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# Upper bound estimate of character sums over Lehmer's numbers

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**Abstract**

Let $p$ be an odd prime. For each integer $a$ with $1 \le a \le p-1$, it is clear that there exists one and only one $\overline{a}$ with $0 \le \overline{a} \le p-1$ such that $a \cdot \overline{a} \equiv 1 \bmod p$. Let $\mathcal{A}$ denote the set of all integers $1 \le a \le p-1$, in which $a$ and $\overline{a}$ are of opposite parity. The main purpose of this paper is using the analytic method and the properties of Kloosterman sums to study the estimate problem of the mean value $\sum_{a\in\mathcal{A}} \chi(a)$, and give a sharp upper bound estimate for it, where $\chi$ denotes any non-principal even character mod $p$.

**MSC:** 11L40

**Keywords:** Lehmer's numbers; character sums; Kloosterman sums; upper bound estimate

## 1 Introduction

Let $q \ge 3$ be an odd number. For each integer $a$ with $1 \le a \le q-1$ and $(a,q) = 1$, it is clear that there exists one and only one $b$ with $0 \le b \le q-1$ such that $ab \equiv 1 \bmod q$. Let $\mathcal{A}(q) = \mathcal{A}$ denote the set of cases, in which $a$ and $b$ are of opposite parity. For $q = p$, an odd prime, professor Lehmer [1] asked to study $|\mathcal{A}|$ or at least to say something nontrivial about it, where $p$ is a prime, and $|\mathcal{A}|$ denote the number of all elements in $\mathcal{A}$. We call such a number a Lehmer's number. It is known that $|\mathcal{A}| \equiv 2$ or $0 \bmod 4$ when $p \equiv \pm 1 \bmod 4$. For general odd number $q \ge 3$, Zhang ([2] and [3]) studied the asymptotic properties of $|\mathcal{A}|$, and obtained a sharp asymptotic formula for it. That is, he proved the asymptotic formula

$$|\mathcal{A}| = \frac{1}{2}\phi(q) + O\big(q^{\frac{1}{2}} \cdot d^2(q) \cdot \ln^2 q\big),$$

where $\phi(q)$ is the Euler function, and $d(q)$ is the Dirichlet divisor function.

Let $M(a,p)$ denote the number of all integers $1 \le b, c \le p-1$ such that $bc \equiv a \bmod p$ and $(2, b+c) = 1$, and $E(a,p) = M(a,p) - \frac{p-1}{2}$. Then Zhang [4] also studied the mean value properties of $E(a,p)$, and proved that

$$\sum_{a=1}^{p-1} E^2(a,p) = \frac{3}{4}p^2 + O\left(p \cdot \exp\left(\frac{3\ln p}{\ln\ln p}\right)\right),$$

where $\exp(y) = e^y$. Some related works can also be found in [5, 6] and [7].

In this paper, we consider the estimate problem of the character sums

$$\sum_{a\in\mathcal{A}}\chi(a)=\sum_{\substack{a=1\\ 2\dagger a+\overline{a}}}^{p-1}\chi(a), \tag{1.1}$$

and give a sharper upper bound estimate for it, where $\chi$ denotes any non-principal even character mod $p$.

About character sums (1.1), it is clear that its value is zero, if $\chi$ is an odd character mod $p$. If $\chi$ is a non-principal even character mod $p$, then how large is the upper bound estimate of (1.1)? About this problem, it seems that none had studied it yet, at least we have not seen any related results. The problem is interesting, because it can help us to understand the deep properties of the character sums over some special sets, for example, Lehmer's numbers.

The main purpose of this paper is using the analytic method and the properties of Kloosterman sums to study this problem, and prove the following result.

**Theorem** *Let $p>3$ be an odd prime. Then for any non-principal character $\chi$ mod $p$, we have the estimate*

$$\sum_{a\in\mathcal{A}}\chi(a)\ll p^{\frac{1}{2}}\cdot\ln^2 p.$$

For general odd number $q\geq 3$, whether there exists a similar upper bound estimate for (1.1) is an interesting problem, we will further study it.

## 2 Several lemmas

In this section, we will give several lemmas, which are necessary in the proof of our theorem. Hereinafter, we will use many properties of character sums, all of which can be found in [8], so they will not be repeated here. First we have the following.

**Lemma 1** *Let $q>2$ be an odd number. Suppose that $\chi$ is an odd character* mod $q$, *then we have the identity*

$$\bigl(1-2\chi(2)\bigr)\sum_{a=1}^{q}a\chi(a)=\chi(2)q\sum_{a=1}^{\frac{q-1}{2}}\chi(a).$$

*Proof* See reference [9]. □

**Lemma 2** *Let $q>2$ be an odd number. Then for any even character $\chi_1$* mod $q$, *we have the identity*

$$\sum_{a\in\mathcal{A}}\chi(a)=-\frac{2}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\bigl(1-2\chi\chi_1(2)\bigr)\bigl(1-2\chi(2)\bigr)\left(\frac{1}{q}\sum_{a=1}^{q}a\chi\chi_1(a)\right)\left(\frac{1}{q}\sum_{a=1}^{q}a\chi(a)\right),$$

*where $\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}$ denotes the summation over all odd characters $\chi$* mod $q$.

*Proof* From the orthogonality relation for character sums mod $q$ and the definition of $\mathcal{A}$ we have

$$\begin{aligned}\sum_{a\in\mathcal{A}}\chi(a) &= \frac{1}{2}{\sum_{a=1}^{q-1}}'\bigl(1-(-1)^{a+\overline{a}}\bigr)\chi_1(a) = \frac{1}{2}\sum_{a=1}^{q-1}\chi_1(a) - \frac{1}{2}{\sum_{a=1}^{q-1}}'(-1)^{a+\overline{a}}\chi_1(a)\\ &= -\frac{1}{2\phi(q)}\sum_{\chi \bmod q}\left(\sum_{a=1}^{q-1}(-1)^a\chi_1\chi(a)\right)\left(\sum_{b=1}^{q-1}(-1)^b\chi(b)\right),\end{aligned} \tag{2.1}$$

where ${\sum'}_{a=1}^{q-1}$ denotes the summation over all $1\le a\le q$ such that $(a,q)=1$.

If $\chi(-1)=1$, then

$$\sum_{b=1}^{q-1}(-1)^b\chi(b)=0. \tag{2.2}$$

If $\chi(-1)=-1$, then

$$\sum_{b=1}^{q-1}(-1)^b\chi(b)=2\chi(2)\sum_{b=1}^{\frac{q-1}{2}}\chi(b). \tag{2.3}$$

Note that if $\chi(-1)=-1$ and $\chi_1(-1)=1$, then $\chi\chi_1(-1)=-1$. So combining (2.1), (2.2), (2.3) and Lemma 1, we have

$$\begin{aligned}\sum_{a\in\mathcal{A}}\chi(a) &= -\frac{1}{2\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\left(2\chi\chi_1(2)\sum_{a=1}^{(q-1)/2}\chi_1\chi(a)\right)\left(2\chi(2)\sum_{b=1}^{(q-1)/2}\chi(b)\right)\\ &= -\frac{2}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\bigl(1-2\chi\chi_1(2)\bigr)\bigl(1-2\chi(2)\bigr)\left(\frac{1}{q}\sum_{a=1}^{q}a\chi\chi_1(a)\right)\left(\frac{1}{q}\sum_{a=1}^{q}a\chi(a)\right).\end{aligned}$$

This proves Lemma 2. □

**Lemma 3** *Let $p$ be an odd prime, let $\chi$ be any character* mod $p$. *Then for any integers $m$ and $n$, we have the estimate*

$$\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma+n\overline{a}}{p}\right)\ll(m,n,p)^{\frac{1}{2}}\cdot p^{\frac{1}{2}},$$

*where $e(y)=e^{2\pi iy}$, $(m,n,p)$ denotes the greatest common divisor of $m$, $n$ and $p$.*

*Proof* See [10] and [11]. □

**Lemma 4** *Let $p$ be an odd prime, let $\chi$ be any even character* mod $p$. *Then for any integer $c$ with $(c,p)=1$, we have the estimate*

$$\left|\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(c)\tau(\chi)\tau(\chi\chi_1)L(1,\overline{\chi\chi_1})L(1,\overline{\chi})\right|\ll p^{\frac{3}{2}}\ln^2 p.$$

*Proof* For any non-principal character $\chi \bmod p$, applying Abel's identity (see Theorem 4.2 of [8]), we have

$$L(1,\overline{\chi\chi_1})L(1,\overline{\chi}) = \sum_{n=1}^{p^3} \frac{\overline{\chi}(n)\sum_{d|n}\overline{\chi}_1(d)}{n} + \int_{p^3}^{\infty} \frac{A(y,\overline{\chi})}{y^2}\,dy, \tag{2.4}$$

where $A(y,\overline{\chi}) = \sum_{p^3<n\le y}\overline{\chi}(n)\sum_{d|n}\overline{\chi}_1(d)$.

From [12], we know that for any real number $y > p^3$, we have the estimate

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \left|A(y,\overline{\chi})\right|^2 \ll y\phi^2(p). \tag{2.5}$$

From (2.4), (2.5), Lemma 3, the orthogonality relation for character sums mod $p$, the definition of Gauss sums, and noting that $|\tau(\chi)| = \sqrt{p}$, we have

$$\begin{aligned}
&\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(c)\tau(\chi)\tau(\chi\chi_1)L(1,\overline{\chi\chi_1})L(1,\overline{\chi}) \\
&\quad= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_1(a)e\left(\frac{a+b}{p}\right)\sum_{n=1}^{p^3}\frac{\sum_{d|n}\overline{\chi}_1(d)}{n}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(abc\overline{n}) \\
&\qquad + O\left(\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|\tau(\chi)\right|\cdot\left|\tau(\chi\chi_1)\right|\int_{p^3}^{\infty}\frac{|A(y,\overline{\chi})|}{y^2}\,dy\right) \\
&\quad= \frac{p-1}{2}\sum_{n=1}^{p^3}\frac{\sum_{d|n}\overline{\chi}_1(d)}{n}\sum_{\substack{a=1\\ abc\equiv n \bmod p}}^{p-1}\sum_{b=1}^{p-1}\chi_1(a)e\left(\frac{a+b}{p}\right) \\
&\qquad - \frac{p-1}{2}\sum_{n=1}^{p^3}\frac{\sum_{d|n}\overline{\chi}_1(d)}{n}\sum_{\substack{a=1\\ abc\equiv -n \bmod p}}^{p-1}\sum_{b=1}^{p-1}\chi_1(a)e\left(\frac{a+b}{p}\right) \\
&\qquad + O\left(p\cdot\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\int_{p^3}^{\infty}\frac{|A(y,\overline{\chi})|}{y^2}\,dy\right) \\
&\quad= \frac{p-1}{2}\sum_{n=1}^{p^3}\frac{\sum_{d|n}\overline{\chi}_1(d)}{n}\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{a+n\overline{c}\cdot\overline{a}}{p}\right) \\
&\qquad - \frac{p-1}{2}\sum_{n=1}^{p^3}\frac{\sum_{d|n}\overline{\chi}_1(d)}{n}\sum_{a=1}^{p-1}\chi_1(a)e\left(\frac{a-n\overline{c}\cdot\overline{a}}{p}\right) + O(p) \\
&\quad= O\left(p^{\frac{3}{2}}\sum_{n=1}^{p^3}\frac{d(n)}{n}\right) = O\left(p^{\frac{3}{2}}\cdot\ln^2 p\right).
\end{aligned}$$

This proves Lemma 4. □

## 3 Proof of the theorems

In this section, we will complete the proof of our theorem. First, if $\chi(-1) = -1$, then from Theorems 12.11 and 12.20 of [8], we have

$$\frac{1}{p}\sum_{b=1}^{p} b\chi(b) = \frac{i}{\pi}\tau(\chi)L(1,\overline{\chi}). \tag{3.1}$$

From (3.1), Lemma 2 and Lemma 4, we have

$$\begin{aligned}
\sum_{a\in\mathcal{A}}\chi(a) &= -\frac{2}{\phi(p)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\bigl(1-2\chi\chi_1(2)\bigr)\bigl(1-2\chi(2)\bigr)\\
&\quad\times\left(\frac{1}{p}\sum_{a=1}^{p}a\chi\chi_1(a)\right)\left(\frac{1}{p}\sum_{a=1}^{p}a\chi(a)\right)\\
&= \frac{2}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\bigl(1-2\chi\chi_1(2)\bigr)\bigl(1-2\chi(2)\bigr)\tau(\chi)\tau(\chi\chi_1)L(1,\overline{\chi\chi_1})L(1,\overline{\chi})\\
&= \frac{2}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\tau(\chi)\tau(\chi\chi_1)L(1,\overline{\chi\chi_1})L(1,\overline{\chi})\\
&\quad-\frac{4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\tau(\chi)\tau(\chi\chi_1)L(1,\overline{\chi\chi_1})L(1,\overline{\chi})\\
&\quad-\frac{4}{\pi^2(p-1)}\chi_1(2)\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\tau(\chi)\tau(\chi\chi_1)L(1,\overline{\chi\chi_1})L(1,\overline{\chi})\\
&\quad+\frac{8}{\pi^2(p-1)}\chi_1(2)\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(4)\tau(\chi)\tau(\chi\chi_1)L(1,\overline{\chi\chi_1})L(1,\overline{\chi})\\
&= O\left(\frac{1}{p}\cdot p^{\frac{3}{2}}\cdot\ln^2 p\right) = O\bigl(p^{\frac{1}{2}}\cdot\ln^2 p\bigr).
\end{aligned}$$

This completes the proof of our theorem.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

DH studied the estimate problem of the mean value $\sum_{a\in\mathcal{A}}\chi(a)$, and gave a sharp upper bound estimate for it. WZ participated in the research and summary of the study.

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