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On a hierarchy of means

Slavko Simic*

*Correspondence:
ssimic@turing.mi.sanu.ac.rs
Mathematical Institute SANU, Kneza Mihaila 36, Belgrade, 11000, Serbia

Abstract

For a class of partially ordered means, we introduce a notion of the (nontrivial) cancelling mean. A simple method is given, which helps to determine cancelling means for the well-known classes of the Hölder and Stolarsky means.

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1 Introduction

A *mean* is a map $M:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ with a property

$$\min(a,b)\le M(a,b)\le\max(a,b)$$

for each $a,b\in\mathbb{R}_+$.

Denote by Ω the class of means which are *symmetric* (in variables a, b), *reflexive* and *homogeneous* (necessarily of order one). We shall consider in the sequel only means from this class.

The set of means can be equipped with a partial ordering defined by $M\le N$ if and only if $M(a,b)\le N(a,b)$ for all $a,b\in\mathbb{R}_+$. Thus, Δ is an *ordered* family of means if for any $M,N\in\Delta$ we have $M\le N$ or $N\le M$.

Most known ordered family of means is the following family Δ_0 of elementary means,

$$\Delta_0: H\le G\le L\le I\le A\le S,$$

where

$$H=H(a,b):=2(1/a+1/b)^{-1};\qquad G=G(a,b):=\sqrt{ab};\qquad L=L(a,b):=\frac{b-a}{\log b-\log a};$$
$$I=I(a,b):=\left(b^b/a^a\right)^{1/(b-a)}/e;\qquad A=A(a,b):=\frac{a+b}{2};\qquad S=S(a,b):=a^{\frac{a}{a+b}}b^{\frac{b}{a+b}},$$

are the harmonic, geometric, logarithmic, identric, arithmetic and Gini mean, respectively.

Another example is the class of Hölder (or Power) means $\{A_s\}$, defined for $s\in\mathbb{R}$ as

$$A_s(a,b):=\left(\frac{a^s+b^s}{2}\right)^{1/s},\qquad A_0=G.$$

It is well known that the inequality $A_s(a,b) < A_t(a,b)$ holds for all $a,b \in \mathbb{R}_+$, $a \neq b$ if and only if $s < t$. This property is used in a number of papers for approximation of a particular mean by means from the class $\{A_s\}$.

Hence (*cf.* [1–3]),

$$G = A_0 < L < A_{1/3}; \qquad A_{2/3} < I < A_1 = A;$$
$$A_{\log_\pi 2} < P < A_{2/3}; \qquad A_{\log_{\pi/2} 2} < T < A_{5/3},$$

where all bounds are best possible and Seiffert means P and T are defined by

$$P = P(a,b) := \frac{a-b}{2\arcsin\frac{a-b}{a+b}}; \qquad T = T(a,b) := \frac{a-b}{2\arctan\frac{a-b}{a+b}}.$$

In the recent paper [4], we introduce a more complex structured class of means $\{\lambda_s\}$, given by

$$\lambda_s(a,b) := \frac{s-1}{s+1}\frac{A_{s+1}^{s+1} - A^{s+1}}{A_s^s - A^s}, \quad s \in \mathbb{R},$$

that is,

$$\lambda_s(a,b) := \begin{cases} \frac{s-1}{s+1}\frac{a^{s+1}+b^{s+1}-2(\frac{a+b}{2})^{s+1}}{a^s+b^s-2(\frac{a+b}{2})^s}, & s \in \mathbb{R}/\{-1,0,1\}; \\ \frac{2\log\frac{a+b}{2}-\log a-\log b}{\frac{1}{2a}+\frac{1}{2b}-\frac{2}{a+b}}, & s = -1; \\ \frac{a\log a+b\log b-(a+b)\log\frac{a+b}{2}}{2\log\frac{a+b}{2}-\log a-\log b}, & s = 0; \\ \frac{(b-a)^2}{4(a\log a+b\log b-(a+b)\log\frac{a+b}{2})}, & s = 1, \end{cases}$$

where the last three formulae are obtained by the proper limit processes at the points –1, 0, 1, respectively. Those means are obviously symmetric and homogeneous of order one.

We also proved that λ_s is monotone increasing in $s \in \mathbb{R}$; therefore $\{\lambda_s\}$ represents an ordered family of means.

Among others, the following approximations are obtained for $a \neq b$:

$$\lambda_{-4} < H < \lambda_{-3}; \qquad \lambda_{-1} < G < \lambda_{-1/2}; \qquad \lambda_0 < L < \lambda_1 < I < \lambda_2 = A; \qquad \lambda_5 < S,$$

and *there is no s*, $s > 5$ *such that the inequality* $S(a,b) \leq \lambda_s(a,b)$ *holds for each* $a,b \in \mathbb{R}^+$.

This last result shows that in a sense, the mean S is 'greater' than any other mean from the class $\{\lambda_s\}$. We shall say that S is the *cancelling mean* for the class $\{\lambda_s\}$.

Definition 1 The mean $S^*(\Delta)$ is a right cancelling mean for an ordered class of means $\Delta \subset \Omega$ if there exists $M \in \Delta$, $M \neq S^*$ such that $S^*(a,b) \geq M(a,b)$, but there is no mean $N \in \Delta$, $N \neq S^*$ such that the inequality $N(a,b) \geq S^*(a,b)$ holds for each $a,b \in \mathbb{R}_+$.

Definition of the left cancelling mean S_* is analogous.

Therefore,

$$S_*(\Delta_0) = H; \qquad S^*(\Delta_0) = S; \qquad S^*(\lambda_s) = S.$$

Of course the left and right cancelling means exist for an arbitrary ordered family of means as $S^*(a,b) = \max(a,b)$, $S_*(a,b) = \min(a,b)$. We call them *trivial*.

The aim of this article is to determine non-trivial cancelling means for some well-known classes of ordered means. We shall also give a simple criteria for the right cancelling mean with a further discussion in the sequel.

As an illustration of problems and methods, which shall be treated in this paper, we prove firstly the following.

1.1 Cancellation theorem for the generalized logarithmic means

The family of generalized logarithmic means $\{L_p\}$ is given by

$$L_p = L_p(a,b) := \left(\frac{a^p - b^p}{p(\log a - \log b)}\right)^{1/p}, \quad p \in \mathbb{R}; \qquad L_0 = G, \qquad L_1 = L.$$

It is a subclass of well-known Stolarsky means (*cf.* [5–7]), hence symmetric, homogeneous and monotone increasing in p. Therefore, it represents an ordered family of means.

Theorem 1.1 *For the class* $\{L_p\}$, *we have*

$$S_*(L_p) = H, \qquad S^*(L_p) = A.$$

Moreover, for $-3 < p < 3$, $a \neq b$,

$$S_*(L_p) = H(a,b) < L_{-3}(a,b) < L_p(a,b) < L_3(a,b) < A(a,b) = S^*(L_p),$$

with those bounds as best possible.

Proof We prove firstly that the inequality $L_3(a,b) < A(a,b)$ holds for all $a, b \in \mathbb{R}_+$, $a \neq b$.

Indeed,

$$\frac{L_3^3}{A^3} = \frac{(\frac{2a}{a+b})^3 - (\frac{2b}{a+b})^3}{3(\log\frac{2a}{a+b} - \log\frac{2b}{a+b})} = \frac{(1+t)^3 - (1-t)^3}{3(\log(1+t) - \log(1-t))} = \frac{3+t^2}{3(1+t^2/3+t^4/5+\cdots)} < 1,$$

where we put $t := \frac{a-b}{a+b}$, $-1 < t < 1$.

Also,

$$\frac{L_p^p}{A^p} = \frac{(1+t)^p - (1-t)^p}{p(\log(1+t) - \log(1-t))}$$

and

$$\lim_{t\to 0}\frac{1}{t^2}\left(\frac{L_p^p}{A^p} - 1\right) = \frac{1}{6}p(p-3).$$

Thus, $p = 3$ is the largest p such that the inequality $L_p(a,b) \le A(a,b)$ holds for each $a, b \in \mathbb{R}_+$, since for $p > 3$ and t sufficiently small (*i.e.*, a is sufficiently close to b) we have that $L_p(a,b) > A(a,b)$.

We shall show now that A is the right cancelling mean for the class $\{L_p\}$.

Indeed, since $\lim_{t\to 1^-}\frac{L_p^p}{A^p}=0$ for fixed positive p, we conclude that the inequality $L_p \ge A$ cannot hold.

Hence by Definition 1, A is the right cancelling mean for the class $\{L_p\}$.

Noting that $H(a,b)=\frac{ab}{A(a,b)}$ and $L_{-p}(a,b)=\frac{ab}{L_p(a,b)}$, we readily get

$$L_{-p}(a,b)\ge L_{-3}(a,b)\ge H(a,b)=S_*(L_p). \qquad \square$$

2 Characteristic number and characteristic function

Let $M=M(a,b)$ be an arbitrary homogeneous and symmetric mean. In order to facilitate determination of a non-trivial right cancelling mean, we introduce here a notion of *characteristic number* $\sigma(M)$ as

$$\sigma(M):=\lim_{a/b\to\infty}\frac{M(a,b)}{A(a,b)}=M\bigl(2,0^+\bigr)=M\bigl(0^+,2\bigr).$$

By homogeneity, we get

$$\frac{M(a,b)}{A(a,b)}=M\left(\frac{2a}{a+b},\frac{2b}{a+b}\right)=M\left(2\frac{\frac{a}{b}}{\frac{a}{b}+1},\frac{2}{\frac{a}{b}+1}\right),$$

and the result follows.

Therefore,

$$\sigma(H)=\sigma(G)=\sigma(L)=0;\qquad \sigma(I)=2/e;\qquad \sigma(A)=1;\qquad \sigma(S)=2$$

and, in general,

$$0\le\sigma(M)\le 2.$$

Some simple reasoning gives the next.

Theorem 2.1 *Let $M,N\in\Omega$. If $M\le N$, then $\sigma(M)\le\sigma(N)$ but if $\sigma(M)>\sigma(N)$, then the inequality $M\le N$ cannot hold, at least when a/b is sufficiently large.*

This assertion is especially important in applications.

Also,

$$\frac{M(a,b)}{A(a,b)}=M\left(\frac{2a}{a+b},\frac{2b}{a+b}\right)=M\left(1-\frac{b-a}{a+b},1+\frac{b-a}{a+b}\right)=M(1-t,1+t),$$

where $t:=\frac{b-a}{a+b}$, $-1<t<1$.

We say that the function $\phi=\phi_M(t):=M(1-t,1+t)$ is a *characteristic function* for M (related to the arithmetic mean). If ϕ is analytic, then, because of $\phi(0)=1$, $\phi(-t)=\phi(t)$, it has a power series representation of the form

$$\phi(t)=\sum_0^\infty a_n t^{2n},\quad a_0=1, 0\le t<1.$$

In this way, a comparison between means reduces to a comparison between their characteristic functions [3, 4, 8].

Obviously,

$$\phi_H(t) = 1 - t^2; \qquad \phi_G(t) = \sqrt{1-t^2}; \qquad \phi_L(t) = \frac{2t}{\log(1+t) - \log(1-t)}; \qquad \phi_A(t) = 1;$$
$$\phi_I(t) = \exp\left(\frac{(1+t)\log(1+t) - (1-t)\log(1-t)}{2t} - 1\right); \tag{1}$$
$$\phi_S(t) = \exp\left(\frac{1}{2}\big((1+t)\log(1+t) + (1-t)\log(1-t)\big)\right).$$

Note that

$$\sigma(M) = \lim_{t\to 1^-} \phi_M(t).$$

We shall give now some applications of the above.

First of all, for an arbitrary mean $M = M(a,b)$, it is not difficult to show that $M_s = M_s(a,b) := (M(a^s, b^s))^{1/s}$ is also a mean for $s \neq 0$. Especially $M_{-1}(a,b) = \frac{ab}{M(a,b)}$ is a mean.

Moreover, it is proved in [9] that the condition $[\log M(x,y)]_{xy} < 0$ is sufficient for M_s to be monotone increasing in $s \in \mathbb{R}$ and, if $M \in \Omega$, then $M_0 = \lim_{s\to 0} M_s = G$.

For the family of means $\{M_s\}$, we can state the following *cancellation* assertion.

Theorem 2.2 *Let* $M \in \Omega$ *with* $[\log M(x,y)]_{xy} < 0$ *and* $0 < \sigma(M) < 2$.

For the ordered class of means

$$M_s = M_s(a,b) := \big(M\big(a^s, b^s\big)\big)^{1/s} \in \Omega, \quad s \neq 0; \qquad M_0 = G,$$

we have

$$S_*(M_s) = a^{\frac{b}{a+b}} b^{\frac{a}{a+b}}; \qquad S^*(M_s) = a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}.$$

Proof For fixed s, $s > 0$, we have $G = M_0 \le M_s$.

But

$$\sigma(M_s) = \big(M\big(0^+, 2^s\big)\big)^{1/s} = 2^{1-1/s}\big(\sigma(M)\big)^{1/s} < 2 = \sigma(S).$$

Therefore, by Theorem 2.1, we conclude that S is the right cancelling mean for $\{M_s\}$.

Also $G = M_0 \ge M_{-s}$. Since

$$\begin{aligned} M_{-s}(a,b) &= \big(M\big(a^{-s}, b^{-s}\big)\big)^{-1/s} = \big(M\big((ab)^{-s}b^s, (ab)^{-s}a^s\big)\big)^{-1/s} \\ &= ab\big(M\big(b^s, a^s\big)\big)^{-1/s} = \frac{ab}{M_s(a,b)} \end{aligned}$$

and

$$a^{\frac{b}{a+b}} b^{\frac{a}{a+b}} = a^{1-\frac{a}{a+b}} b^{1-\frac{b}{a+b}} = \frac{ab}{S(a,b)},$$

it easily follows that $a^{\frac{b}{a+b}} b^{\frac{a}{a+b}} = S_*(M_s)$.

Another consequence is the *cancellation* assertion for the family of Hölder means $A_r = A_r(a,b) := (A(a^r,b^r))^{1/r} = (\frac{a^r+b^r}{2})^{1/r}$, $A_0 = G$. Since $[\log A(x,y)]_{xy} = -\frac{1}{(x+y)^2} < 0$, we obtain (as is already stated) that A_r are monotone increasing with r. □

Theorem 2.3 *For* $-2 \le r \le 2$ *we have*

$$S_*(A_r) = a^{\frac{b}{a+b}} b^{\frac{a}{a+b}} \le A_{-2}(a,b) \le A_r(a,b) \le A_2(a,b) \le a^{\frac{a}{a+b}} b^{\frac{b}{a+b}} = S^*(A_r),$$

where given constants are best possible.

Proof We have

$$\frac{A_r(a,b)}{S(a,b)} = \frac{A_r(a,b)/A(a,b)}{S(a,b)/A(a,b)} = \frac{\phi_{A_r}(t)}{\phi_S(t)}$$

and

$$\begin{aligned} f_r(t) &:= \log \frac{\phi_{A_r}(t)}{\phi_S(t)} \\ &= \frac{1}{r}\log\left(\frac{(1+t)^r + (1-t)^r}{2}\right) - \frac{1}{2}\big((1+t)\log(1+t) + (1-t)\log(1-t)\big), \quad 0 < t < 1. \end{aligned}$$

Denote

$$g(t) := 2f_2(t) = 2\log\frac{\phi_{A_2}(t)}{\phi_S(t)} = \log\big(1+t^2\big) - (1+t)\log(1+t) - (1-t)\log(1-t).$$

Since

$$g'(t) = \frac{2t}{1+t^2} - \log(1+t) + \log(1-t)$$

and

$$g''(t) = \frac{2}{1+t^2} - \frac{4t^2}{(1+t^2)^2} - \frac{1}{1+t} - \frac{1}{1-t} = -\frac{8t^2}{(1+t^2)(1-t^4)} < 0,$$

we clearly have $g'(t) < g'(0) = 0$ and $g(t) < g(0) = 0$.

Therefore, the inequality $A_2(a,b) \le S(a,b)$ holds for all $a, b \in \mathbb{R}_+$.

Also, since

$$\lim_{t\to 0^+} \frac{f_r(t)}{t^2} = \frac{1}{2}(r-2),$$

we conclude that $r = 2$ is the best possible upper bound for $A_r \le S$ to hold.

Values for $S_*(A_r)$ and $S^*(A_r)$ follow from Theorem 2.2. □

3 Cancellation theorem for the class of Stolarsky means

There are plenty of papers (*cf.* [5–7]) studying different properties of the so-called Stolarsky (or extended) two-parametric mean value, defined for positive values of x, y,

$x \neq y$ by the following

$$I_{r,s} = I_{r,s}(x,y) := \begin{cases} (\frac{r(x^s-y^s)}{s(x^r-y^r)})^{1/(s-r)}, & rs(r-s) \neq 0, \\ \exp(-\frac{1}{s} + \frac{x^s \log x - y^s \log y}{x^s - y^s}), & r = s \neq 0, \\ (\frac{x^s-y^s}{s(\log x - \log y)})^{1/s}, & s \neq 0, r = 0, \\ \sqrt{xy}, & r = s = 0, \\ x, & y = x > 0. \end{cases}$$

In this form, it was introduced by Stolarsky in [6].

Most of the classical two variable means are special cases of the class $\{I_{r,s}\}$. For example, $I_{1,2} = A$, $I_{0,0} = I_{-1,1} = G$, $I_{-2,-1} = H$, $I_{0,1} = L$, $I_{1,1} = I$, *etc.*

The main properties of the Stolarsky means are given in the following assertion.

Proposition 3.1 *Means $I_{r,s}(x,y)$ are*

a. *symmetric in both parameters, i.e., $I_{r,s}(x,y) = I_{s,r}(x,y)$;*
b. *symmetric in both variables, i.e., $I_{r,s}(x,y) = I_{r,s}(y,x)$;*
c. *homogeneous of order one, that is $I_{r,s}(tx,ty) = tI_{r,s}(x,y)$, $t > 0$;*
d. *monotone increasing in either r or s;*
e. *monotone increasing in either x or y;*
f. *logarithmically convex for $r,s \in \mathbb{R}_-$ and logarithmically concave for $r,s \in \mathbb{R}_+$.*

Theorem 3.2 *For $-3 \leq r \leq s \leq 3$, we have*

$$S_*(I_{r,s}) = a^{\frac{b}{a+b}} b^{\frac{a}{a+b}} \leq I_{-3,-3}(a,b) \leq I_{r,s}(a,b) \leq I_{3,3}(a,b) \leq a^{\frac{a}{a+b}} b^{\frac{b}{a+b}} = S^*(I_{r,s}),$$

where given constants are best possible.

Proof We prove firstly that $I_{3,3}(a,b) \leq S(a,b)$, and that $s = 3$ is the largest constant such that the inequality $I_{s,s}(a,b) \leq S(a,b)$ holds for all $a,b \in \mathbb{R}_+$. For this aim, we need a notion of Lehmer means l_r defined by

$$l_r = l_r(a,b) := \frac{a^{r+1} + b^{r+1}}{a^r + b^r}.$$

They are continuous and strictly increasing in $r \in \mathbb{R}$ (*cf.* [8]). □

Lemma 3.3 [8] *$L(a,b) > l_{-\frac{1}{3}}(a,b)$ for all $a,b > 0$ with $a \neq b$, and $l_{-\frac{1}{3}}(a,b)$ is the best possible lower Lehmer mean bound for the logarithmic mean $L(a,b)$.*

We also need the following interesting identity, which is new to our modest knowledge.

Lemma 3.4 *For all $s \in \mathbb{R}/\{0\}$, we have*

$$\log \frac{I_{s,s}(a,b)}{S(a,b)} = \frac{1}{s}\left(\frac{l_{-\frac{1}{s}}(a^s,b^s)}{L(a^s,b^s)} - 1\right).$$

Proof Indeed, by the definition of $I_{s,s}$, we get

$$\begin{aligned}\log\frac{I_{s,s}(a,b)}{S(a,b)} &= -\frac{1}{s}+\frac{a^s\log a-b^s\log b}{a^s-b^s}-\frac{a\log a+b\log b}{a+b}\\ &= -\frac{1}{s}+ab\frac{a^{s-1}+b^{s-1}}{a+b}\frac{\log a-\log b}{a^s-b^s}\\ &= \frac{1}{s}\left(\frac{(a^s)^{1-1/s}+(b^s)^{1-1/s}}{(a^s)^{-1/s}+(b^s)^{-1/s}}\frac{\log a^s-\log b^s}{a^s-b^s}-1\right)\\ &= \frac{1}{s}\left(\frac{l_{-\frac{1}{s}}(a^s,b^s)}{L(a^s,b^s)}-1\right).\end{aligned}$$

Now, putting $s = 3$ in the identity above and applying Lemma 3.3, the proof follows immediately.

Therefore, by Property d of Proposition 3.1, for $r,s \in [-3,3]$, we get

$$I_{r,s} \le I_{3,3} \le S.$$

Also, since for fixed s, $s > 3$,

$$\sigma(I_{s,s}) = 2e^{-1/s} < 2 = \sigma(S),$$

it follows by Theorem 2.1 that the mean S is the right cancelling mean for $\{I_{s,s}\}$.

Similarly,

$$I_{r,s} \ge I_{-3,-3},$$

and the left hand side of Theorem 3.2 follows from easy-checkable relations

$$I_{-s,-s}(a,b) = \frac{ab}{I_{s,s}(a,b)}, \qquad a^{\frac{b}{a+b}}b^{\frac{a}{a+b}} = \frac{ab}{S(a,b)}.$$ □

4 Discussion and some open questions

Obviously, the right cancelling mean $S^*(\Delta)$ (respectively, the left cancelling mean $S_*(\Delta)$) is not unique. For instance, $T(a,b) = \frac{1}{2}(S^*(\Delta) + \max(a,b))$, $T \in \Omega$ is also cancelling mean for the class Δ.

Therefore, the mean S is not an exclusive right cancelling mean in the assertions above. Moreover, we can construct a whole class of means, which may replace the mean S as the right cancelling mean.

Theorem 4.1 *For $r > -1$, each term of the family of means K_r,*

$$K_r = K_r(a,b) := \left(\frac{a^{r+1}+b^{r+1}}{a+b}\right)^{1/r}, \qquad K_0 = S,$$

can be taken as the right cancelling mean for the class $\{M_s\}$.

Proof We shall prove first that K_r is monotone increasing in $r \in \mathbb{R}$. For this aim, consider the weighted arithmetic mean $A_{p,q}(x,y) := px + qy$, where p, q are arbitrary positive numbers such that $p + q = 1$. Since

$$\left[\log A_{p,q}(x,y)\right]_{xy} = -\frac{pq}{(px+qy)^2},$$

we conclude that

$$\tilde{A}_r(p,q;a,b) := \left(pa^r + qb^r\right)^{1/r}$$

is monotone increasing in $r \in \mathbb{R}$.

Hence, the relation

$$\tilde{A}_r\left(\frac{a}{a+b}, \frac{b}{a+b}; a, b\right) = K_r(a,b),$$

yields the proof.

Now, since for fixed $r > -1$,

$$M_0 = G \le A = K_{-1} \le K_r,$$

and $\sigma(K_r) = 2$, it follows that K_r is the right cancelling mean for the class $\{M_s\}$, analogously to the proof of Theorem 2.2. □

Finally, we propose two open questions concerning the matter above.

Q 1 Does there exist $\min(S^*(A_s))$?

Denote by $\{K_r'\}$ the subset of $\{K_r\}$ with $r > -1$, *i.e.*, $\sigma(K_r') = 2$. Then $\max(S_*(K_r')) = K_{-1} = A$.

Q 2 Does there exist a non-trivial right cancelling mean for the class $\{K_r'\}$?

Competing interests

The author declares that they have no competing interests.

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