t)} g(s)\, ds \quad \text{for } u \in X.$$

Corollary 3.3 *Assume that* (H2) *is satisfied and there exist three nonnegative constants* c_1, c_2, d, *with* $c_1 < \sqrt{2}d < 2d < c_2$, *such that*

$$\frac{G(c_2) - G(d)}{c_2^2 - 4d^2} < \frac{G(d) - G(c_1)}{4d^2 - c_1^2}. \tag{3.9}$$

Then, for each $\lambda \in ((4d^2 - c_1^2)/4\|\alpha\|_1(G(d) - G(c_1)), (c_2^2 - 4d^2)/4\|\alpha\|_1(G(c_2) - G(d)))$, *where* $\|\alpha\|_1 = \int_0^1 |\alpha(t)|\, dt$, *problem* (3.8) *admits at least one weak solution* $u(t)$, $t \in [0,1]$, *such that* $\frac{\sqrt{6}}{6}c_1 < \|u\|_X < \frac{\sqrt{2}}{2}c_2$.

Proof Let $f(t,u) = \alpha(t)g(u)$ for all $(t,u) \in [0,1] \times \mathbb{R}$. It is clear that $F(t,u) = \alpha(t)G(u)$ for all $(t,u) \in [0,1] \times X$. Moreover, G is a nondecreasing function about u. So, we have

$$a(c_2, d) = 4\|\alpha\|_1 \frac{G(c_2) - G(d)}{c_2^2 - 4d^2} < 4\|\alpha\|_1 \frac{G(d) - G(c_1)}{4d^2 - c_1^2} = b(c_1, d).$$

So, for each $\lambda \in ((4d^2 - c_1^2)/4\|\alpha\|_1(G(d) - G(c_1)), (c_2^2 - 4d^2)/4\|\alpha\|_1(G(c_2) - G(d)))$, problem (3.8) admits at least one weak solution $u(t)$, $t \in [0,1]$, such that $\frac{\sqrt{6}}{6}c_1 < \|u\|_X < \frac{\sqrt{2}}{2}c_2$. □

Corollary 3.4 *Assume that* (H2) *is satisfied and there exist two positive constants* c, d, *with* $c > 2d$, *such that*

$$\frac{G(c)}{c^2} < \frac{G(d)}{4d^2}. \tag{3.10}$$

Then, for each $\lambda \in (d^2/G(d)\|\alpha\|_1, c^2/4G(c)\|\alpha\|_1)$, *problem* (3.8) *admits at least one weak solution* $u(t)$ *such that* $|u(t)| < c$ *for all* $t \in [0,1]$.

Proof Let $f(t,u) = \alpha(t)g(u)$ for all $(t,u) \in [0,1] \times \mathbb{R}$. It is clear that $F(t,u) = \alpha(t)G(u)$ for all $(t,u) \in [0,1] \times X$. Moreover, G is a nondecreasing function about u. So, we have

$$\frac{\int_0^1 \max_{|u(t)| \le c} F(t, u(t))\, dt}{c^2} = \frac{\|\alpha\|_1 G(c)}{c^2} < \frac{\|\alpha\|_1 G(d)}{4d^2} = \frac{\int_0^1 F(t, d)\, dt}{4d^2},$$

$$\frac{d^2}{\int_0^1 F(t, d)\, dt} = \frac{d^2}{G(d)\|\alpha\|_1}, \qquad \frac{c^2}{4\int_0^1 \max_{|u(t)| \le c} F(t, u(t))\, dt} = \frac{c^2}{4G(c)\|\alpha\|_1}.$$

Therefore, by Theorem 3.2, for each $\lambda \in (d^2/G(d)\|\alpha\|_1, c^2/4G(c)\|\alpha\|_1)$, problem (3.8) admits at least one weak solution $u(t)$ such that $|u(t)| < c$ for all $t \in [0,1]$. □

4 An example

In this section, we give an example to illustrate our main results.

Example 4.1 Consider the boundary value problem

$$\begin{cases} -u''(t) + u = 2\lambda t e^{-u}, & t \ne t_1, t \in [0,1], \\ \Delta u'(t_1) = \frac{1}{2}u(t_1), & t_1 = \frac{1}{2}, \\ u(0) - u(1) = u'(0) - u'(1) = 0. \end{cases} \tag{4.1}$$

Compared to problem (3.8), $\alpha(t) = 2t$, $g(u) = e^{-u}$, $I_j(u) = \frac{1}{2}u$. Clearly, (H2) is satisfied and $\alpha \in L^1([0,1])$ such that $\alpha(t) \geq 0$ a.e. $t \in [0,1]$, $\alpha \not\equiv 0$, and $g : R \to R$ is a nonnegative continuous function. Choose $d = 0.5$, $c = 2$. By simple calculations, we obtain

$$\frac{G(c)}{c^2} \approx 0.2162 < \frac{G(d)}{4d^2} \approx 0.3934,$$

$$\frac{d^2}{G(d)\|\alpha\|_1} \approx 0.6355, \qquad \frac{c^2}{4G(c)\|\alpha\|_1} \approx 1.1566.$$

Applying Corollary 3.4, when $\lambda \in (0.6355, 1.1566)$, system (4.1) has at least one weak solution u such that $|u(t)| < 2$ for all $t \in [0,1]$.

Authors' contributions
JX drafted the manuscript and joined discussion with ZL. ZL read the draft and provided comments. Both authors read and approved the final version of the manuscript.

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