Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Complete moment convergence for randomly weighted sums of martingale differences

Wenzhi Yang[1], Yiwei Wang[2], Xinghui Wang[1] and Shuhe Hu[1*]

*Correspondence: hushuhe@263.net
[1]School of Mathematical Science, Anhui University, Hefei, China
Full list of author information is available at the end of the article

Abstract

In this article, we obtain the complete moment convergence for randomly weighted sums of martingale differences. Our results generalize the corresponding ones for the nonweighted sums of martingale differences to the case of randomly weighted sums of martingale differences.

MSC: 60G50; 60F15

Keywords: complete convergence; randomly weighted sums; martingale differences

1 Introduction

The concept of complete convergence was introduced by Hsu and Robbins [1], *i.e.*, a sequence of random variables $\{X_n, n \geq 1\}$ is said to *converge completely* to a constant C if $\sum_{n=1}^{\infty} P(|X_n - C| \geq \varepsilon) < \infty$ for all $\varepsilon > 0$. In view of Borel-Cantelli lemma, this implies that $X_n \to C$ almost surely (a.s.). The converse is true if $\{X_n, n \geq 1\}$ is independent. Hsu and Robbins [1] obtained that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös [2] proved the converse. The result of Hsu-Robbins-Erdös is a fundamental theorem in probability theory, and it has been generalized and extended in several directions by many authors. Baum and Katz [3] gave the following generalization to establish a rate of convergence in the sense of Marcinkiewicz-Zygmund-type strong law of large numbers.

Theorem 1.1 *Let $\alpha > 1/2$, $\alpha p > 1$ and $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables. Assume that $EX_1 = 0$ if $\alpha \leq 1$. Then the following statements are equivalent*

(i) $E|X_1|^p < \infty$;
(ii) $\sum_{n=1}^{\infty} n^{\alpha p-2} P(\max_{1 \leq k \leq n} |\sum_{i=1}^{k} X_i| > \varepsilon n^{\alpha}) < \infty$ *for all* $\varepsilon > 0$.

Many authors have extended Theorem 1.1 for the i.i.d. case to some dependent cases. For example, Shao [4] investigated the moment inequalities for the φ-mixing random variables and gave its application to the complete convergence for this stochastic process; Yu [5] obtained the complete convergence for weighted sums of martingale differences; Ghosal and Chandra [6] gave the complete convergence of martingale arrays; Stoica [7, 8] investigated the Baum-Katz-Nagaev-type results for martingale differences and the rate of convergence in the strong law of large numbers for martingale differences; Wang *et al.* [9] also studied the complete convergence and complete moment convergence for martingale differences, which generalized some results of Stoica [7, 8]; Yang *et al.* [10] obtained the complete convergence for the moving average process of martingale differences and so forth. For other works about convergence analysis, one can refer to Gut [11], Chen *et al.* [12], Sung [13–16], Sung and Volodin [17], Hu *et al.* [18] and the references therein.

Recently, Thanh and Yin [19] studied the complete convergence for randomly weighted sums of independent random elements in Banach spaces. On the other hand, Cabrera *et al.* [20] investigated some theorems on conditional mean convergence and conditional almost sure convergence for randomly weighted sums of dependent random variables. Inspired by the papers above, we will investigate the complete moment convergence for randomly weighted sums of martingale differences in this paper, which implies the complete convergence and Marcinkiewicz-Zygmund-type strong law of large numbers for this stochastic process. We generalize the results of Stoica [7, 8] and Wang *et al.* [9] for the nonweighted sums of martingale differences to the case of randomly weighted sums of martingale differences. For the details, one can refer to the main results presented in Section 2. The proofs of the main results are presented in Section 3.

Recall that the sequence $\{X_n, n \geq 1\}$ is stochastically dominated by a nonnegative random variable X if

$$\sup_{n\geq 1} P\big(|X_n| > t\big) \leq CP(X > t) \quad \text{for some positive constant } C \text{ and for all } t \geq 0.$$

Throughout the paper, let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $x^+ = xI\ (x \geq 0)$, $I(B)$ be the indicator function of set B and $C, C_1, C_2, \ldots$ denote some positive constants not depending on n, which may be different in various places.

To prove the main results of the paper, we need the following lemmas.

Lemma 1.1 (*cf.* Hall and Heyde [21], Theorem 2.11) *If $\{X_i, \mathcal{F}_i, 1 \leq i \leq n\}$ is a martingale difference and $p > 0$, then there exists a constant C depending only on p such that*

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leq C\left\{E\left(\sum_{i=1}^{n} E\big(X_i^2|\mathcal{F}_{i-1}\big)\right)^{p/2} + E\Big(\max_{1\leq i\leq n}|X_i|^p\Big)\right\}, \quad n \geq 1.$$

Lemma 1.2 (*cf.* Sung [13], Lemma 2.4) *Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be sequences of random variables. Then for any $n \geq 1$, $q > 1$, $\varepsilon > 0$ and $a > 0$,*

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(X_i + Y_i)\right| - \varepsilon a\right)^+ \leq \left(\frac{1}{\varepsilon^q} + \frac{1}{q-1}\right)\frac{1}{a^{q-1}}E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^q\right) + E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} Y_i\right|\right).$$

Lemma 1.3 (*cf.* Wang *et al.* [9], Lemma 2.2) *Let $\{X_n, n \geq 1\}$ be a sequence of random variables stochastically dominated by a nonnegative random variable X. Then for any $n \geq 1$,*

$a > 0$ and $b > 0$, the following two statements hold

$$E\big[|X_n|^a I\big(|X_n| \le b\big)\big] \le C_1\big\{E\big[X^a I(X \le b)\big] + b^a P(X > b)\big\}$$

and

$$E\big[|X_n|^a I\big(|X_n| > b\big)\big] \le C_2 E\big[X^a I(X > b)\big],$$

where C_1 and C_2 are positive constants.

2 Main results

Theorem 2.1 *Let $\alpha > 1/2$, $1 < p < 2$, $1 \le \alpha p < 2$ and $\{X_n, \mathcal{F}_n, n \ge 1\}$ be a martingale difference sequence stochastically dominated by a nonnegative random variable X with $EX^p < \infty$. Assume that $\{A_n, n \ge 1\}$ is a random sequence, and it is independent of $\{X_n, n \ge 1\}$. If*

$$\sum_{i=1}^{n} EA_i^2 = O(n), \tag{2.1}$$

then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} A_i X_i\right| - \varepsilon n^{\alpha}\right)^{+} < \infty \tag{2.2}$$

and for $\alpha p > 1$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} E\left(\sup_{k\ge n}\left|\frac{\sum_{i=1}^{k} A_i X_i}{k^{\alpha}}\right| - \varepsilon\right)^{+} < \infty. \tag{2.3}$$

Theorem 2.2 *Let $\alpha > 1/2$, $p \ge 2$ and $\{X_n, \mathcal{F}_n, n \ge 1\}$ be a martingale difference sequence stochastically dominated by a nonnegative random variable X with $EX^p < \infty$. Let $\{A_n, n \ge 1\}$ be a random sequence, which is independent of $\{X_n, n \ge 1\}$. Denote $\mathcal{G}_0 = \{\emptyset, \Omega\}$ and $\mathcal{G}_n = \sigma(X_1, \ldots, X_n)$, $n \ge 1$. For some $q > \frac{2(\alpha p-1)}{2\alpha-1}$, we assume that $E[\sup_{n\ge 1} E(X_n^2|\mathcal{G}_{n-1})]^{q/2} < \infty$ and*

$$\sum_{i=1}^{n} E|A_i|^q = O(n). \tag{2.4}$$

Then for every $\varepsilon > 0$, (2.2) *and* (2.3) *hold.*

Meanwhile, for the case $p = 1$, we have the following theorem.

Theorem 2.3 *Let $\alpha > 0$ and $\{X_n, \mathcal{F}_n, n \ge 1\}$ be a martingale difference sequence stochastically dominated by a nonnegative random variable X with $E[X \ln(1 + X)] < \infty$. Assume that* (2.1) *holds and $\{A_n, n \ge 1\}$ is a random sequence, which is independent of $\{X_n, n \ge 1\}$. Then for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} A_i X_i\right| - \varepsilon n^{\alpha}\right)^{+} < \infty \tag{2.5}$$

and for $\alpha > 1$,

$$\sum_{n=1}^{\infty} n^{\alpha-2} E\left(\sup_{k\geq n}\left|\frac{\sum_{i=1}^{k} A_i X_i}{k^{\alpha}}\right| - \varepsilon\right)^{+} < \infty. \tag{2.6}$$

In particular, for $\alpha > 0$, *it has*

$$\sum_{n=1}^{\infty} n^{\alpha-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} A_i X_i\right| > \varepsilon n^{\alpha}\right) < \infty, \tag{2.7}$$

and for $\alpha > 1$, *it has*

$$\sum_{n=1}^{\infty} n^{\alpha-2} P\left(\sup_{k\geq n}\left|\frac{\sum_{i=1}^{k} A_i X_i}{k^{\alpha}}\right| > \varepsilon\right) < \infty. \tag{2.8}$$

On the other hand, for $\alpha \geq 1$ and $EX < \infty$, we have the following theorem.

Theorem 2.4 *Let* $\alpha \geq 1$ *and* $\{X_n, \mathcal{F}_n, n \geq 1\}$ *be a martingale difference sequence stochastically dominated by a nonnegative random variable* X *with* $EX < \infty$. *Denote* $\mathcal{G}_0 = \{\emptyset, \Omega\}$ *and* $\mathcal{G}_n = \sigma(X_1, \ldots, X_n)$, $n \geq 1$. *Let* (2.1) *hold, and let* $\{A_n, n \geq 1\}$ *be a random sequence, which is independent of* $\{X_n, n \geq 1\}$. *We assume* (i) *under the case of* $\alpha = 1$, *there exists a* $\delta > 0$ *such that*

$$\lim_{n\to\infty} \frac{\max_{1\leq i\leq n} E[|X_i|^{1+\delta} | \mathcal{G}_{i-1}]}{n^{\delta}} = 0, \quad a.s.$$

and (ii) *under the case of* $\alpha > 1$, *it has for any* $\lambda > 0$ *that*

$$\lim_{n\to\infty} \frac{\max_{1\leq i\leq n} E[|X_i| | \mathcal{G}_{i-1}]}{n^{\lambda}} = 0, \quad a.s.$$

Then for $\alpha \geq 1$ *and every* $\varepsilon > 0$, *it has* (2.7). *In addition, for* $\alpha > 1$, *it has* (2.8).

Remark 2.1 If the conditions of Theorem 2.1 or Theorem 2.2 hold, then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} A_i X_i\right| > \varepsilon n^{\alpha}\right) < \infty, \tag{2.9}$$

and for $\alpha p > 1$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\sup_{k\geq n}\left|\frac{\sum_{i=1}^{k} A_i X_i}{k^{\alpha}}\right| > \varepsilon\right) < \infty. \tag{2.10}$$

In fact, it can be checked that for every $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} A_i X_i\right| - \varepsilon n^{\alpha}\right)^{+} \\
&\quad = \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_0^{\infty} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} A_i X_i\right| - \varepsilon n^{\alpha} > t\right) dt
\end{aligned}$$

$$\geq \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_0^{\varepsilon n^\alpha} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} A_i X_i\right| - \varepsilon n^\alpha > t\right) dt$$
$$\geq \varepsilon \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} A_i X_i\right| > 2\varepsilon n^\alpha\right). \tag{2.11}$$

So (2.2) implies (2.9).

On the other hand, by the proof of Theorem 12.1 of Gut [11] and the proof of (3.2) in Yang *et al.* [10], for $\alpha p > 1$, it is easy to see that

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\sup_{k\geq n}\left|\frac{\sum_{i=1}^{k} A_i X_i}{k^\alpha}\right| > 2^{2\alpha}\varepsilon\right) \leq C_1 \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} A_i X_i\right| > \varepsilon n^\alpha\right).$$

Thus (2.10) follows from (2.9).

Remark 2.2 In Theorem 2.1, if $\alpha = 1/p$, then for every $\varepsilon > 0$, we get by (2.9) that

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} A_i X_i\right| > \varepsilon n^{1/p}\right) < \infty. \tag{2.12}$$

By using (2.12), one can easily get the Marcinkiewicz-Zygmund-type strong law of large numbers of randomly weighted sums of martingale difference as following

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} \sum_{i=1}^{n} A_i X_i = 0, \quad \text{a.s.}$$

If $A_n = a_n$ is non-random (the case of constant weighted), $n \geq 1$, then one can get the results of Theorems 2.1-2.4 for the non-random weighted sums of martingale differences.

Meanwhile, it can be seen that our condition $E[\sup_{n\geq 1} E(X_n^2|\mathcal{G}_{n-1})]^{q/2} < \infty$ in Theorem 2.2 is weaker than the condition $\sup_{n\geq 1} E(X_n^2|\mathcal{F}_{n-1}) \leq C$, a.s. in Theorem 1.4, Theorem 1.5 and Theorem 1.7 of Wang *et al.* [9]. In fact, it follows from $\mathcal{G}_{n-1} \subseteq \mathcal{F}_{n-1}$ that

$$E(X_n^2|\mathcal{G}_{n-1}) = E\left[E(X_n^2|\mathcal{F}_{n-1})|\mathcal{G}_{n-1}\right] \leq E\left[\sup_{n\geq 1} E(X_n^2|\mathcal{F}_{n-1})|\mathcal{G}_{n-1}\right].$$

If $\sup_{n\geq 1} E(X_n^2|\mathcal{F}_{n-1}) \leq C$, a.s., then it has $E[\sup_{n\geq 1} E(X_n^2|\mathcal{G}_{n-1})]^{q/2} < \infty$. For $\alpha \geq 1$ and $E[X \ln(1+X)] < \infty$, Wang *et al.* [9] obtained the result of (2.7) (see Theorem 1.6 of Wang *et al.* [9]). Therefore, by Theorems 2.1-2.4 in this paper, we generalize Theorems 1.4-1.7 of Wang *et al.* [9] for the nonweighted sums of martingale differences to the case of randomly weighted sums of martingale differences.

On the other hand, let the hypothesis that $\{A_n, n \geq 1\}$ is independent of $\{X_n, n \geq 1\}$ be replaced by that A_n is $\mathcal{F}_{n-1}$-measurable and A_n is independent of X_n for each $n \geq 1$ in Theorem 2.1, and the other conditions of Theorem 2.1 hold, one can get (2.2) and (2.3) (the proof is similar to the one of Theorem 2.1). Let A_n be $\mathcal{F}_{n-1}$-measurable, A_n be independent of X_n for each $n \geq 1$, $E[\sup_{n\geq 1} E(X_n^2|\mathcal{F}_{n-1})]^{q/2} < \infty$ and other conditions of Theorem 2.2 hold, one can also obtain (2.2) and (2.3). We can obtain some similar results if we only require A_n is $\mathcal{F}_{n-1}$-measurable for all $n \geq 1$ (without any independence hypothesis). This case would have many interesting applications (see Huang and Guo [22], Thanh *et al.* [23] and the references therein).

3 The proofs of main results

Proof of Theorem 2.1 Let $\mathcal{G}_0 = \{\emptyset, \Omega\}$, for $n \geq 1$, $\mathcal{G}_n = \sigma(X_1, \ldots, X_n)$ and

$$X_{ni} = X_i I(|X_i| \leq n^\alpha), \quad 1 \leq i \leq n.$$

It can be seen that

$$A_i X_i = A_i X_i I(|X_i| > n^\alpha) + [A_i X_{ni} - E(A_i X_{ni}|\mathcal{G}_{i-1})] + E(A_i X_{ni}|\mathcal{G}_{i-1}), \quad 1 \leq i \leq n.$$

So, by Lemma 1.2 with $a = n^\alpha$, for $q > 1$, one has that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} A_i X_i\right| - \varepsilon n^\alpha\right)^+ \\
&\quad\leq C_1 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}[A_i X_{ni} - E(A_i X_{ni}|\mathcal{G}_{i-1})]\right|^q\right) \\
&\qquad+ \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}[A_i X_i I(|X_i| > n^\alpha) + E(A_i X_{ni}|\mathcal{G}_{i-1})]\right|\right) \\
&\quad\leq C_1 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}[A_i X_{ni} - E(A_i X_{ni}|\mathcal{G}_{i-1})]\right|^q\right) \\
&\qquad+ \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} A_i X_i I(|X_i| > n^\alpha)\right|\right) \\
&\qquad+ \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} E(A_i X_{ni}|\mathcal{G}_{i-1})\right|\right) \\
&\quad:= H_1 + H_2 + H_3.
\end{aligned} \tag{3.1}$$

Obviously, it follows from Hölder's inequality and (2.1) that

$$\sum_{i=1}^{n} E|A_i| \leq \left(\sum_{i=1}^{n} EA_i^2\right)^{1/2} \left(\sum_{i=1}^{n} 1\right)^{1/2} = O(n). \tag{3.2}$$

By the fact that $\{A_n, n \geq 1\}$ is independent of $\{X_n, n \geq 1\}$, we can check by Markov's inequality, Lemma 1.3, (3.2) and $EX^p < \infty$ $(p > 1)$ that

$$\begin{aligned}
H_2 &\leq \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^{n} E|A_i| E[|X_i| I(|X_i| > n^\alpha)] \\
&\leq \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E[X I(X > n^\alpha)] \\
&= \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} \sum_{m=n}^{\infty} E[X I(m^\alpha < X \leq (m+1)^\alpha)]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} E\bigl[XI\bigl(m^{\alpha} < X \le (m+1)^{\alpha}\bigr)\bigr] \sum_{n=1}^{m} n^{\alpha p-1-\alpha} \\
&\le C_2 \sum_{m=1}^{\infty} m^{\alpha p-\alpha} E\bigl[XI\bigl(m^{\alpha} < X \le (m+1)^{\alpha}\bigr)\bigr] \\
&\le C_3 EX^p < \infty. \qquad (3.3)
\end{aligned}
$$

On the other hand, one can see that $\{X_n, \mathcal{G}_n, n \ge 1\}$ is also a martingale difference, since $\{X_n, \mathcal{F}_n, n \ge 1\}$ is a martingale difference. Combining with the fact that $\{A_n, n \ge 1\}$ is independent of $\{X_n, n \ge 1\}$, we have that

$$
\begin{aligned}
E(A_n X_n | \mathcal{G}_{n-1}) &= E\bigl[E(A_n X_n | \mathcal{G}_n) | \mathcal{G}_{n-1}\bigr] = E\bigl[X_n E(A_n | \mathcal{G}_n) | \mathcal{G}_{n-1}\bigr] \\
&= EA_n E[X_n | \mathcal{G}_{n-1}] = 0, \quad \text{a.s., } n \ge 1.
\end{aligned}
$$

Consequently, by the proof of (3.3), it follows that

$$
\begin{aligned}
H_3 &= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} E\bigl[A_i X_i I\bigl(|X_i| \le n^{\alpha}\bigr) | \mathcal{G}_{i-1}\bigr]\right|\right) \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} E\bigl[A_i X_i I\bigl(|X_i| > n^{\alpha}\bigr) | \mathcal{G}_{i-1}\bigr]\right|\right) \\
&\le \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^{n} E|A_i| E\bigl[|X_i| I\bigl(|X_i| > n^{\alpha}\bigr)\bigr] \\
&\le C_4 \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E\bigl[XI\bigl(X > n^{\alpha}\bigr)\bigr] \le C_5 EX^p < \infty. \qquad (3.4)
\end{aligned}
$$

Next, we turn to prove $H_1 < \infty$. It can be found that for fixed real numbers $a_1, \ldots, a_n$,

$$
\bigl\{a_i X_{ni} - E(a_i X_{ni} | \mathcal{G}_{i-1}), \mathcal{G}_i, 1 \le i \le n\bigr\}
$$

is also a martingale difference. Note that $\{A_1, A_2, \ldots, A_n\}$ is independent of $\{X_{n1}, X_{n2}, \ldots, X_{nn}\}$. So, by Markov's inequality, (2.1), (3.1) with $q = 2$, Lemma 1.1 with $p = 2$ and Lemma 1.3, we get that

$$
\begin{aligned}
H_1 &= C_1 \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} E\left\{E \max_{1\le k\le n} \sum_{i=1}^{k} \bigl[a_i X_{ni} - E(a_i X_{ni} | \mathcal{G}_{i-1})\bigr]^2 \middle| A_1 = a_1, \ldots, A_n = a_n\right\} \\
&\le C_2 E\left(\sum_{i=1}^{n} E(a_i X_{ni})^2 \middle| A_1 = a_1, \ldots, A_n = a_n\right) \\
&= C_2 \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} \sum_{i=1}^{n} E(A_i X_{ni})^2 = C_2 \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} \sum_{i=1}^{n} EA_i^2 EX_{ni}^2 \\
&\le C_3 \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} E\bigl[X^2 I\bigl(X \le n^{\alpha}\bigr)\bigr] + C_4 \sum_{n=1}^{\infty} n^{\alpha p-1} P\bigl(X > n^{\alpha}\bigr) \\
&=: C_3 H_{11} + C_4 H_{12}. \qquad (3.5)
\end{aligned}
$$

By the condition $EX^p < \infty$ with $p < 2$, it follows

$$\begin{aligned} H_{11} &= \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} \sum_{i=1}^{n} E\big[X^2 I\big((i-1)^\alpha < X \le i^\alpha\big)\big] \\ &= \sum_{i=1}^{\infty} E\big[X^2 I\big((i-1)^\alpha < X \le i^\alpha\big)\big] \sum_{n=i}^{\infty} n^{\alpha p-1-2\alpha} \\ &\le C_5 \sum_{i=1}^{\infty} E\big[X^p X^{2-p} I\big((i-1)^\alpha < X \le i^\alpha\big)\big] i^{\alpha p-2\alpha} \le C_6 EX^p < \infty. \end{aligned} \tag{3.6}$$

From (3.3), it has

$$H_{12} \le \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E\big[XI\big(X > n^\alpha\big)\big] \le CEX^p < \infty. \tag{3.7}$$

Consequently, by (3.1) and (3.3)-(3.7), we obtain (2.2) immediately.

For $\alpha p > 1$, we turn to prove (2.3). Denote $S_k = \sum_{i=1}^{k} A_i X_i$, $k \ge 1$. It can be seen that $\alpha p < 2 < 2 + \alpha$. So, similar to the proof of (3.4) in Yang *et al.* [10], we can check that

$$\begin{aligned} \sum_{n=1}^{\infty} n^{\alpha p-2} E\left(\sup_{k\ge n}\left|\frac{S_k}{k^\alpha}\right| - \varepsilon 2^{2\alpha}\right)^+ &\le C_1 \sum_{l=1}^{\infty} 2^{l(\alpha p-1-\alpha)} \int_0^\infty P\Big(\max_{1\le k\le 2^l} |S_k| > \varepsilon 2^{\alpha(l+1)} + s\Big)\,ds \\ &\le C_1 2^{2+\alpha-\alpha p} \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\Big(\max_{1\le k\le n} |S_k| - \varepsilon n^\alpha\Big)^+. \end{aligned}$$

Combining with (2.2), we get (2.3) finally. □

Proof of Theorem 2.2 To prove Theorem 2.2, we use the same notation as that in the proof of Theorem 2.1. For $p \ge 2$, it is easy to see that $q > 2(\alpha p - 1)/(2\alpha - 1) \ge 2$. Consequently, for any $1 \le s \le 2$, by Hölder's inequality and (2.4), we get

$$\sum_{i=1}^{n} E|A_i|^s \le \left(\sum_{i=1}^{n} E|A_i|^q\right)^{s/q} \left(\sum_{i=1}^{n} 1\right)^{1-s/q} = O(n). \tag{3.8}$$

By (3.1), (3.3) and (3.4), one can find that $H_2 < \infty$ and $H_3 < \infty$. So we need to prove that $H_1 < \infty$ under the conditions of Theorem 2.2. For $p \ge 2$, noting that $\{A_1, A_2, \ldots, A_n\}$ is independent of $\{X_{n1}, X_{n2}, \ldots, X_{nn}\}$, similar to the proof of (3.5), one has by Lemma 1.1 that

$$\begin{aligned} H_1 &= C_1 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\big[A_i X_{ni} - E(A_i X_{ni}|\mathcal{G}_{i-1})\big]\right|^q\right) \\ &\le C_2 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} E\left(\sum_{i=1}^{n} E\big\{\big[A_i X_{ni} - E(A_i X_{ni}|\mathcal{G}_{i-1})\big]^2|\mathcal{G}_{i-1}\big\}\right)^{q/2} \\ &\quad + C_3 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} \sum_{i=1}^{n} E\big|A_i X_{ni} - E(A_i X_{ni}|\mathcal{G}_{i-1})\big|^q \\ &=: C_2 H_{11} + C_3 H_{12}. \end{aligned} \tag{3.9}$$

Obviously, for $1 \le i \le n$, it has

$$\begin{aligned} &E\{[A_iX_{ni} - E(A_iX_{ni}|\mathcal{G}_{i-1})]^2|\mathcal{G}_{i-1}\} \\ &\quad = E[A_i^2X_i^2I(|X_i| \le n^\alpha)|\mathcal{G}_{i-1}] - [E(A_iX_iI(|X_i| \le n^\alpha)|\mathcal{G}_{i-1})]^2 \\ &\quad \le E[A_i^2X_i^2I(|X_i| \le n^\alpha)|\mathcal{G}_{i-1}] \le EA_i^2E(X_i^2|\mathcal{G}_{i-1}), \quad \text{a.s.} \end{aligned}$$

Combining (3.8) with $E[\sup_{i\ge1} E(X_i^2|\mathcal{G}_{i-1})]^{q/2} < \infty$, we obtain that

$$\begin{aligned} H_{11} &\le \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} \left(\sum_{i=1}^{n} EA_i^2\right)^{q/2} E\left(\sup_{i\ge1} E(X_i^2|\mathcal{G}_{i-1})\right)^{q/2} \\ &\le C_4 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha+q/2} < \infty, \end{aligned} \tag{3.10}$$

following from the fact that $q > 2(\alpha p - 1)/(2\alpha - 1)$. Meanwhile, by C_r inequality, Lemma 1.3 and (2.4),

$$\begin{aligned} H_{12} &\le C_5 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} \sum_{i=1}^{n} E|A_i|^q E[|X_i|^q I(|X_i| \le n^\alpha)] \\ &\le C_6 \sum_{n=1}^{\infty} n^{\alpha p-1-q\alpha} E[X^q I(X \le n^\alpha)] + C_7 \sum_{n=1}^{\infty} n^{\alpha p-1} P(X > n^\alpha) \\ &\le C_6 \sum_{n=1}^{\infty} n^{\alpha p-1-q\alpha} E[X^q I(X \le n^\alpha)] + C_7 \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E[XI(X > n^\alpha)] \\ &=: C_6 H_{11}^* + C_7 H_{12}^*. \end{aligned} \tag{3.11}$$

By the condition $p \ge 2$ and $\alpha > 1/2$, we have that $2(\alpha p - 1)/(2\alpha - 1) - p \ge 0$, which implies that $q > p$. So, one gets by $EX^p < \infty$ that

$$\begin{aligned} H_{11}^* &= \sum_{n=1}^{\infty} n^{\alpha p-1-q\alpha} \sum_{i=1}^{n} E[X^q I((i-1)^\alpha < X \le i^\alpha)] \\ &= \sum_{i=1}^{\infty} E[X^q I((i-1)^\alpha < X \le i^\alpha)] \sum_{n=i}^{\infty} n^{\alpha p-1-q\alpha} \\ &\le C_8 \sum_{i=1}^{\infty} E[X^p X^{q-p} I((i-1)^\alpha < X \le i^\alpha)] i^{\alpha p-q\alpha} \le C_8 EX^p < \infty. \end{aligned} \tag{3.12}$$

By the proof of (3.3), it follows

$$H_{12}^* = \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E[XI(X > n^\alpha)] \le C_9 EX^p < \infty. \tag{3.13}$$

Therefore, by (3.9)-(3.13), it has $H_1 < \infty$. Consequently, it completes the proof of (2.2).

Finally, by the fact that $\alpha p > 1$, similar to the proof of (3.4) in Yang *et al.* [10], it is easy to see that (2.3) holds for the case $\alpha p < 2 + \alpha$ and the case $\alpha p \ge 2 + \alpha$. □

Proof of Theorem 2.3 Similar to the proof of Theorem 2.1, by Lemma 1.2, it can be checked that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} A_i X_i\right| - \varepsilon n^{\alpha}\right)^{+} \\
&\quad\le C_1 \sum_{n=1}^{\infty} n^{-2-\alpha} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\left[A_i X_{ni} - E(A_i X_{ni}|\mathcal{G}_{i-1})\right]\right|^{2}\right) \\
&\qquad + \sum_{n=1}^{\infty} n^{-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} A_i X_i I\left(|X_i| > n^{\alpha}\right)\right|\right) \\
&\qquad + \sum_{n=1}^{\infty} n^{-2} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} E(A_i X_{ni}|\mathcal{G}_{i-1})\right|\right) \\
&\quad := J_1 + J_2 + J_3.
\end{aligned} \tag{3.14}$$

Similarly to the proof of (3.3), we have

$$\begin{aligned}
J_2 &\le C_1 \sum_{n=1}^{\infty} n^{-1} E\left[XI\left(X > n^{\alpha}\right)\right] \\
&= C_1 \sum_{n=1}^{\infty} n^{-1} \sum_{m=n}^{\infty} E\left[XI\left(m^{\alpha} < X \le (m+1)^{\alpha}\right)\right] \\
&= C_1 \sum_{m=1}^{\infty} E\left[XI\left(m^{\alpha} < X \le (m+1)^{\alpha}\right)\right] \sum_{n=1}^{m} n^{-1} \\
&\le C_2 \sum_{m=1}^{\infty} \ln(1+m) E\left[XI\left(m^{\alpha} < X \le (m+1)^{\alpha}\right)\right] \\
&\le C_3 E\left[X \ln(1+X)\right] < \infty.
\end{aligned} \tag{3.15}$$

Meanwhile, by the proofs of (3.4) and (3.15), we get

$$J_3 \le C_1 \sum_{n=1}^{\infty} n^{-1} E\left[XI\left(X > n^{\alpha}\right)\right] \le C_2 E\left[X \ln(1+X)\right] < \infty. \tag{3.16}$$

On the other hand, by the proof of (3.5), it can be checked that for $\alpha > 0$,

$$\begin{aligned}
J_1 &\le C_2 \sum_{n=1}^{\infty} n^{-2-\alpha} \sum_{i=1}^{n} E(A_i X_{ni})^2 = C_2 \sum_{n=1}^{\infty} n^{-2-\alpha} \sum_{i=1}^{n} EA_i^2 EX_{ni}^2 \\
&\le C_3 \sum_{n=1}^{\infty} n^{-1-\alpha} E\left[X^2 I\left(X \le n^{\alpha}\right)\right] + C_4 \sum_{n=1}^{\infty} n^{\alpha-1} P\left(X > n^{\alpha}\right) \\
&\le C_3 \sum_{n=1}^{\infty} n^{-1-\alpha} \sum_{i=1}^{n} E\left[X^2 I\left((i-1)^{\alpha} < X \le i^{\alpha}\right)\right] + C_4 \sum_{n=1}^{\infty} n^{-1} E\left[XI\left(X > n^{\alpha}\right)\right] \\
&\le C_3 \sum_{i=1}^{\infty} E\left[X^2 I\left((i-1)^{\alpha} < X \le i^{\alpha}\right)\right] \sum_{n=i}^{\infty} n^{-1-\alpha} + C_5 E\left[X \ln(1+X)\right]
\end{aligned}$$

$$\le C_4 \sum_{i=1}^{\infty} E\big[X^2 I\big((i-1)^\alpha < X \le i^\alpha\big)\big] i^{-\alpha} + C_5 E\big[X \ln(1+X)\big]$$
$$\le C_6 EX + C_5 E\big[X \ln(1+X)\big] < \infty. \tag{3.17}$$

Therefore, by (3.14)-(3.17), one gets (2.5) immediately. Similar to the proof of (2.3), it is easy to have (2.6). Obviously, by the proof of (2.11) in Remark 2.2, (2.7) also holds under the conditions of Theorem 2.3. Finally, by the proof of Theorem 12.1 of Gut [11] and the proof of (3.2) in Yang *et al.* [10], for $\alpha > 1$, it is easy to get (2.8). □

Proof of Theorem 2.4 For $n \ge 1$, we also denote $X_{ni} = X_i I(|X_i| \le n^\alpha)$, $1 \le i \le n$. It is easy to see that

$$P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} A_i X_i\right| > \varepsilon n^\alpha\right) \le \sum_{i=1}^{n} P\big(|X_i| > n^\alpha\big) + P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} A_i X_{ni}\right| > \varepsilon n^\alpha\right). \tag{3.18}$$

For the case of $\alpha = 1$, there exists a $\delta > 0$ such that $\lim_{n\to\infty} \frac{\max_{1\le i\le n} E[|X_i|^{1+\delta}|\mathcal{G}_{i-1}]}{n^\delta} = 0$, a.s. So by $E(A_n X_n|\mathcal{G}_{n-1}) = 0$, a.s., $n \ge 1$, we can check that

$$\begin{aligned}
\frac{1}{n^\alpha}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} E(A_i X_{ni}|\mathcal{G}_{i-1})\right|\right) &= \frac{1}{n}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} E\big[A_i X_i I\big(|X_i| \le n\big)|\mathcal{G}_{i-1}\big]\right|\right) \\
&= \frac{1}{n}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} E\big[A_i X_i I\big(|X_i| > n\big)|\mathcal{G}_{i-1}\big]\right|\right) \\
&\le \frac{1}{n}\sum_{i=1}^{n} E|A_i| E\big[|X_i| I\big(|X_i| > n\big)|\mathcal{G}_{i-1}\big] \\
&\le \frac{1}{n^{1+\delta}}\sum_{i=1}^{n} E|A_i| E\big[|X_i|^{1+\delta}|\mathcal{G}_{i-1}\big] \\
&\le \frac{K}{n^\delta}\max_{1\le i\le n} E\big[|X_i|^{1+\delta}|\mathcal{G}_{i-1}\big] \to 0, \quad \text{a.s.},
\end{aligned}$$

as $n \to \infty$.

Otherwise, for the case of $\alpha > 1$, it is assumed that $\lim_{n\to\infty} \frac{\max_{1\le i\le n} E[|X_i||\mathcal{G}_{i-1}]}{n^\lambda} = 0$, a.s., for any $\lambda > 0$. Consequently, for any $\alpha > 1$, it follows that

$$\begin{aligned}
\frac{1}{n^\alpha}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} E(A_i X_{ni}|\mathcal{G}_{i-1})\right|\right) &= \frac{1}{n^\alpha}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} E\big[A_i X_i I\big(|X_i| \le n\big)|\mathcal{G}_{i-1}\big]\right|\right) \\
&\le \frac{1}{n^\alpha}\sum_{i=1}^{n} E|A_i| E\big[|X_i| I\big(|X_i| \le n\big)|\mathcal{G}_{i-1}\big] \\
&\le \frac{K_1}{n^{\alpha-1}}\max_{1\le i\le n} E[|X_i||\mathcal{G}_{i-1}] \to 0, \quad \text{a.s.},
\end{aligned}$$

as $n \to \infty$. Meanwhile,

$$\sum_{n=1}^{\infty} n^{\alpha-2}\sum_{i=1}^{n} P\big(|X_i| > n^\alpha\big) \le K_1 \sum_{n=1}^{\infty} n^{\alpha-1} P\big(X > n^\alpha\big) \le K_2 EX < \infty. \tag{3.19}$$

By (3.18) and (3.19), to prove (2.7), it suffices to show that

$$I_3 = \sum_{n=1}^{\infty} n^{\alpha-2} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\bigl[A_i X_{ni} - E(A_i X_{ni}|\mathcal{G}_{i-1})\bigr]\right| > \frac{\varepsilon n^{\alpha}}{2}\right) < \infty.$$

Obviously, by Markov's inequality and the proofs of (3.5), (3.6), (3.19), one can check that

$$\begin{aligned} I_3 &\le \frac{4}{\varepsilon^2}\sum_{n=1}^{\infty} n^{-2-\alpha} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\bigl[A_i X_{ni} - E(A_i X_{ni}|\mathcal{G}_{i-1})\bigr]\right|^2\right) \\ &\le K_1 \sum_{n=1}^{\infty} n^{-1-\alpha} E\bigl[X^2 I\bigl(X \le n^{\alpha}\bigr)\bigr] + K_2 \sum_{n=1}^{\infty} n^{\alpha-1} P\bigl(X > n^{\alpha}\bigr) \\ &\le K_3 EX < \infty. \end{aligned}$$

On the other hand, by proof of Theorem 12.1 of Gut [11] and the proof of (3.2) in Yang *et al.* [10], we can easily obtain (2.8) for $\alpha > 1$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1] School of Mathematical Science, Anhui University, Hefei, China. [2] Department of System Engineering and Engineering Management, The Chinese University of Hong Kong, Hong Kong, China.

Acknowledgements
The authors are most grateful to editor prof. Soo Hak Sung and two anonymous referees for their careful reading and insightful comments, which helped to significantly improve an earlier version of this paper. Supported by the NNSF of China (11171001, 11201001), Natural Science Foundation of Anhui Province (1208085QA03, 1308085QA03), Talents Youth Fund of Anhui Province Universities (2012SQRL204) and Doctoral Research Start-up Funds Projects of Anhui University.

Received: 31 March 2013 Accepted: 6 August 2013 Published: 21 August 2013

References

1. Hsu, PL, Robbins, H: Complete convergence and the law of large numbers. Proc. Natl. Acad. Sci. USA **33**(2), 25-31 (1947)
2. Erdös, P: On a theorem of Hsu and Robbins. Ann. Math. Stat. **20**(2), 286-291 (1949)
3. Baum, LE, Katz, M: Convergence rates in the law of large numbers. Trans. Am. Math. Soc. **120**(1), 108-123 (1965)
4. Shao, QM: A moment inequality and its application. Acta Math. Sin. **31A**(6), 736-747 (1988)
5. Yu, KF: Complete convergence of weighted sums of martingale differences. J. Theor. Probab. **3**(2), 339-347 (1990)
6. Ghosal, S, Chandra, TK: Complete convergence of martingale arrays. J. Theor. Probab. **11**(3), 621-631 (1998)
7. Stoica, G: Baum-Katz-Nagaev type results for martingales. J. Math. Anal. Appl. **336**(2), 1489-1492 (2007)
8. Stoica, G: A note on the rate of convergence in the strong law of large numbers for martingales. J. Math. Anal. Appl. **381**(2), 910-913 (2011)
9. Wang, XJ, Hu, SH, Yang, WZ, Wang, XH: Convergence rates in the strong law of large numbers for martingale difference sequences. Abstr. Appl. Anal. **2012**, Article ID 572493 (2012)
10. Yang, WZ, Hu, SH, Wang, XJ: Complete convergence for moving average process of martingale differences. Discrete Dyn. Nat. Soc. **2012**, Article ID 128492 (2012)
11. Gut, A: Probability: A Graduate Course. Springer, New York (2005)
12. Chen, PY, Hu, T-C, Volodin, A: Limiting behaviour of moving average processes under φ-mixing assumption. Stat. Probab. Lett. **79**(1), 105-111 (2009)
13. Sung, SH: Moment inequalities and complete moment convergence. J. Inequal. Appl. **2009**, Article ID 271265 (2009)
14. Sung, SH: Complete convergence for weighted sums of ρ^*-mixing random variables. Discrete Dyn. Nat. Soc. **2010**, Article ID 630608 (2010)
15. Sung, SH: Convergence of moving average processes for dependent random variables. Commun. Stat., Theory Methods **40**(13), 2366-2376 (2011)
16. Sung, SH: Complete qth moment convergence for arrays of random variables. J. Inequal. Appl. **2013**, Article ID 24 (2013)

17. Sung, SH, Volodin, A: A note on the rate of complete convergence for weighted sums of arrays of Banach space valued random elements. Stoch. Anal. Appl. **29**(2), 282-291 (2011)
18. Hu, T-C, Rosalsky, A, Volodin, A: A complete convergence theorem for row sums from arrays of rowwise independent random elements in Rademacher type p Banach spaces. Stoch. Anal. Appl. **30**(2), 343-353 (2012)
19. Thanh, LV, Yin, G: Almost sure and complete convergence of randomly weighted sums of independent random elements in Banach spaces. Taiwan. J. Math. **15**(4), 1759-1781 (2011)
20. Cabrera, MO, Rosalsky, A, Volodin, A: Some theorems on conditional mean convergence and conditional almost sure convergence for randomly weighted sums of dependent random variables. Test **21**(2), 369-385 (2012)
21. Hall, P, Heyde, CC: Martingale Limit Theory and Its Application. Academic Press, New York (1980)
22. Huang, DW, Guo, L: Estimation of nonstationary ARMAX models based on the Hannan-Rissanen method. Ann. Stat. **18**(4), 1729-1756 (1990)
23. Thanh, LV, Yin, G, Wang, LY: State observers with random sampling times and convergence analysis of double-indexed and randomly weighted sums of mixing processes. SIAM J. Control Optim. **49**(1), 106-124 (2011)

doi:10.1186/1029-242X-2013-396
Cite this article as: Yang et al.: **Complete moment convergence for randomly weighted sums of martingale differences.** *Journal of Inequalities and Applications* 2013 **2013**:396.