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Poincaré inequalities and the sharp maximal inequalities with L^{φ}-norms for differential forms

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Abstract

This paper is concerned with the Poincaré inequalities and the sharp maximal inequalities for differential forms with L^{φ}-norm, where φ satisfies nonstandard growth conditions. These results can be used to estimate the norms of classical operators and analyze integral properties of differential forms.

Keywords: Poincaré inequalities; sharp maximal inequalities; nonstandard growth conditions; differential forms

1 Introduction

In this paper, we consider the functional

$$I(\Omega, v) = \int_{\Omega} \varphi(|dv|)\,dx, \tag{1}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a Young function satisfying the nonstandard growth condition

$$p\varphi(t) \le t\varphi'(t) \le q\varphi(t), \quad 1 < p \le q < \infty. \tag{2}$$

This condition was first used in [1]; the author used this condition to get the higher integrability of the gradient of minimizers. We can find that the first inequality in (2) is equivalent to $\frac{\varphi(t)}{t^p}$ that is increasing, and the second inequality in (2) is equivalent to $\triangle_2$-condition, *i.e.*, for each $t > 0$, $\varphi(2t) \le K\varphi(t)$, where $K > 1$, and $\frac{\varphi(t)}{t^q}$ is decreasing with t. Also, condition (2) implies that $\varphi(t)$ satisfies

$$c_1 t^p - c_2 \le \varphi(t) \le c_3\left(t^q + 1\right), \tag{3}$$

but the exponents p, q in (2) may not be the best ones in order for (3) to hold. The following example can be found in [2], the convex function

$$\varphi(t) = \begin{cases} et^3 & \text{if } 0 \le t \le e, \\ t^{4+\sin\log\log t} & \text{if } t \ge e \end{cases}$$

satisfies (2) with $p = 4 - \sqrt{2}$ and $q = 4 + \sqrt{2}$ and satisfies (3) with $p = 3$ and $q = 5$. Moreover, if $4 - \sqrt{2} < r < 4 + \sqrt{2}$, $\frac{\varphi(t)}{t^r}$ is neither strictly increasing nor decreasing. Particularly, $\varphi(t) = t^p$ satisfies (2) because of $t\varphi'(t) = p\varphi(t)$, and this makes inequalities with the norm $\|\cdot\|_p$ become a special case of Theorem 2.3, for more details see [1, 2].

We assume that Ω is a bounded convex domain in $\mathbb{R}^n$ ($n \geq 2$). Both B and σB are balls or cubes with the same center, $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$. $|E|$ is used to denote the n-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^n$, and all integrals involved in this paper are the Lebesgue integrals. Let $e_1, \dots, e_n$ be the standard unit basis of $\mathbb{R}^n$. For $l = 0, 1, \dots, n$, the linear space of l-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = \{i_i, i_2, \dots, i_l\}$, $1 \leq i_1 < i_2 < \cdots < i_l \leq n$, is denoted by $\Lambda^l = \Lambda^l(\mathbb{R}^n)$. The Grassmann algebra $\Lambda = \Lambda(\mathbb{R}^n) = \bigoplus_{l=0}^n \Lambda^l(\mathbb{R}^n)$ is a graded algebra with respect to the exterior products. A differential l-form ω on Ω is a Schwartz distribution on Ω with values in $\Lambda^l(\mathbb{R}^n)$, and it can be denoted by

$$u(x) = \sum_I u_I(x)\,dx_I = \sum_{1\leq i_1 \leq \cdots \leq i_l \leq n} u_{i_1 i_2 \cdots i_l}(x)\,dx_{i_1} \wedge \cdots \wedge dx_{i_l}.$$

The exterior derivative $d : \mathcal{D}'(\Omega, \Lambda^l) \to \mathcal{D}'(\Omega, \Lambda^{l+1})$ is expressed by

$$du(x) = \sum_{j=1}^n \sum_I \frac{\partial u_{i_1 \cdots i_l}}{\partial x_j}\,dx_j \wedge dx_{i_1} \wedge \cdots \wedge dxi_l.$$

A differential l-form $u \in \mathcal{D}'(\Omega, \Lambda^l)$ is a closed form if $du = 0$ in Ω. $L^p(\Omega, \Lambda^l)$ is the space of l-forms $u(x) = \sum_I u_I(x)\,dx_I$ with $u_I \in L^p(\Omega, \mathbb{R})$ for all ordered l-tuples I, it is a Banach space endowed with the norm $\|u\|_{p,\Omega} = (\int_\Omega |u|^p\,dx)^{1/p}$. Similarly, $W^{1,p}(\Omega, \Lambda^l)$ are those differential l-forms on Ω, whose coefficients are in $W^{1,p}(\Omega, \mathbb{R})$, and it is a Banach space endowed with the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$.

The following result was obtained in [3]. For $y \in \Omega$, there corresponds a linear operator $K_y : C^\infty(\Omega, \Lambda^l) \to C^\infty(\Omega, \Lambda^l)$ defined by $(K_y u)(x; \xi_1, \dots, \xi_{l-1}) = \int_0^1 t^{l-1} u(tx + y - ty; x - y, \xi_1, \dots, \xi_{l-1})\,dt$ and a decomposition $u = d(K_y u) + K_y(du)$. A homotopy operator $T : C^\infty(\Omega, \Lambda^l) \to C^\infty(\Omega, \Lambda^l)$ is defined by averaging K_y over all points y in Ω, *i.e.*, $Tu = \int_\Omega \varphi(y) K_y u\,dy$, where $\varphi(y) \in C_0^\infty(\Omega)$ is normalized by $\int_\Omega \varphi(y)\,dy = 1$. Then, there is a decomposition $u = d(Tu) + T(du)$. The l-form $u_\Omega \in \mathcal{D}'(\Omega, \Lambda^l)$ is defined by

$$u_\Omega = \frac{1}{|\Omega|} \int_\Omega u(y)\,dy, \quad l = 0 \quad \text{and} \quad u_\Omega = d(Tu), \quad l = 1, 2, \dots, n$$

for all $u \in L^p(\Omega, \Lambda^l)$, $1 \leq p < \infty$. For any differential forms, $u \in L^s_{\mathrm{loc}}(B, \Lambda^l)$, $l = 1, 2, \dots, n$, $1 < s < \infty$, we have

$$\|\nabla Tu\|_{s,B} \leq C|B|\|u\|_{s,B}, \qquad \|Tu\|_{s,B} \leq C|B|(\operatorname{diam} B)\|u\|_{s,B}. \tag{4}$$

The rest of this paper is organized as follows. In Section 2, Poincaré inequalities for differential forms with Orlicz norm are obtained. In Section 3, the sharp maximal inequalities with Orlicz norms applied to k-quasiminimizer are obtained. In Section 4, using the methods that appeared in Section 2 and Section 3, we get some estimates of classical operators.

2 Poincaré inequalities

It is well known that the Poincaré inequality played an important role in studying the partial differential equations (PDEs) and the potential theory. Some versions of Poincaré inequalities for functions and differential forms with L^p-norm have been obtained, and in recent years, these inequalities for differential forms and operators with Luxemburg norm have been established, we refer reader to [4–13] and the references therein. First, we introduce some existing definitions and lemmas.

A continuously increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ is called an Orlicz function. The Orlicz space $L^\varphi(\Omega)$ consists of all measurable functions f on Ω satisfying $\int_\Omega \varphi(\frac{|f|}{\lambda})\,dx \leq \infty$ for some $\lambda = \lambda(f) > 0$, and it is equipped with the nonlinear Luxemburg functional

$$\|f\|_{\varphi(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \varphi\left(\frac{|f|}{\lambda}\right) dx \leq 1\right\}. \tag{5}$$

A convex Orlicz function φ is called a Young function. If φ is a Young function, then $\|\cdot\|_{\varphi(\Omega)}$ defines a norm in $L^\varphi(\Omega)$, which is called the Orlicz norm or Luxemburg norm, for more details see [5, 8–10, 12].

Lemma 2.1 [3] *Let $u \in \mathcal{D}'(B, \Lambda^l)$ and $du \in L^p(B, \Lambda^{l+1})$. Then, $u - u_B$ is in $L^{\frac{np}{n-p}}(B, \Lambda^l)$ and*

$$\left(\int_B |u - u_B|^{\frac{np}{n-p}}\,dx\right)^{\frac{n-p}{np}} \leq c(n,p)\left(\int_B |du|\,dx\right)^{\frac{1}{p}} \tag{6}$$

for B is a ball or cube in Ω, $l = 0, 1, \ldots, n$ and $1 < p < n$.

Lemma 2.2 [1] *Suppose φ is a continuous function satisfying* (2) *with $q(n-p) < np$, $1 < p \leq q < \infty$. For any $t > 0$, setting*

$$F(t) = \int_0^t \left(\frac{\varphi(s^{\frac{1}{q}})}{s}\right)^{\frac{n+q}{q}} ds, \qquad H(t) = \frac{(\varphi(t^{\frac{1}{q}}))^{\frac{n+q}{q}}}{t^{\frac{n}{q}}}.$$

Then, $F(t)$ is a concave function, and there exists a constant C, such that

$$H(t) \leq F(t) \leq CH(t) \quad \forall t > 0. \tag{7}$$

Next, we start with the main result of this section.

Theorem 2.3 *Suppose $u \in \mathcal{D}'(\Omega, \Lambda^l)$ and $du \in L^p(\Omega, \Lambda^{l+1})$, φ is a Young function satisfying* (2) *with $q(n-p) < np$, $1 < p \leq q < \infty$. Then, there exists a constant C, independent of u, such that*

$$\int_B \varphi\big(|u - u_B|\big)\,dx \leq C\int_B \varphi\big(|du|\big)\,dx, \tag{8}$$

where B is a ball in Ω.

Proof Applying the Hölder inequality, we have

$$\begin{aligned}
\fint_B \varphi\big(|u-u_B|\big)\,dx &= \fint_B \frac{\varphi(|u-u_B|)}{|u-u_B|^{\frac{nq}{n+q}}}|u-u_B|^{\frac{nq}{n+q}}\,dx \\
&\le \left(\fint_B \frac{\varphi^{\frac{n+q}{q}}(|u-u_B|)}{|u-u_B|^n}\,dx\right)^{\frac{q}{n+q}}\left(\fint_B |u-u_B|^q\,dx\right)^{\frac{n}{n+q}}.
\end{aligned} \tag{9}$$

Because of (7) and the concavity of F, which appeared in Lemma 2.2, (9) becomes

$$\begin{aligned}
\fint_B \varphi\big(|u-u_B|\big)\,dx &\le \left(\fint_B H\big(|u-u_B|^q\big)\,dx\right)^{\frac{q}{n+q}}\left(\fint_B |u-u_B|^q\,dx\right)^{\frac{n}{n+q}} \\
&\le \left(\fint_B F\big(|u-u_B|^q\big)\,dx\right)^{\frac{q}{n+q}}\left(\fint_B |u-u_B|^q\,dx\right)^{\frac{n}{n+q}} \\
&\le F^{\frac{q}{n+q}}\left(\fint_B |u-u_B|^q\,dx\right)\left(\fint_B |u-u_B|^q\,dx\right)^{\frac{n}{n+q}} \\
&\le c_1(n,q)H^{\frac{q}{n+q}}\left(\fint_B |u-u_B|^q\,dx\right)\left(\fint_B |u-u_B|^q\,dx\right)^{\frac{n}{n+q}} \\
&= c_1(n,q)\frac{\varphi((\fint_B |u-u_B|^q\,dx)^{\frac{1}{q}})}{(\fint_B |u-u_B|^q\,dx)^{\frac{n}{n+q}}}\left(\fint_B |u-u_B|^q\,dx\right)^{\frac{n}{n+q}} \\
&= c_1(n,q)\varphi\left(\left(\fint_B |u-u_B|^q\,dx\right)^{\frac{1}{q}}\right).
\end{aligned} \tag{10}$$

If $1 < p < n$, by assumption, we have $q < \frac{np}{n-p}$. It follows from Lemma 2.1 that

$$\begin{aligned}
\left(\fint_B |u-u_B|^q\,dx\right)^{\frac{1}{q}} &\le \left(\fint_B |u-u_B|^{\frac{np}{n-p}}\,dx\right)^{\frac{n-p}{np}} \\
&\le c_2(n,p)|B|^{\frac{1}{n}}\left(\fint_B |du|^p\,dx\right)^{\frac{1}{p}}.
\end{aligned} \tag{11}$$

If $p \ge n$, note that $(\fint_B |u-u_B|^s\,dx)^{\frac{1}{s}}$ increases with s and $\frac{ns}{n-s} \to \infty$ as $s \to n$, thus, there exists $1 < p_0 < n$, such that $q < \frac{np_0}{n-p_0}$. Then, we have

$$\begin{aligned}
\left(\fint_B |u-u_B|^q\,dx\right)^{\frac{1}{q}} &\le \left(\fint_B |u-u_B|^{\frac{np_0}{n-p_0}}\,dx\right)^{\frac{n-p_0}{np_0}} \\
&\le c_3(n,q)|B|^{\frac{1}{n}}\left(\fint_B |du|^{p_0}\,dx\right)^{\frac{1}{p_0}} \\
&\le c_3(n,q)|B|^{\frac{1}{n}}\left(\fint_B |du|^p\,dx\right)^{\frac{1}{p}}.
\end{aligned} \tag{12}$$

Combining (11) and (12), we get

$$\left(\fint_B |u-u_B|^q\,dx\right)^{\frac{1}{q}} \le c_4(n,p,q)|B|^{\frac{1}{n}}\left(\fint_B |du|^p\,dx\right)^{\frac{1}{p}}. \tag{13}$$

Since φ is increasing and satisfies $\triangle_2$-condition, (10) becomes

$$\fint_B \varphi\big(|u-u_B|\big)\,dx \le c_5(n,p,q,B)\varphi\bigg(\bigg(\fint_B |du|^p\,dx\bigg)^{\frac{1}{p}}\bigg). \tag{14}$$

Setting $\Phi(t)=\int_0^t \frac{\varphi(s)}{s}\,dx$, it follows from (2) that $\frac{\varphi(t)}{t^q}$ is decreasing with t, so

$$\int_0^t \frac{\varphi(s)}{s}\,dx = \int_0^t \frac{\varphi(s)}{s^q}s^{q-1}\,dx \ge \frac{\varphi(t)}{t^q}\cdot\frac{1}{q}s^q\bigg|_0^t = \frac{1}{q}\varphi(t).$$

Similarly, we have $\Phi(t)\le\frac{1}{p}\varphi(t)$. Thus,

$$\frac{1}{q}\varphi(t)\le\Phi(t)\le\frac{1}{p}\varphi(t). \tag{15}$$

Let $\Psi(t)=\Phi(t^{\frac{1}{p}})$, $(\Phi(t^{\frac{1}{p}}))'=\frac{1}{p}\frac{\varphi(t^{\frac{1}{p}})}{t}$ is increasing, so Ψ is a convex function.

For all $v\in L^1(B)$, by Jensen's inequality $\Psi(\fint_B v\,dx)\le\fint_B\Psi(v)\,dx$, we get

$$\Phi\bigg(\bigg(\fint_B v\,dx\bigg)^{\frac{1}{p}}\bigg)\le\fint_B\Phi\big(v^{\frac{1}{p}}\big)\,dx.$$

Replace v with $|du|^p$, we have

$$\Phi\bigg(\bigg(\fint_B |du|^p\,dx\bigg)^{\frac{1}{p}}\bigg)\le\fint_B\Phi\big(|du|\big)\,dx. \tag{16}$$

Combining (15) and (16), (14) becomes

$$\begin{aligned}\fint_B \varphi\big(|u-u_B|\big)\,dx &\le c_5(n,p,q,B)\Phi\bigg(\bigg(\fint_B |du|^p\,dx\bigg)^{\frac{1}{p}}\bigg)\\ &\le c_5(n,p,q,B)\fint_B\Phi\big(|du|\big)\,dx\\ &\le c_6(n,p,q,B)\fint_B\varphi\big(|du|\big)\,dx.\end{aligned}$$

This completes the proof. □

Since φ is a Young function satisfying $\triangle_2$-condition, from the proof of Theorem 2.3, we have

$$\int_B\varphi\bigg(\frac{|u-u_B|}{\lambda}\bigg)\,dx\le C\int_B\varphi\bigg(\frac{|du|}{\lambda}\bigg)\,dx$$

for any constant $\lambda>0$. According to (5), Poincaré-type inequality with Luxemburg norm

$$\|u-u_B\|_{\varphi(B)}\le C\|du\|_{\varphi(B)} \tag{17}$$

holds under the conditions in Theorem 2.3.

Next, we extend the local Poincaré-type inequality into the global case in L^ϕ-averaging domain, which are extension of John domains and L^s-averaging domain, for more details see [5, 8, 9].

Definition 2.4 [8] Let ϕ be an increasing convex function on $[0,\infty)$ with $\phi(0) = 0$, we call a proper subdomain $\Omega \subset \mathbb{R}^n$ a L^ϕ-averaging domain if $|\Omega| < \infty$ and there exists a constant C, such that

$$\int_\Omega \phi\big(\tau|u - u_{B_0}|\big)\,dx \le C \sup_{B\subset\Omega} \int_B \phi\big(\sigma|u - u_B|\big)\,dx$$

for some ball $B_0 \subset \Omega$ and all x such that $\phi(|u|) \in L^I_{\mathrm{loc}}(\Omega)$, where τ, σ are constants with $0 < \tau < \infty$, $0 < \sigma < \infty$ and the supremum is over all balls $B \subset \Omega$.

Similarly to the process of Theorem 3.2 in [9], we have the following theorem.

Theorem 2.5 *Suppose that $u \in \mathcal{D}'(\Omega, \Lambda^l)$ and $du \in L^p(\Omega, \Lambda^{l+1})$, φ is a Young function satisfying* (2) *with $q(n-p) < np$, $1 < p \le q < \infty$, Ω is any bounded L^φ-averaging domain. Then, there exists a constant C, independent of u, such that*

$$\int_\Omega \varphi\big(|u - u_B|\big)\,dx \le C \int_\Omega \varphi\big(|du|\big)\,dx, \tag{18}$$

where B is a ball or cube in Ω.

Choosing $\varphi(t) = t^p \log^\alpha(e + t)$, $p\varphi(t) \le t\varphi'(t) \le (p + \alpha)\varphi(t)$ can be checked easily, from Theorem 2.5, we can get Poincaré inequality with $L^p(\log^\alpha L)$-norm.

Corollary 2.6 *Suppose that u is a smooth differential form, $\varphi(t) = t^p \log^\alpha(e+t)$, $1 < p < \infty$. Ω is a bounded L^φ-averaging domain. Then, there exists a constant C, independent of u, such that*

$$\int_\Omega |u - u_B|^p \log^\alpha\big(|u - u_B|\big)\,dx \le C \int_\Omega |du|^p \log^\alpha\big(|du|\big)\,dx. \tag{19}$$

3 Inequalities for the sharp maximal operator applied to minimizers

Let B_0 be a fixed open ball in a bounded domain Ω. For differential form $u \in L^s(B_0, \Lambda^l)$, $1 \le s < \infty$, $l = 0, 1, \ldots, n$, the Hardy-Littlewood maximal operator of order s is defined by

$$M_s u = M_s u(x) = \sup\left\{\left(\fint_B |u|^s\,dx\right)^{1/s} : x \in B \subset B_0\right\},$$

where the supremum is taken over all parallel open subcubes B of B_0 containing the point x. Write $M_1 u(x) = Mu(x)$ if $s = 1$. Similarly, the sharp maximal operator $M_s^\sharp$ is defined by

$$M_s^\sharp u = M_s^\sharp u(x) = \sup\left\{\left(\fint_B |u - u_B|^s\,dx\right)^{1/s} : x \in B \subset B_0\right\}.$$

We say that a differential form $u \in W^{1,1}(\Omega, \Lambda^l)$ is a k-quasiminimizer for the functional

$$I(\Omega, u) = \int_\Omega \varphi(|du|)\,dx \tag{20}$$

if and only if for every $\psi \in W^{1,1}(\Omega, \Lambda^l)$ with compact support

$$I(\operatorname{supp}\psi; u) \le kI(\operatorname{supp}\psi; u + \psi), \tag{21}$$

where $k > 1$ is a constant.

The following lemmas will be used in this section.

Lemma 3.1 [14] *If B_0 is a cube in Ω, $f \in L^s(B_0)$, $f \ge 0$. Then, for any $t > 0$,*

$$\bigl|\{x \in B_0 : M_s f(x) > t\}\bigr| \le \frac{c(n)}{t^s}\int_{\{f>t/2\}} \bigl|f(x)\bigr|^s\,dx. \tag{22}$$

Lemma 3.2 [10] *Let u be a k-quasiminimizer for the functional* (20), *and let φ be a Young function satisfying $\triangle_2$-condition. Then, for any ball $B_R \subset \Omega$ with radius R, there exists a constant C, independent of u such that*

$$\int_{B_{R/2}} \varphi(|du|)\,dx \le C\int_{B_R} \varphi\left(\frac{|u-c|}{R}\right)dx, \tag{23}$$

where c is any closed form.

The main result of this section is the following theorem.

Theorem 3.3 *Suppose u is a k-quasiminimizer for the functional* (20), *and φ is a Young function satisfying* (2) *with $q(n-p) < np$, $1 < p \le q < \infty$, $M_s^\sharp$ is the sharp maximal operator, $B_0 \subset \Omega$ is a ball satisfying $2B_0 \subset \Omega$. Then, for $1 \le s < q-1$, there exists a constant C, independent of u such that*

$$\int_{B_0} \varphi(|M_s^\sharp u|)\,dx \le C\int_{2B_0} \varphi(|u-c|)\,dx, \tag{24}$$

where c is a closed form.

Proof Similarly to (10), we obtain

$$\fint_{B_0} \varphi(|M_s^\sharp u|)\,dx \le c_1\varphi\left(\left(\fint_{B_0} |M_s^\sharp u|^q\,dx\right)^{\frac{1}{q}}\right).$$

Denote $f = |u - u_B|$, then $M_s f = M_s^\sharp u$. It follows from the standard representation theorem and Lemma 3.1 that,

$$\begin{aligned}\fint_{B_0} |M_s^\sharp u|^q\,dx &= \fint_{B_0} |M_s f|^q\,dx \\ &= \frac{q}{|B_0|}\int_0^\infty t^{q-1}\bigl|\{x \in B_0 : M_s f > t\}\bigr|\,dt\end{aligned}$$

$$\begin{aligned}
&\leq \frac{q}{|B_0|}\int_0^{\infty} t^{q-1}\cdot\frac{c}{t^s}\int_{\{f>t/2\}} f^s\,dx\,dt \\
&= \frac{cq}{|B_0|}\int_0^{\infty} t^{q-s-1}\int_{\{f>t/2\}} f^s\,dx\,dt. \qquad (25)
\end{aligned}$$

By Fubini's theorem, (25) becomes

$$\begin{aligned}
\fint_{B_0} |M_s f|^q\,dx &\leq \frac{cq}{|B_0|}\int_{B_0} f^s\,dx\int_0^{2f(x)} t^{q-s-1}\,dt \\
&\leq c_2\fint_{B_0} f^q\,dx. \qquad (26)
\end{aligned}$$

That is,

$$\left(\fint_{B_0}\left|M_s^{\sharp}u\right|^q\,dx\right)^{\frac{1}{q}} \leq c_3\left(\fint_{B_0}|u-u_B|^q\,dx\right)^{\frac{1}{q}}.$$

For similar discussion of $(\fint_{\Omega}|u-u_{\Omega}|^q\,dx)^{\frac{1}{q}}$ in Theorem 2.3, we have

$$\left(\fint_{B_0}\left|M_s^{\sharp}u\right|^q\,dx\right)^{\frac{1}{q}} \leq c_4|B_0|^{\frac{1}{n}}\left(\fint_{B_0}|du|^p\,dx\right)^{\frac{1}{p}}.$$

Then,

$$\begin{aligned}
\fint_{B_0}\varphi\left(M_s^{\sharp}u\right) &\leq c_5\varphi\left(\left(\fint_{B_0}|du|^p\,dx\right)^{\frac{1}{p}}\right) \\
&\leq c_6\fint_{B_0}\varphi\left(|du|\right)dx. \qquad (27)
\end{aligned}$$

Since φ satisfying $\triangle_2$-condition, from Lemma 3.2, we get

$$\fint_{B_0}\varphi\left(\left|M_s^{\sharp}u\right|\right)dx \leq c_7\fint_{2B_0}\varphi\left(|u-c|\right)dx,$$

where c is a closed form. This completes the proof. □

Similarly to (17), the sharp maximal inequality with Luxemburg norm

$$\left\|M_s^{\sharp}u\right\|_{\varphi(B_0)} \leq C\|u-c\|_{\varphi(B_0)} \qquad (28)$$

holds under the conditions described in Theorem 3.3.

4 Application

As an application, we develop some estimates for homotopy operator T, Green's operator G and the composition of the operators $T\circ G$. Inequalities for the other class operators and composition operators can be obtained similarly.

Let $\mathcal{W}(\Omega,\Lambda^l) = \{u\in L^1(\Omega,\Lambda^l) : u \text{ has generalized gradient}\}$. The harmonic l-fields are defined by $\mathcal{H}(\Omega,\Lambda^l) = \{u\in\mathcal{W}(\Omega,\Lambda^l) : du = d^*u = 0, u\in L^p(\Omega,\Lambda^l) \text{ for some } 1 < p < \infty\}$.

The orthogonal complement of $\mathcal{H}$ in $L^1(\Omega,\Lambda^l)$ is defined by $\mathcal{H}^\perp(\Omega,\Lambda^l) = \{u \in L^1(\Omega,\Lambda^l) : \langle u,h\rangle = 0$ for all $h \in \mathcal{H}\}$. Then, the Green's operator G is defined as $G : C^\infty(\Omega,\Lambda^l) \to \mathcal{H}^\perp \cap C^\infty(\Omega,\Lambda^l)$ by assigning $G(u)$ to be the unique element of $\mathcal{H}^\perp \cap C^\infty(\Omega,\Lambda^l)$ satisfying Poisson's equation $\triangle G(u) = u - \mathcal{H}(u)$, where $\mathcal{H}$ is the harmonic projection operator that maps $C^\infty(\Omega,\Lambda^l)$ on $\mathcal{H}$, so that $\mathcal{H}(u)$ is the harmonic of u. See [4] for more properties of Green's operator.

Next, we prove the estimate for homotopy operator T.

Theorem 4.1 *Suppose that u is a smooth differential form, φ is a Young function satisfying* (2) *with $q(n-p) < np$, $1 < p \le q < \infty$. T is a homotopy operator. Then, there exists a constant C, independent of u, such that*

$$\int_B \varphi\bigl(\bigl|T(u) - \bigl(T(u)\bigr)_B\bigr|\bigr)\,dx \le C\int_B \varphi\bigl(|u|\bigr)\,dx \tag{29}$$

for all balls $B \subset \Omega$.

Proof Since $|dv| \le c|\nabla v|$ holds for each differential form v, combining (4) and the similar process of (10), we have

$$\begin{aligned}\int_B \varphi\bigl(\bigl|T(u) - \bigl(T(u)\bigr)_B\bigr|\bigr)\,dx &\le c_1\varphi\biggl(\biggl(\int_B \bigl|T(u) - \bigl(T(u)\bigr)_B\bigr|^q\biggr)\,dx\biggr)^{\frac{1}{q}} \\ &\le c_1\varphi\biggl(c_2\biggl(\int_B \bigl|d\bigl(T(u)\bigr)\bigr|^p\,dx\biggr)^{\frac{1}{p}}\biggr) \\ &\le c_3\varphi\biggl(\biggl(\int_B \bigl|\nabla\bigl(T(u)\bigr)\bigr|^p\,dx\biggr)^{\frac{1}{p}}\biggr) \\ &\le c_3\varphi\biggl(c_4\biggl(\int_B |u|^p\,dx\biggr)^{\frac{1}{p}}\biggr) \\ &\le c_5\int_B \varphi\bigl(|u|\bigr)\,dx. \qquad \square\end{aligned}$$

For Green's operator, we have the following estimate for Green's operator applied to k-quasiminimizer.

Theorem 4.2 *Suppose that the smooth differential form u is a k-quasiminimizer for the functional* (20), *and φ is a Young function satisfying* (2) *with $q(n-p) < np$, $1 < p \le q < \infty$, G is a Green's operator. Then, for each ball $B \subset \Omega$ with $2B \subset \Omega$, there exists a constant C, independent of u, such that*

$$\int_B \varphi\bigl(\bigl|G(u) - \bigl(G(u)\bigr)_B\bigr|\bigr)\,dx \le C\int_{\sigma B} \varphi\bigl(|u-c|\bigr)\,dx, \tag{30}$$

where $\sigma > 1$ is a constant, c is a closed form.

Proof Since

$$\bigl\|G(u)\bigr\|_{p,B} \le c\|u\|_{p,B} \tag{31}$$

holds for differential form u and G commutes with differential operator d, we have

$$\begin{aligned}\int_B \varphi\big(\big|G(u)-\big(G(u)\big)_B\big|\big)\,dx &\le c_1\varphi\left(\left(\int_B \big|G(u)-\big(G(u)\big)_B\big|^q\,dx\right)^{\frac{1}{q}}\right)\\ &\le c_1\varphi\left(c_2\left(\int_B \big|G(u)-\big(G(u)\big)_B\big|^p\,dx\right)^{\frac{1}{p}}\right)\\ &\le c_3\varphi\left(\left(\int_B \big|d\big(G(u)\big)\big|^p\,dx\right)^{\frac{1}{p}}\right)\\ &= c_3\varphi\left(c_4\left(\int_B \big|G(du)\big|^p\,dx\right)^{\frac{1}{p}}\right)\\ &\le c_5\int_B \varphi\big(|du|\big)\,dx.\end{aligned}$$

It follows from Lemma 3.2 that

$$\int_B \varphi\big(\big|G(u)-\big(G(u)\big)_B\big|\big)\,dx \le c_6\int_{\sigma B}\varphi\big(|u-c|\big)\,dx.$$

This completes the proof of Theorem 4.2. □

Corollary 4.3 *Suppose that u is a smooth differential form, φ is a Young function satisfying* (2) *and $q(n-p) < np$, $1 < p \le q < \infty$. When T is a homotopy operator and G is a Green's operator, there exists a constant C, independent of u such that*

$$\int_B \varphi\big(\big|T\big(G(u)\big)-\big(T\big(G(u)\big)\big)_B\big|\big)\,dx \le C\int_B \varphi\big(|u|\big)\,dx \tag{32}$$

for all balls $B \subset \Omega$.

Proof From (31) and Theorem 4.1, we have

$$\begin{aligned}\int_B \varphi\big(\big|T\big(G(u)\big)-\big(T\big(G(u)\big)\big)_B\big|\big)\,dx &\le c_1\varphi\left(\left(\int_B \big|G(u)\big|^p\,dx\right)^{\frac{1}{p}}\right)\\ &\le c_1\varphi\left(c_2\left(\int_B |u|^p\,dx\right)^{\frac{1}{p}}\right)\\ &\le c_3\int_B \varphi\big(|u|\big)\,dx.\end{aligned}$$

This ends the proof of Corollary 4.3. □

Remark 4.4 In 2004, Buckley and Keoskela first introduced a function class $G(p,q,C)$ in [15]. After that, some mathematicians devoted themselves to study the inequalities with L^ψ-norm for differential form and operators, where ψ lies in $G(p,q,C)$. We find that t^p and $t^p\log^\alpha(e+t)$ are both of function class $G(p,q,C)$ and satisfying condition (2) in this paper. But it is still open which of the conditions of function class $G(p,q,C)$ and (2) is stronger, or they are not inclusive of each other.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this paper. They read and approved the final manuscript.

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