Yazlik and Taskara *Journal of Inequalities and Applications* 2013, **2013**:394
http://www.journalofinequalitiesandapplications.com/content/2013/1/394

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

On the norms of an r-circulant matrix with the generalized k-Horadam numbers

Yasin Yazlik[1*] and Necati Taskara[2]

*Correspondence: yyazlik@nevsehir.edu.tr
[1]Department of Mathematics, Faculty of Science and Art, Nevsehir University, Nevsehir, Turkey
Full list of author information is available at the end of the article

Abstract

In this paper, we present new upper and lower bounds for the spectral norm of an r-circulant matrix $H = C_r(H_{k,0}, H_{k,1}, H_{k,2}, \dots, H_{k,n-1})$ whose entries are the generalized k-Horadam numbers. Furthermore, we obtain new formulas to calculate the eigenvalues and determinant of the matrix H.

MSC: 11B39; 15A60; 15A15

Keywords: generalized k-Horadam sequence; r-circulant matrix; spectral norm; eigenvalue

1 Introduction and preliminaries

The r-circulant matrices [1–3] have been one of the most interesting research areas in the field of computation mathematics. It is known that these matrices have a wide range of applications in signal processing, coding theory, image processing, digital image disposal, linear forecast and design of self-regress.

Although there are many works about special matrices and their norms, we can mainly depict the following ones since they will be needed for our results. In [4], Solak defined the $n \times n$ circulant matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, where $a_{ij} \equiv F_{(\mathrm{mod}(j-i,n))}$ and $b_{ij} \equiv L_{(\mathrm{mod}(j-i,n))}$, and then investigated the upper and lower bounds for A and B, respectively. In [5], they defined circulant matrices involving k-Lucas and generalized k-Fibonacci numbers and also investigated the upper and lower bounds for the norms of these matrices. Later on, Shen and Cen in [6], found some other upper and lower bounds for the spectral norms of r-circulant matrices in the form $A = C_r(F_0, F_1, \dots, F_{n-1})$, $B = C_r(L_0, L_1, \dots, L_{n-1})$, where F_n is the nth Fibonacci number and L_n is the nth Lucas number. In [7], the authors found some other upper and lower bounds for the spectral norms of r-circulant matrices associated with the k-Fibonacci and k-Lucas numbers, respectively. It is clear that the same study about similar subject can also be done for different numbers. For instance, in [8], authors defined a circulant matrix whose entries are the generalized k-Horadam numbers and then computed the spectral norm, eigenvalues and determinant of the matrix.

At this point, we can keep going to present the fundamental material which is about the definition of the generalized k-Horadam sequence $\{H_{k,n}\}_{n\in\mathbb{N}}$. In fact, by [9], it was defined as the form

$$H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}, \qquad H_{k,0} = a, \qquad H_{k,1} = b \quad (a, b \in \mathbb{R}), \tag{1}$$

Springer

where, $n \geq 0$, $k \in \mathbb{R}^+$ and $f^2(k) + 4g(k) > 0$. Obviously, if we choose suitable values on $f(k)$, $g(k)$, a and b for (1), then this sequence reduces to the special all second-order sequences in the literature. For example, by taking $f(k) = g(k) = 1$, $a = 0$ and $b = 1$, the well-known Fibonacci sequence is obtained.

Binet's formula allows us to express the generalized k-Horadam number in function of the roots α and β of the characteristic equation $x^2 - f(k)x - g(k) = 0$. Binet's formula related to the sequence $\{H_{k,n}\}_{n\in\mathbb{N}}$ has the form

$$H_{k,n} = \frac{X\alpha^n - Y\beta^n}{\alpha - \beta}, \tag{2}$$

where $X = b - a\beta$ and $Y = b - a\alpha$.

Lemma 1 [9] *Let the entries of each matrix* $X_n = \begin{pmatrix} H_{k,n-1} & H_{k,n} \\ H_{k,n} & H_{k,n+1} \end{pmatrix}$ *be the generalized k-Horadam numbers. For* $n \geq 1$, *we get*

$$|X_n| = \bigl(-g(k)\bigr)^{n-1}\bigl(a^2g(k) + abf(k) - b^2\bigr).$$

Definition 2 For any given $c_0, c_1, c_2, \ldots, c_{n-1} \in \mathbb{C}$, the *r-circulant matrix* $C_r = (c_{ij})_{n\times n}$ is defined by

$$C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_2 & rc_3 & rc_4 & \cdots & c_0 & c_1 \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}.$$

Let $C_r = (c_0, c_1, c_2, \ldots, c_{n-1})$ denote an r-circulant matrix.

It is obvious that the matrix C_r turns into a classical circulant matrix for $r = 1$. Let us take any $A = [a_{ij}] \in M_{n,n}(\mathbb{C})$ (which could be a circulant matrix as well). The well-known Frobenius (or Euclidean) norm of the matrix A is given by

$$\|A\|_F = \left[\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right]^{\frac{1}{2}},$$

and also the spectral norm of A is presented by

$$\|A\|_2 = \sqrt{\max_{1\leq i\leq n}\lambda_i\bigl(A^HA\bigr)},$$

where $\lambda_i(A^HA)$ are the eigenvalues of A^HA such that A^H is the conjugate transpose of A. Then it is quite well known that

$$\frac{1}{\sqrt{n}}\|A\|_F \leq \|A\|_2 \leq \|A\|_F.$$

The following lemma will be needed in the proof of Theorem 5 below.

Lemma 3 [10] *For any matrices* $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$ *and* $B = [b_{ij}] \in M_{m,n}(\mathbb{C})$, *we have*

$$\|A \circ B\|_2 \leq r_1(A)c_1(B),$$

where $A \circ B$ *is the Hadamard product and* $r_1(A) = \max_{1\leq i\leq n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2}$, $c_1(B) = \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2}$.

Throughout this paper, the r-circulant matrix, whose entries are the generalized k-Horadam numbers, will be denoted by $H = C_r(H_{k,0}, H_{k,1}, \ldots, H_{k,n-1})$.

In this paper, we first give lower and upper bounds for the spectral norms of H. In particular, by taking $r = 1$, we obtain lower and upper bounds for the spectral norms of the circulant matrix associated with the generalized k-Horadam numbers. Afterwards, we also formulate the eigenvalues and determinant of the matrix H defined above.

2 Main results

Let us first consider the following lemma which states the sum square of the generalized k-Horadam numbers.

Lemma 4 *For* $n \geq 1$, *we have*

$$\sum_{i=0}^{n-1} H_{k,i}^2 = \frac{H_{k,n}^2 - g^2(k)H_{k,n-1}^2 + M}{f^2(k) + 2g(k) - g^2(k) - 1},$$

where $M = (b - af(k))^2 - a^2 - 2XY(\frac{1-(-g(k))^n}{1+g(k)})$, $X = b - a\beta$ *and* $Y = b - a\alpha$.

Proof Let $A = \sum_{i=0}^{n-1} H_{k,i}^2$. By using (1), we have

$$\begin{aligned} A &= \sum_{i=0}^{n-1} \left(\frac{H_{k,i+1} - g(k)H_{k,i-1}}{f(k)} \right)^2 \\ &= \frac{\sum_{i=0}^{n-1} H_{k,i+1}^2 + g^2(k)\sum_{i=0}^{n-1} H_{k,i-1}^2 - 2g(k)\sum_{i=0}^{n-1} H_{k,i+1}H_{k,i-1}}{f^2(k)}. \end{aligned}$$

By considering (1), we obtain

$$\begin{aligned} Af^2(k) &= A\bigl(1 + g^2(k) - 2g(k)\bigr) + H_{k,n}^2 - g^2(k)H_{k,n-1} - a^2 \\ &\quad + \bigl(b - af(k)\bigr)^2 - 2XY\left(\frac{1 - (-g(k))^n}{1 + g(k)} \right) \end{aligned}$$

which is desired. □

The following theorem gives us the upper and lower bounds for the spectral norms of the matrix H.

Theorem 5 *Let* $H = C_r(H_{k,0}, H_{k,1}, H_{k,2}, \ldots, H_{k,n-1})$ *be an r-circulant matrix. Then we have* *if* $|r| \geq 1$, *then*

$$\sqrt{\sum_{i=0}^{n-1} H_{k,i}^2} \leq \|H\|_2 \leq \sqrt{\left(a^2\left(1-|r|^2\right)+|r|^2\sum_{i=0}^{n-1} H_{k,i}^2\right)\left(1-a^2+\sum_{i=0}^{n-1} H_{k,i}^2\right)},$$

if $|r| < 1$, *then*

$$|r|\sqrt{\sum_{i=0}^{n-1} H_{k,i}^2} \leq \|H\|_2 \leq \sqrt{n\sum_{i=0}^{n-1} H_{k,i}^2},$$

where $r \in \mathbb{C}$, $\sum_{i=0}^{n-1} H_{k,i}^2 = \frac{H_{k,n}^2 - g^2(k)H_{k,n-1}^2 + M}{f^2(k)+2g(k)-g^2(k)-1}$, $M = (b-af(k))^2 - a^2 - 2XY(\frac{1-(-g(k))^n}{1+g(k)})$.

Proof The matrix H is of the form

$$H = \begin{bmatrix} H_{k,0} & H_{k,1} & H_{k,2} & \cdots & H_{k,n-1} \\ rH_{k,n-1} & H_{k,0} & H_{k,1} & \cdots & H_{k,n-2} \\ rH_{k,n-2} & rH_{k,n-1} & H_{k,0} & \cdots & H_{k,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rH_{k,1} & rH_{k,2} & rH_{k,3} & \cdots & H_{k,0} \end{bmatrix} \tag{3}$$

and, by the definition of Frobenius norm, we clearly have

$$\|H\|_F^2 = \sum_{i=0}^{n-1}(n-i)H_{k,i}^2 + \sum_{i=1}^{n-1} i|r|^2 H_{k,i}^2.$$

If $|r| \geq 1$, then we obtain

$$\|H\|_F^2 \geq \sum_{i=0}^{n-1}(n-i)H_{k,i}^2 + \sum_{i=1}^{n-1} iH_{k,i}^2 = n\sum_{i=0}^{n-1} H_{k,i}^2.$$

Also, by considering Lemma 4, we can write

$$\|H\|_F^2 \geq n\left[\frac{H_{k,n}^2 - g^2(k)H_{k,n-1}^2 + M}{f^2(k)+2g(k)-g^2(k)-1}\right].$$

It follows that

$$\frac{\|H\|_F}{\sqrt{n}} \geq \sqrt{\frac{H_{k,n}^2 - g^2(k)H_{k,n-1}^2 + M}{f^2(k)+2g(k)-g^2(k)-1}},$$

where $M = (b-af(k))^2 - a^2 - 2XY(\frac{1-(-g(k))^n}{1+g(k)})$. Then by (1), we have

$$\|H\|_2 \geq \sqrt{\frac{H_{k,n}^2 - g^2(k)H_{k,n-1}^2 + M}{f^2(k)+2g(k)-g^2(k)-1}}.$$

Similarly, for $|r| < 1$, we can write

$$\begin{aligned}\|H\|_F^2 &= \sum_{i=0}^{n-1}(n-i)H_{k,i}^2 + \sum_{i=1}^{n-1} i|r|^2 H_{k,i}^2 \\ &\geq \sum_{i=0}^{n-1}(n-i)|r|^2 H_{k,i}^2 + \sum_{i=1}^{n-1} i|r|^2 H_{k,i}^2 = n|r|^2 \sum_{i=0}^{n-1} H_{k,i}^2.\end{aligned}$$

Again, by considering Lemma 4 and (1), we get

$$\frac{\|H\|_F}{\sqrt{n}} \geq |r| \sqrt{\sum_{i=1}^{n-1} H_{k,i}^2}.$$

It follows that

$$\|H\|_2 \geq |r| \sqrt{\frac{H_{k,n}^2 - g^2(k)H_{k,n-1}^2 + M}{f^2(k) + 2g(k) - g^2(k) - 1}}.$$

Now, for $|r| \geq 1$, we give the upper bound for the spectral norm of the matrix H as in the following. Let the matrices B and C be as

$$B = \begin{pmatrix} H_{k,0} & 1 & 1 & \cdots & 1 \\ rH_{k,n-1} & H_{k,0} & 1 & \cdots & 1 \\ rH_{k,n-2} & rH_{k,n-1} & H_{k,0} & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rH_{k,1} & rH_{k,2} & rH_{k,3} & \cdots & H_{k,0} \end{pmatrix} \quad \text{and}$$

$$C = \begin{pmatrix} 1 & H_{k,1} & H_{k,2} & \cdots & H_{k,n-1} \\ 1 & 1 & H_{k,1} & \cdots & H_{k,n-2} \\ 1 & 1 & 1 & \cdots & H_{k,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

such that $H = B \circ C$. Then we obtain

$$\begin{aligned} r_1(B) &= \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{H_{k,0}^2 + |r|^2 \sum_{j=1}^{n-1} H_{k,j}^2} \\ &= \sqrt{a^2\left(1 - |r|^2\right) + |r|^2 \frac{H_{k,n}^2 - g^2(k)H_{k,n-1}^2 + M}{f^2(k) + 2g(k) - g^2(k) - 1}}, \\ c_1(C) &= \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{n} |c_{ij}|^2} = \sqrt{1 + \sum_{i=1}^{n-1} H_{k,i}^2} \\ &= \sqrt{1 - a^2 + \frac{H_{k,n}^2 - g^2(k)H_{k,n-1}^2 + M}{f^2(k) + 2g(k) - g^2(k) - 1}}, \end{aligned}$$

where $M = (b - af(k))^2 - a^2 - 2XY(\frac{1-(-g(k))^n}{1+g(k)})$, $X = b - a\beta$ and $Y = b - a\alpha$. By considering Lemma 3, we can write

$$\|H\|_2 \le r_1(B) \circ c_1(C) = \sqrt{a^2\left(1 - |r|^2\right) + |r|^2 \sum_{i=0}^{n-1} H_{k,i}^2} \sqrt{1 - a^2 + \sum_{i=0}^{n-1} H_{k,i}^2}.$$

For $|r| < 1$, we give the upper bound for the spectral norm of the matrix H as in the following. On the other hand, if the matrices D and E are

$$D = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ r & 1 & 1 & \cdots & 1 \\ r & r & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r & r & r & \cdots & 1 \end{pmatrix} \quad \text{and}$$

$$E = \begin{pmatrix} H_{k,0} & H_{k,1} & H_{k,2} & \cdots & H_{k,n-1} \\ H_{k,n-1} & H_{k,0} & H_{k,1} & \cdots & H_{k,n-2} \\ H_{k,n-2} & H_{k,n-1} & H_{k,0} & \cdots & H_{k,n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{k,1} & H_{k,2} & H_{k,3} & \cdots & H_{k,0} \end{pmatrix}$$

such that $H = D \circ E$, then we obtain

$$r_1(D) = \max_{1 \le i \le n} \sqrt{\sum_{j=1}^{n} |d_{ij}|^2} = \sqrt{n},$$

$$c_1(E) = \max_{1 \le j \le n} \sqrt{\sum_{i=1}^{n} |e_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} H_{k,i}^2} = \sqrt{\frac{H_{k,n}^2 - g^2(k) H_{k,n-1}^2 + M}{f^2(k) + 2g(k) - g^2(k) - 1}},$$

where $M = (b - af(k))^2 - a^2 - 2XY(\frac{1-(-g(k))^n}{1+g(k)})$, $X = b - a\beta$ and $Y = b - a\alpha$. By Lemma 3, then we have

$$\|H\|_2 \le r_1(D) \circ c_1(E) = \sqrt{n \frac{H_{k,n}^2 - g^2(k) H_{k,n-1}^2 + M}{f^2(k) + 2g(k) - g^2(k) - 1}},$$

as required. □

Lemma 6 *Let H be an $n \times n$ r-circulant matrix. Then we have*

$$\lambda_j(H) = \sum_{p=0}^{n-1} a_p r^{\frac{p}{n}} w^{-jp},$$

where $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$, $j = 0, 1, \ldots, n-1$.

Considering Lemma 6, we obtain the following theorem which gives us the eigenvalues of the matrix in (3).

Theorem 7 *Let $H = C_r(H_{k,0}, H_{k,1}, H_{k,2}, \dots, H_{k,n-1})$ be an r-circulant matrix. Then the eigenvalues of H are written by*

$$\lambda_j(H) = \frac{rH_{k,n} + (g(k)rH_{k,n-1} - b + af(k))r^{\frac{1}{n}}w^{-j} - H_{k,0}}{g(k)r^{\frac{2}{n}}w^{-2j} + f(k)r^{\frac{1}{n}}w^{-j} - 1},$$

where $H_{k,n}$ is the nth generalized k-Horadam number and $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$, $j = 0, 1, \dots, n-1$.

Proof By Lemma 6, we have $\lambda_j(H) = \sum_{i=0}^{n-1} H_{k,i}r^{\frac{i}{n}}w^{-ji}$. Moreover, by (2), $\sum_{i=0}^{n-1} H_{k,i}r^{\frac{i}{n}}w^{-ji} = \sum_{i=0}^{n-1} \frac{X\alpha^n - Y\beta^n}{\alpha-\beta} r^{\frac{i}{n}}w^{-ji}$. Hence, we obtain

$$\begin{aligned}
\sum_{i=0}^{n-1} H_{k,i}r^{\frac{i}{n}}w^{-ji} &= \frac{1}{\alpha-\beta}\left[X\sum_{i=0}^{n-1}\left(\alpha r^{\frac{1}{n}}w^{-j}\right)^i - Y\sum_{i=0}^{n-1}\left(\beta r^{\frac{1}{n}}w^{-j}\right)^i\right] \\
&= \frac{1}{\alpha-\beta}\left[\frac{X[(\alpha r^{\frac{1}{n}}w^{-j})^n - 1]}{\alpha r^{\frac{1}{n}}w^{-j} - 1} - \frac{Y[(\beta r^{\frac{1}{n}}w^{-j})^n - 1]}{\beta r^{\frac{1}{n}}w^{-j} - 1}\right] \\
&= \frac{rH_{k,n} + (g(k)rH_{k,n-1} - b + af(k))r^{\frac{1}{n}}w^{-j} - H_{k,0}}{g(k)r^{\frac{2}{n}}w^{-2j} + f(k)r^{\frac{1}{n}}w^{-j} - 1}
\end{aligned}$$

which completes the proof. □

Theorem 8 *The determinant of $H = C_r(H_{k,0}, H_{k,1}, H_{k,2}, \dots, H_{k,n-1})$ is formulated by*

$$\det(H) = \frac{(H_{k,0} - rH_{k,n})^n - (g(k)rH_{k,n-1} - b + af(k))^n r}{(1 - r\alpha^n)(1 - r\beta^n)}.$$

Proof It is clear that $\det(H) = \prod_{j=0}^{n-1}\lambda_j(H)$. By Theorem 7, we get

$$\prod_{j=0}^{n-1} \frac{(H_{k,0} - rH_{k,n}) - (g(k)rH_{k,n-1} - b + af(k))r^{\frac{1}{n}}w^{-j}}{(\alpha r^{\frac{1}{n}}w^{-j} - 1)(\beta r^{\frac{1}{n}}w^{-j} - 1)}.$$

Also, by considering the well-known identity $\prod_{k=0}^{n-1}(x - yw^k) = x^n - y^n$, we can write

$$\det(H) = \frac{(H_{k,0} - rH_{k,n})^n - (g(k)rH_{k,n-1} - b + af(k))^n r}{(1 - r\alpha^n)(1 - r\beta^n)}.$$

Thus the proof is completed, as desired. □

Remark 9 We should note that choosing suitable values on r, $f(k)$, $g(k)$, a and b in Theorems 5, 7 and 8, lower and upper bounds of the spectral norm, eigenvalues and determinant of r-circulant matrices for the special all second-order sequences are actually obtained.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

All authors completed the paper together. All authors read and approved the final manuscript.

Author details

[1]Department of Mathematics, Faculty of Science and Art, Nevsehir University, Nevsehir, Turkey. [2]Department of Mathematics, Faculty of Science, Selcuk University, Konya, Turkey.

Acknowledgements

The authors thank the referees for their helpful comments and suggestions concerning the presentation of this paper. The authors are also thankful to TUBITAK.

Received: 14 June 2013 Accepted: 1 August 2013 Published: 20 August 2013

References

1. Dong, C: The nonsingularity on the symmetric *r*-circulant matrices. In: Inter. Conf. on Computer Tech. and Development - ICCTD (2009). doi:10.1109/ICCCS.2009.33
2. Peirce, AP, Spottiswoode, S, Napier, JAL: The spectral boundary element method: a new window on boundary elements in rock mechanics. Int. J. Rock Mech. Min. Sci. Geomech. Abstr. **29**(4), 379-400 (1992)
3. Zhao, G: The improved nonsingularity on the *r*-circulant matrices in signal processing. In: Inter. Conf. on Computer Tech. and Development - ICCTD, Kota Kinabalu, pp. 564-567 (2009)
4. Solak, S: On the norms of circulant matrices with the Fibonacci and Lucas numbers. Appl. Math. Comput. **160**, 125-132 (2005)
5. Uslu, K, Taskara, N, Uygun, S: The relations among *k*-Fibonacci, *k*-Lucas and generalized *k*-Fibonacci and *k*-Lucas numbers and the spectral norms of the matrices of involving these numbers. Ars Comb. **102**, 183-192 (2011)
6. Shen, S, Cen, J: On the bounds for the norms of *r*-circulant matrices with the Fibonacci and Lucas numbers. Appl. Math. Comput. **216**, 2891-2897 (2010)
7. Shen, S, Cen, J: On the spectral norms of *r*-circulant matrices with the *k*-Fibonacci and *k*-Lucas numbers. Int. J. Contemp. Math. Sci. **5**(12), 569-578 (2010)
8. Yazlik, Y, Taskara, N: Spectral norm, eigenvalues and determinant of circulant matrix involving generalized *k*-Horadam numbers. Ars Comb. **104**, 505-512 (2012)
9. Yazlik, Y, Taskara, N: A note on generalized *k*-Horadam sequence. Comput. Math. Appl. **63**, 36-41 (2012)
10. Horn, RA, Johnson, CR: Topics in Matrix Analysis. Cambridge University Press, Cambridge (1991)

doi:10.1186/1029-242X-2013-394
Cite this article as: Yazlik and Taskara: **On the norms of an *r*-circulant matrix with the generalized *k*-Horadam numbers.** *Journal of Inequalities and Applications* 2013 **2013**:394.