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Estimates for operators on weighted Morrey spaces and their applications to nondivergence elliptic equations

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Abstract
In this paper, we study the norm inequalities for sublinear operators and their commutators on weighted Morrey spaces. As application, the regularity in the weighted Morrey spaces of strong solutions to nondivergence elliptic equations with *VMO* coefficients is considered.

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Keywords: weighted Morrey space; sublinear operator; nondivergence elliptic equation; commutator

1 Introduction and main results

It is well known that Morrey first introduced the classical Morrey spaces to investigate the local behavior of solutions to second-order elliptic partial differential equations (PDEs) in [1]. In recent years there has been an explosion of interest in the study of the boundedness of operators on Morrey-type spaces. It was found that many properties of solutions to PDEs are concerned with the boundedness of some operators on Morrey-type spaces. In fact, the better inclusion between Morrey and Hölder spaces permits to obtain higher regularity of the solutions to different elliptic and parabolic boundary problems. Given $f \in L^p_{loc}(\mathbb{R}^n)$ and $1 \le p \le q < \infty$, Morrey spaces are defined by (*cf.* [2])

$$M_{p,q}(\mathbb{R}^n) = \left\{ f : \|f\|_{M_{p,q}(\mathbb{R}^n)} = \sup_B \left(\frac{1}{|B|^{1-\frac{p}{q}}} \int_B |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty \right\},$$

where the supremum is taken over all the balls in $\mathbb{R}^n$. Obviously, $M_{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. For more connections between Morrey spaces and some other function spaces, see [3].

We will introduce two important operators including the Hardy-Littlewood maximal operator and the Calderón-Zygmund singular integral operator. Given $f \in L_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy, \quad x \in \mathbb{R}^n.$$

The Calderón-Zygmund singular integral operator is defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y) \, dy,$$

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where K is the general Calderón-Zygmund kernel satisfying the following conditions:

$$\left|K(x)\right| \le \frac{C}{|x|^n}, \qquad \left|\nabla K(x)\right| \le \frac{C}{|x|^{n+1}}, \quad x \ne 0$$

and

$$\int_{r\le|x|\le R} K(x)\,dx = 0, \quad 0 < r < R < \infty.$$

The operators M and T play a key role in harmonic analysis since the operator M could control much crucial quantitative information concerning the given functions despite their larger size, while the operator T, with Hilbert transform as its prototype, is closely connected with PDEs, operator theory and other fields; see [4] for more details. In 1987, Chiarenza and Frasca [2] obtained the boundedness of M and T on $M_{p,q}(\mathbb{R}^n)$. For the boundedness of operators in Morrey spaces on homogeneous spaces, see [5]. For some results on the boundedness for the multilinear singular integral operators on Morrey-type spaces, see [6].

Weighted inequalities arise naturally in Fourier analysis, but their use is best justified by the variety of applications in which they appear. For example, the theory of weights is of great importance in the study of boundary value problems for Laplace's equations on Lipschitz domains. Other applications of weighted inequalities include extrapolation theory, vector-valued inequalities, and estimates for certain classes of nonlinear mathematical physics equations (see [4]). It is worth pointing out that many authors are interested in the weighted norm inequalities when the weight function belongs to the Muckenhoupt classes. Let $w(x) \ge 0$ and $w(x) \in L_{loc}(\mathbb{R}^n)$. We say that $w \in A_p$ (the Muckenhoupt class) for $1 < p < \infty$ if there is a constant $C > 0$ such that

$$\sup_{B\subset\mathbb{R}^n}\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}\,dx\right)^{p-1} < \infty,$$

where $1/p + 1/p' = 1$. $w \in A_1$ if there is a constant $C > 0$ such that $Mw(x) \le Cw(x)$.

For any nonnegative locally integrable function w and any Lebesgue measurable function f, the norm of the weighted Lebesgue space was defined by the norm

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n}\left|f(x)\right|^p w(x)\,dx\right)^{1/p}, \quad 1 \le p < \infty.$$

If $w \equiv 1$, we denote $\|f\|_{L^p(w)}$ simply by $\|f\|_{L^p(\mathbb{R}^n)}$. It is well known that M and T are bounded operators on $L^p(w)$ with $w \in A_p$ $(1 < p < \infty)$. For the boundedness of sublinear operators on $L^p(w)$, see [7].

Komori and Shirai [8] introduced a version of the weighted Morrey space $M_{p,\lambda}(w)$, which is a natural generalization of the weighted Lebesgue space $L^p(w)$. Let $1 \le p < \infty$, $0 < \lambda < 1$ and w be a weight function. Then the spaces $M_{p,\lambda}(w)$ are defined by

$$M_{p,\lambda}(w) = \left\{f : \|f\|_{M_{p,\lambda}(w)} = \sup_{B\subset\mathbb{R}^n}\left(\frac{1}{w(B)^\lambda}\int_B\left|f(x)\right|^p w(x)\,dx\right)^{1/p} < \infty\right\},$$

where $w(B) = \int_B w(x)\,dx$. It is obvious that if $w \equiv 1$, then $M_{p,1-p/q}(w) = M_{p,q}(\mathbb{R}^n)$. For $w \in A_p$ $(1 \le p < \infty)$, $M_{p,0}(w) = L^p(w)$ and $M_{p,1}(w) = L^\infty(w)$. In [8], the authors investigated the boundedness of the operators M and T on $M_{p,\lambda}(w)$ with $w \in A_p$. Wang and Liu [9] studied the boundedness of the Bochner-Riesz means on $M_{p,\lambda}(w)$ with $w \in A_p$ $(1 \le p < \infty)$. In [10], we discussed the norm inequalities for oscillatory singular integral operators on this space. It is of interest to know whether there is a criterion for the boundedness of operators on $M_{p,\lambda}(w)$, which is the motivation of this paper. The goal of this paper is to extend some known results in [8–10] and to establish the boundedness of some sublinear operators and their commutators on the weighted Morrey spaces under some size conditions. These conditions were first proposed by Li and Yang [11] and are satisfied by most of the operators in harmonic analysis. As applications, the strong solutions of nondivergence elliptic equations with *VMO* coefficients will be given.

Let $D_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $A_k = D_k \setminus D_{k-1}$ for $k \in Z$ and χ_E be the characteristic function of the set E. Then we can formulate our main theorems as follows.

Theorem 1.1 *Suppose that a sublinear operator $\mathcal{T}$ satisfies the size conditions*

$$\left|\mathcal{T}f(x)\right| \le C|x|^{-n}\|f\|_{L^1(\mathbb{R}^n)}, \tag{1.1}$$

when $\operatorname{supp} f \subseteq A_k$, $|x| \ge 2^{k+1}$ *with* $k \in Z$ *and*

$$\left|\mathcal{T}f(x)\right| \le C2^{-kn}\|f\|_{L^1(\mathbb{R}^n)}, \tag{1.2}$$

when $\operatorname{supp} f \subseteq A_k$, $|x| \le 2^{k-1}$ *with* $k \in Z$. *Let* $0 < \lambda < 1$. *Then we have*:

(a) *If $\mathcal{T}$ is bounded on $L^p(w)$ with $w \in A_p$, then $\mathcal{T}$ is bounded on $M_{p,\lambda}(w)$, where* $1 < p < \infty$.

(b) *If $\mathcal{T}$ is bounded from $L^1(w)$ to $L^{1,\infty}(w)$ with $w \in A_1$, then there exists a constant $C > 0$ such that for all $\mu > 0$ and all B,*

$$w\left(\left\{x \in B : \mathcal{T}f(x) > \mu\right\}\right) \le C\mu^{-1}\|f\|_{M_{1,\lambda}(w)}w(B)^{\lambda}.$$

It is easy to check that both M and T satisfy the hypotheses of Theorem 1.1 (see [11, p.488]). Therefore, when $\mathcal{T}$ is M or T, Theorem 1.1 agrees with [8, Theorem 3.2] and [8, Theorem 3.2], respectively. If $\mathcal{T}$ is the Bochner-Riesz means, Theorem 1.1 is [9, Theorem 1, Theorem 2].

Corollary 1.2 *Let $1 < p < \infty$, $0 < \lambda < 1$ and $w \in A_p$. Suppose that a sublinear operator $\overline{\mathcal{T}}$ satisfies the condition*

$$\left|\overline{\mathcal{T}}f(x)\right| \le C\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\,dy, \quad x \notin \operatorname{supp} f \tag{1.3}$$

for any integrable function f with compact support. If $\overline{\mathcal{T}}$ is bounded on $L^p(w)$, then $\overline{\mathcal{T}}$ is bounded on $M_{p,\lambda}(w)$.

If $w \equiv 1$, Corollary 1.2 is [12, Theorem 2.1]. Theorem 1.1 is one of the main results of this paper. It is easy to see that condition (1.3) implies size conditions (1.1) and (1.2) since $|x-y| > |x|/2$ when $|x| \ge 2^{k+1}$, $\operatorname{supp} f \subseteq A_k$ while $\operatorname{supp} f \subseteq A_k$, $|x| \le 2^{k-1}$ imply that $|x-y| > |y|/2$. So, the proof of Corollary 1.2 is straightforward; see also Theorem 3 in [13].

Condition (1.3) was first introduced by Soria and Weiss [14]. It is worth pointing out that (1.3) is satisfied by many operators in harmonic analysis, such as the Calderón-Zygmund singular integral operator T, the Carleson maximal operator, Fefferman's singular multiplier operator, Fefferman's singular integral operator, oscillatory integral of Ricci and Stein [15], Bochner-Riesz means at the critical index, singular integral operators with oscillating kernels and so on. For more details, see [16, Remark 5] and [12, p.427]. Bandaliev [16] studied the boundedness of a certain sublinear operator which satisfies (1.3) on weighted variable Lebesgue spaces. For the boundedness of a certain sublinear operator which satisfies (1.3) on product Hardy spaces, see [17].

The bounded mean oscillation function space BMO was first introduced by John and Nirenberg [18] in the study of regular solutions of elliptic PDEs. A locally integrable function f will be said to belong to BMO if

$$\|f\|_{BMO} = \sup_{B \ni x} \frac{1}{|B|} \int_B \big|f(y) - f_B\big|\, dy < \infty,$$

where $f_B = \frac{1}{|B|} \int_B f(y)\, dy$.

The commutator formed by an operator N and a symbol function b is usually defined by $N_b f$. The boundedness of N_b is worse than N (for example, the singularity, see also [19]). Coifman, Rochberg and Weiss [20] first studied the boundedness of N_b in their study of certain factorization theorems for generalized Hardy spaces. They showed that $N_b f$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in BMO$ when $N = T$. Since then many works concerning the topic of commutators of different operators with BMO functions have come into existence. For some of other works, see [21] and [22]. In [8], the authors proved the weighted boundedness for M_b and T_b on $M_{p,\lambda}(w)$ with $b \in BMO(\mathbb{R}^n)$. We shall extend the corresponding results of the sublinear operator to their commutators. For a sublinear operator $\overline{\mathcal{T}}$, we will make the following assumption on its commutator $\overline{\mathcal{T}}_b$:

$$\big|\overline{\mathcal{T}}_b f(x)\big| \le C \int_{\mathbb{R}^n} \frac{|b(x) - b(y)||f(y)|}{|x - y|^n}\, dy, \quad x \notin \operatorname{supp} f.$$

Theorem 1.3 *Let* $1 < p < \infty$, $0 < \lambda < 1$, $w \in A_p$, *and let a sublinear operator* $\overline{\mathcal{T}}$ *satisfy* (1.3). *If* $\overline{\mathcal{T}}_b$ *is bounded on* $L^p(w)$ *with* $b \in BMO(\mathbb{R}^n)$, *then* $\overline{\mathcal{T}}_b$ *is bounded on* $M_{p,\lambda}(w)$.

When $\overline{\mathcal{T}} = T$, Theorem 1.3 agrees with [12, Theorem 2.2] and [8, Theorem 3.4].

Let $0 < \alpha < n$. Then the fractional maximal operator and the fractional integral are defined by

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/n}} \int_B \big|f(y)\big|\, dy, \quad x \in \mathbb{R}^n$$

and

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}}\, dy,$$

respectively. An early impetus to the study of fractional integrals originated from the problem of fractional derivation (see [23]). Besides their contributions to harmonic analysis, the fractional integrals also play an essential role in many fields. The Hardy-Littlewood-Sobolev inequality about the fractional integral is still an indispensable tool to establish time-space estimates for the heat semigroup of nonlinear evolution equations; see [24]. For the fractional case, weighted Morrey spaces with two weights, which are also introduced by Komori and Shirai in [8], will be needed. Suppose that $w(x)$ is a nonnegative locally integrable function on $\mathbb{R}^n$. We say that $w \in A_{(p,q)}$ $(1 < p, q < \infty)$ if there exists a constant $C > 0$ such that

$$\sup_{B\subset\mathbb{R}^n}\left(\frac{1}{|B|}\int_B w(x)^q\,dx\right)^{1/q}\left(\frac{1}{|B|}\int_B w(x)^{-p'}\,dx\right)^{1/p'} \le C$$

and $w \in A_{(1,q)}$ $(1 < q < \infty)$ if there exists a constant $C > 0$ such that

$$\sup_{B\subset\mathbb{R}^n}\left(\frac{1}{|B|}\int_B w(x)^q\,dx\right)^{1/q}\left(ess\sup_B \frac{1}{w(x)}\right) \le C.$$

The boundedness of I_α on $M_{p,q}(\mathbb{R}^n)$ was first established by Adams in [25]. In [26], the authors obtained the corresponding boundedness on weighted Lebesgue spaces for I_α with $w \in A_{(p,q)}$ $(1 \le p, q < \infty)$. Let $1 \le p < \infty$, $0 < \lambda < 1$. For two weights w_1 and w_2, the weighted Morrey spaces with two weights are defined by

$$M_{p,\lambda}(w_1, w_2) = \left\{f : \|f\|_{M_{p,k}(w_1,w_2)} = \sup_B\left(\frac{1}{w_2(B)^\lambda}\int_B |f(x)|^p w_1(x)\,dx\right)^{\frac{1}{p}} < \infty\right\}.$$

If $w_1 = w_2 = w$, then we define $M_{p,\lambda}(w, w) = M_{p,\lambda}(w)$. For the corresponding boundedness of M_α and I_α on $M_{p,\lambda}(w_1, w_2)$, see also [8].

We can get similar results for fractional integrals following the line of Theorem 1.1-Theorem 1.3.

Theorem 1.4 *Let $0 < \alpha < n$ and $0 < \lambda < 1$. Suppose that a sublinear operator $\mathcal{T}_\alpha$ satisfies the size conditions*

$$|\mathcal{T}_\alpha f(x)| \le C|x|^{-(n-\alpha)}\|f\|_{L^1(\mathbb{R}^n)}$$

when $\operatorname{supp} f \subseteq A_k$, $|x| \ge 2^{k+1}$ *with* $k \in Z$ *and*

$$|\mathcal{T}_\alpha f(x)| \le C2^{-k(n-\alpha)}\|f\|_{L^1(\mathbb{R}^n)}$$

when $\operatorname{supp} f \subseteq A_k$, $|x| \le 2^{k-1}$ *with* $k \in Z$. *Then we have*:

(a) *If $\mathcal{T}_\alpha$ maps $L^p(w^p)$ into $L^q(w^q)$ with $w \in A_{(p,q)}$, then $\mathcal{T}_\alpha$ is bounded from $M_{p,\lambda}(w^p, w^q)$ to $M_{q,q\lambda/p}(w^q)$, where $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $1 < p < q < \infty$.*

(b) *If $\mathcal{T}_\alpha$ is bounded from $L^1(w)$ to $L^{q,\infty}(w^q)$ with $w \in A_{(1,q)}$, then there exists a constant $C > 0$ such that for all $\mu > 0$ and any ball $B \subset \mathbb{R}^n$,*

$$w(\{x \in B : \mathcal{T}_\alpha f(x) > \mu\})^{1/q} \le C\mu^{-1}\|f\|_{M_{1,\lambda}(w,w^q)}w(B)^\lambda,$$

where $1 < q < \infty$.

The fractional maximal operator M_α satisfies the hypotheses of Theorem 1.4 since the pointwise inequality $M_\alpha f(x) \leq I_\alpha(|f|)(x)$ holds for $0 < \alpha < n$ (see [11, Remark 2.1]). If we take $\mathcal{T}_\alpha = M_\alpha$ and $\mathcal{T}_\alpha = I_\alpha$, then Theorem 1.4 agrees with [8, Theorem 3.5] and [8, Theorem 3.6], respectively.

Theorem 1.5 *Let p, q, α, w, λ be as in Theorem* 1.4, *and let a sublinear operator $\overline{\mathcal{T}}_\alpha$ satisfy*

$$\left|\overline{\mathcal{T}}_\alpha f(x)\right| \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy, \quad x \notin \operatorname{supp} f \tag{1.4}$$

for any integral function f with compact support. If $\overline{\mathcal{T}}_{\alpha,b}$ maps $L^p(w^p)$ into $L^q(w^q)$ with $b \in BMO(\mathbb{R}^n)$, then $\overline{\mathcal{T}}_{\alpha,b}$ is bounded from $M_{p,\lambda}(w^p, w^q)$ to $M_{q,q\lambda/p}(w^q)$.

We remark that both the fractional integral I_α and oscillatory fractional integral of Ricci and Stein [15] are examples of operators which satisfy (1.4). For the corresponding boundedness in unweighted cases of the sublinear operators satisfying (1.4) on Herz spaces, we refer the reader to [27] and [11]. Theorem 1.5 reduces to Theorem 3.7 in [8] when $\overline{\mathcal{T}}_\alpha = I_\alpha$.

Remark 1.1 As another extension of Hilbert transform, a variety of operators related to the singular integrals for Calderón-Zygmund with homogeneous kernels, but lacking the smoothness required in the classical theory, have been studied. In this case, the kernel of the operator has no regularity, and so the operator is called rough integral operator. For some classical survey works about operators with homogeneous kernels, see [4] and [28] for example. Lu, Yang and Zhou studied certain sublinear operators mentioned above with rough kernels on the generalized Morrey space in [29]. In [30], Shi and Fu obtained the boundedness of these sublinear operators with rough kernels on weighted Morrey spaces.

We end this section with the outline of this paper. Section 2 contains the proofs of Theorem 1.1, Theorem 1.3, Theorem 1.4 and Theorem 1.5. In Section 3, by means of the theories of sublinear operators and their commutators obtained in Section 2, we establish the regularity in weighted Morrey spaces of strong solutions to nondivergence elliptic equations with *VMO* coefficients.

2 Proofs of the main results

As in [30], our methods are adopted from [12] in the case of the Lebesgue measure and from [8] dealing with the classical operators. Before the proof of Theorem 1.1, we give some properties of A_p weights, which were also stated in Chapter 9 of [4]. For $\lambda > 1$, let λB denote the ball with the same center as B and radius λ times the radius of B.

Lemma 2.1 *Let $1 \leq p < \infty$ and $w \in A_p$. Then the following statements are true.*

(a) *There exists a constant C such that*

$$w(2B) \leq Cw(B). \tag{2.1}$$

(b) *There exists a constant $C > 1$ such that*

$$w(2B) \geq Cw(B). \tag{2.2}$$

(c) *There exist two constants* C *and* $r > 1$ *such that the following reverse Hölder inequality holds for every ball* $B \subset \mathbb{R}^n$:

$$\left(\frac{1}{|B|}\int_B w(x)^r\,dx\right)^{1/r} \le C\left(\frac{1}{|B|}\int_B w(x)\,dx\right). \tag{2.3}$$

(d) *For all* $\lambda > 1$, *we have*

$$w(\lambda B) \le C\lambda^{np} w(B).$$

(e) *There exist two constants* C *and* $\delta > 0$ *such that for any measurable set* $Q \subset B$,

$$\frac{w(Q)}{w(B)} \le C\left(\frac{|Q|}{|B|}\right)^{\delta}. \tag{2.4}$$

Proof of Theorem 1.1 Let $1 < p < \infty$, $w \in A_p$ and $0 < \lambda < 1$. We first give the proof of (a), for which it suffices to show that

$$\frac{1}{w(B)^{\lambda}}\int_B \left|\mathcal{T}f(x)\right|^p w(x)\,dx \le C\|f\|^p_{M_{p,\lambda}(w)}. \tag{2.5}$$

Without loss of generality, we can assume $r = 1$ for a fixed ball $B = B(x_0, r)$ and decompose $f = f\chi_{2B} + f\chi_{(2B)^c} =: f_1 + f_2$ to get

$$\begin{aligned}
&\frac{1}{w(B)^{\lambda}}\int_B \left|\mathcal{T}f(x)\right|^p w(x)\,dx \\
&\quad\le \frac{C}{w(B)^{\lambda}}\int_B \left|\mathcal{T}f_1(x)\right|^p w(x)\,dx + \frac{C}{w(B)^{\lambda}}\int_B \left|\mathcal{T}f_2(x)\right|^p w(x)\,dx \\
&\quad=: I + II.
\end{aligned}$$

Using the fact that $\mathcal{T}$ is bounded on $L^p(w)$, we have

$$I \le \frac{C}{w(B)^{\lambda}}\int_{\mathbb{R}^n} \left|\mathcal{T}f_1(x)\right|^p w(x)\,dx \le \frac{C}{w(B)^{\lambda}}\int_{2B} \left|f(x)\right|^p w(x)\,dx \le C\|f\|^p_{M_{p,\lambda}(w)}. \tag{2.6}$$

We are now in a position to estimate the term II. It follows from $w \in A_p$ that

$$\begin{aligned}
\int_{(2B)^c} \left|f(y)\right|\,dy &\le C\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB} \left|f(y)\right|\,dy \\
&\le C\sum_{k=1}^{\infty}\left(\int_{2^{k+1}B} \left|f(y)\right|^p w(y)\,dy\right)^{1/p}\left(\int_{2^{k+1}B} w(y)^{-p'/p}\,dy\right)^{1/p'} \\
&\le C\|f\|_{M_{p,\lambda}(w)}\sum_{k=1}^{\infty}\frac{|2^{k+1}B|}{w(2^{k+1}B)^{(1-\lambda)/p}}.
\end{aligned}$$

By assumption (1.2), we have

$$\begin{aligned} II &\le \frac{C}{w(B)^\lambda}\sum_{k=1}^{\infty}2^{-knp}\int_B \|f_2\|_{L^1(D_k)}^p w(x)\,dx \\ &\le \frac{C}{w(B)^{\lambda-1}}\sum_{k=1}^{\infty}2^{-knp}\left(\int_{D_k}\big|f(y)\big|\,dy\right)^p \\ &\le C\|f\|_{M_{p,\lambda}(w)}^p\left(\sum_{k=1}^{\infty}\frac{w(B)^{(1-\lambda)/p}}{w(2^{k+1}B)^{(1-\lambda)/p}}\right)^p \le C\|f\|_{M_{p,\lambda}(w)}^p. \end{aligned} \tag{2.7}$$

Here we have used (2.2) in the last inequality. Combining (2.6) with (2.7), we get (2.5).

We can now proceed analogously to the proof of part (b). We will show the following inequality:

$$\sup_{\mu>0}\frac{\mu}{w(B)^\lambda}w\big(\big\{x\in B:\big|\mathcal{T}f(x)\big|>\mu\big\}\big)\le C\|f\|_{M_{1,\lambda}(w)}^p.$$

Decompose $f = f\chi_{2B} + f\chi_{(2B)^c} =: f_1 + f_2$ with B as that of (a) to obtain

$$\begin{aligned} & w\big(\big\{x\in B:\big|\mathcal{T}f(x)\big|>\mu\big\}\big) \\ &\quad\le w\big(\big\{x\in B:\big|\mathcal{T}f_1(x)\big|>\mu/2\big\}\big)+w\big(\big\{x\in B:\big|\mathcal{T}f_2(x)\big|>\mu/2\big\}\big) \\ &\quad=: J + JJ. \end{aligned}$$

An application of (2.1) and the weighted weak (1,1) type estimates for $\mathcal{T}$ yield that

$$J\le w\big(\big\{x\in\mathbb{R}^n:\big|\mathcal{T}f_1(x)\big|>\mu/2\big\}\big)\le C\mu^{-1}\|f\|_{M_{1,\lambda}}(w)w(B)^\lambda.$$

For the term JJ, an elementary estimate shows

$$JJ\le\frac{C}{\mu}\int_{\{x\in B:|\mathcal{T}f_2(x)|>\mu/2\}}\big|\mathcal{T}f_2(x)\big|w(x)\,dx.$$

On the other hand, a further use of (1.2) yields

$$\big|\mathcal{T}f_2(x)\big|\le C\sum_{k=1}^{\infty}2^{-kn}\int_{D_k}\big|f(y)\big|\,dy\le C\sum_{k=1}^{\infty}2^{-kn}\int_{2^{k+1}B}\big|f(y)\big|\,dy,$$

from which it follows that

$$\begin{aligned} JJ &\le \frac{C}{\mu}\sum_{k=1}^{\infty}2^{-kn}\int_{2^{k+1}B}\big|f(y)\big|w(y)\,dy\le\frac{C}{\mu}\|f\|_{M_{1,\lambda}}(w)\sum_{k=1}^{\infty}2^{kn(\lambda-1)}w(B)^\lambda \\ &\le \frac{C}{\mu}\|f\|_{M_{1,\lambda}}(w)w(B)^\lambda. \end{aligned}$$

We have thus completed the proof of (b). □

The proof of Theorem 1.3 depends heavily on the following remarks about *BMO* functions.

Lemma 2.2 [31, Theorem 3.8] (see also [4, Proposition 7.1.5]) *Let* $1 \le p < \infty$, $b \in BMO(\mathbb{R}^n)$. *Then, for any ball* $B \subset \mathbb{R}^n$, *the following statements are true*:

(a) *There exist constants* C_1, C_2 *such that for all* $\alpha > 0$,

$$\left|\left\{x \in B : \left|b(x) - b_B\right| > \alpha\right\}\right| \le C_1|B|e^{-C_2\alpha/\|b\|_{BMO(\mathbb{R}^n)}}. \tag{2.8}$$

Inequality (2.8) *is called John-Nirenberg inequality.*

(b)

$$|b_{2^\lambda B} - b_B| \le 2^n\lambda\|b\|_{BMO(\mathbb{R}^n)}. \tag{2.9}$$

Lemma 2.3 [4, Proposition 7.1.2] (see also [32, Theorem 5]) *Let* $w \in A_\infty$ *and* $1 < p < \infty$. *Then the following statements are equivalent*:

(a) $\|b\|_{BMO(\mathbb{R}^n)} \sim \sup_B(\frac{1}{|B|}\int_B |b(x) - b_B|^p\,dx)^{\frac{1}{p}}$;

(b) $\|b\|_{BMO(\mathbb{R}^n)} \sim \sup_B \inf_{a\in\mathbb{R}} \frac{1}{|B|}\int_B |b(x) - a|\,dx$;

(c) $\|b\|_{BMO(w)} = \sup_B \frac{1}{w(B)}\int_B |b(x) - b_{B,w}|w(x)\,dx$, *where* $BMO(w) = \{b : \|b\|_{BMO(w)} < \infty\}$ *and* $b_{B,w} = \frac{1}{w(B)}\int_B b(y)w(y)\,dy$.

Lemma 2.4 *Let* $b \in BMO(\mathbb{R}^n)$, $w \in A_p$, $B = B(x_0, r)$ *be a fixed ball*, $0 < \lambda < 1$ *and* $1 < p < \infty$. *Then the inequality*

$$\left(\int_{|x_0-y|>2r} \frac{|f(y)|}{|x_0 - y|^n}\left|b_{B,w} - b(y)\right|dy\right)^p w(B)^{1-\lambda} \le C\|f\|^p_{M_{p,\lambda}(w)}\|b\|^p_{BMO(\mathbb{R}^n)} \tag{2.10}$$

holds for every $y \in (2B)^c$ *and* $f \in M_{p,\lambda}(w)$, *where* $(2B)^c = \mathbb{R}^n \setminus 2B$.

Proof The proof of Lemma 2.4 has a root in [30], which we adopted here for the completeness of this paper. Applying Hölder's inequality to the left-hand side of (2.10), we obtain

$$\begin{aligned}
&\left(\int_{|x_0-y|>2r} \frac{|f(y)|}{|x_0 - y|^n}\left|b_{B,w} - b(y)\right|dy\right)^p w(B)^{1-\lambda} \\
&\quad\le \left(\sum_{j=1}^\infty \int_{2^jr<|x_0-y|<2^{j+1}r} \frac{|f(y)|}{|x_0 - y|^n}\left|b_{B,w} - b(y)\right|dy\right)^p w(B)^{1-\lambda} \\
&\quad\le \left(\sum_{j=1}^\infty \frac{1}{|2^jB|}\int_{2^{j+1}B}\left|f(y)\right|\left|b_{B,w} - b(y)\right|dy\right)^p w(B)^{1-\lambda} \\
&\quad\le C\left[\sum_{j=1}^\infty \frac{1}{|2^jB|}\left(\int_{2^{j+1}B}\left|f(y)\right|^p w(y)\,dy\right)^{1/p}\left(\int_{2^{j+1}B}\left|b_{B,w} - b(y)\right|^{p'} w(y)^{1-p'}\,dy\right)^{1/p'}\right]^p \\
&\qquad\times w(B)^{1-\lambda} \\
&\quad\le C\|f\|^p_{M_{p,\lambda}(w)}\left[\sum_{j=1}^\infty \frac{w(2^{j+1}B)^{\frac{\lambda}{p}}}{|2^jB|}\left(\int_{2^{j+1}B}\left|b_{B,w} - b(y)\right|^{p'} w(y)^{1-p'}\,dy\right)^{1/p'}\right]^p w(B)^{1-\lambda}.
\end{aligned}$$

For the simplicity, we define

$$\mathcal{I} = \left(\int_{2^{j+1}B}\left|b_{B,w} - b(y)\right|^{p'} w(y)^{1-p'}\,dy\right)^{1/p'}.$$

By an elementary estimate, we have

$$\begin{aligned}\mathcal{I} &\le \left(\int_{2^{j+1}B}\left(\left|b_{2^{j+1}B,w^{1-p'}}-b(y)\right|+\left|b_{2^{j+1}B,w^{1-p'}}-b_{B,w}\right|\right)^{p'}w(y)^{1-p'}\,dy\right)^{1/p'}\\ &\le \left(\int_{2^{j+1}B}\left|b_{2^{j+1}B,w^{1-p'}}-b(y)\right|w(y)^{1-p'}\,dy\right)^{\frac{1}{p'}}+\left|b_{2^{j+1}B,w^{1-p'}}-b_{B,w}\right|w^{1-p'}\left(2^{j+1}B\right)^{1/p'}\\ &=: \mathcal{I}_1+\mathcal{I}_2.\end{aligned}$$

For the term $\mathcal{I}_1$, we use the fact that if $w \in A_p$, then $w^{1-p'} \in A_{p'}$. By Lemma 2.3,

$$\mathcal{I}_1 \le C\|b\|_{BMO(w^{1-p'})}w^{1-p'}\left(2^{j+1}B\right)^{1/p'} \le Cw^{1-p'}\left(2^{j+1}B\right)^{1/p'}. \tag{2.11}$$

To deal with $\mathcal{I}_2$, by (2.9), we have

$$\begin{aligned}&\left|b_{2^{j+1}B,w^{1-p'}}-b_{B,w}\right|\\ &\quad\le \left|b_{2^{j+1}B,w^{1-p'}}-b_{2^{j+1}B}\right|+\left|b_{2^{j+1}B}-b_B\right|+\left|b_B-b_{B,w}\right|\\ &\quad\le \frac{1}{w^{1-p'}(2^{j+1}B)}\int_{2^{j+1}B}\left|b(y)-b_{2^{j+1}B}\right|w(y)^{1-p'}\,dy+2^n(j+1)\|b\|_{BMO(\mathbb{R}^n)}\\ &\qquad+\frac{1}{w(B)}\int_B\left|b(y)-b_B\right|w(y)\,dy\\ &\quad=: \mathcal{I}_{21}+\mathcal{I}_{22}+\mathcal{I}_{23}.\end{aligned}$$

Using (2.4) and (2.8), we obtain that

$$\begin{aligned}\mathcal{I}_{23} &= \frac{1}{w(B)}\int_0^\infty w\left(\left\{x\in B:\left|b(y)-b_B\right|>\alpha\right\}\right)d\alpha\\ &\le C\int_0^\infty e^{-C_2\alpha\delta/\|b\|_{BMO(\mathbb{R}^n)}}\,d\alpha \le C,\end{aligned}$$

and analogously, $\mathcal{I}_{21} \le C$.

Hence

$$\mathcal{I}_2 \le C\left(2^n(j+1)+2\right)w^{1-p'}\left(2^{j+1}B\right)^{1/p'}. \tag{2.12}$$

As a by-product of (2.11) and (2.12), we have

$$\mathcal{I} \le C(j+1)w^{1-p'}\left(2^{j+1}B\right)^{1/p'}.$$

Then, the proof of (2.10) is concluded from (2.2) and the following observation:

$$\begin{aligned}&\left[\sum_{j=1}^{\infty}\frac{w(2^{j+1}B)^{\lambda/p}}{|2^jB|}\left(\int_{2^{j+1}B}\left|b(y)-b_{B,w}\right|^{p'}w(y)^{1-p'}\,dy\right)^{1/p'}\right]^p w(B)^{1-\lambda}\\ &\quad\le C\left[\sum_{j=1}^{\infty}\frac{(j+1)w(B)^{(1-\lambda)/p}}{w(2^{j+1}B)^{(1-\lambda)/p}}\right]^p = C.\end{aligned}$$

□

Proof of Theorem 1.3 It is sufficient to show that for a fixed ball $B = B(x_0, 1)$,

$$\frac{1}{w(B)^\lambda}\int_B \left|\overline{\mathcal{T}}_b f(x)\right|^p w(x)\,dx \le C\|f\|^p_{M_{p,\lambda}(w)}. \tag{2.13}$$

Decompose $f = f\chi_{2B} + f\chi_{(2B)^c} =: f_1 + f_2$. Then

$$\begin{aligned}\int_B \left|\overline{\mathcal{T}}_b f(x)\right|^p w(x)\,dx &\le C\left(\int_B \left|\overline{\mathcal{T}}_b f_1(x)\right|^p w(x)\,dx + \int_B \left|\overline{\mathcal{T}}_b f_2(x)\right|^p w(x)\,dx\right)\\ &=: K + KK.\end{aligned}$$

The $L^p(w)$ boundedness of $\overline{\mathcal{T}}_b$ allows us to get

$$K \le C\int_{2B} \left|f(x)\right|^p w(x)\,dx \le C\|f\|^p_{M_{p,\lambda}(w)} w(B)^\lambda. \tag{2.14}$$

Making use of (1.3), we have

$$\begin{aligned}\left|\overline{\mathcal{T}}_b f_2(x)\right|^p &\le C\left(\int_{\mathbb{R}^n} \frac{|f_2(y)||b(x)-b(y)|}{|x-y|^n}\,dy\right)^p\\ &\le C\left(\int_{|x_0-y|>2} \frac{|f(y)|}{|x_0-y|^n}\left\{\left|b(x)-b_{B,w}\right| + \left|b_{B,w}-b(y)\right|\right\}dy\right)^p.\end{aligned}$$

Hence,

$$\begin{aligned}KK &\le C\left(\int_{|x_0-y|>2} \frac{|f(y)|}{|x_0-y|^n}\,dy\right)^p \int_B \left|b(x)-b_{B,w}\right|^p w(x)\,dx\\ &\quad + C\left(\int_{|x_0-y|>2} \frac{|f(y)|}{|x_0-y|^n}\left|b(y)-b_{B,w}\right|dy\right)^p w(B)\\ &=: KK_1 + KK_2.\end{aligned}$$

From Lemma 2.4, we can obtain $KK_2 \le C\|f\|^p_{M_{p,\lambda}(w)} w(B)^\lambda$.

To estimate KK_1, we take into account (2.1), (2.3) and Lemma 2.3. Indeed,

$$\begin{aligned}KK_1 &= C\left(\sum_{j=1}^\infty \int_{2^j<|x_0-y|<2^{j+1}} \frac{|f(y)|}{|x_0-y|^n}\,dy\right)^p \int_B \left|b(x)-b_{B,w}\right|^p w(x)\,dx\\ &\le C\sum_{j=1}^\infty \frac{1}{|2^jB|}\left(\frac{1}{w(2^{j+1}B)^\lambda}\int_{2^{j+1}B} \left|f(y)\right|^p w(y)\,dy\right)^{1/p}\\ &\quad \times w\left(2^{j+1}B\right)^{\lambda/p}\left(\int_{2^{j+1}B} w(y)^{-1/p-1}\,dy\right)^{(p-1)/p} \int_B \left|b(x)-b_{B,w}\right|^p w(x)\,dx\\ &\le C\|f\|_{M_{p,\lambda}(w)}\left(\sum_{j=1}^\infty \frac{|2^{j+1}B|^{-\frac{1}{p}}}{|2^jB|}\left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B} w(y)\,dy\right)^{-1/p} w\left(2^{j+1}B\right)^{\lambda/p}\right)^p\\ &\quad \times \int_B \left|b(x)-b_{B,w}\right|^p w(x)\,dx\end{aligned}$$

$$\le C\|f\|^p_{M_{p,\lambda}(w)}\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{j=1}^{\infty}\left(\frac{w(B)^{(1-\lambda)/p}}{w(2^{j+1}B)^{(1-k)/p}}\right)^p w(B)^{\lambda}$$
$$\le C\|f\|^p_{M_{p,\lambda}(w)}w(B)^{\lambda}.$$

Therefore,

$$KK \le C\|f\|^p_{M^{p,\lambda}(w)}w(B)^{\lambda}. \tag{2.15}$$

Combining (2.14) with (2.15), we obtain (2.13). Therefore, we finish the proof of Theorem 1.3. □

Proof of Theorem 1.4 We can use similar arguments as in the proof of Theorem 1.1. For the proof of (a), it suffices to show that

$$\frac{1}{w^q(B)^{q\lambda/p}}\int_B\left|\mathcal{T}_\alpha f(x)\right|^q w(x)^q\,dx \le C\|f\|^q_{M_{p,\lambda}(w^p,w^q)}.$$

For a fixed ball $B = B(x_0, 1)$, we decompose $f = f\chi_{2B} + f\chi_{(2B)^c} =: f_1 + f_2$. Since $\mathcal{T}_\alpha$ is a sublinear operator, we get

$$\begin{aligned}
&\frac{1}{w^q(B)^{q\lambda/p}}\int_B\left|\mathcal{T}_\alpha f(x)\right|^q w(x)^q\,dx\\
&\quad\le \frac{C}{w^q(B)^{q\lambda/p}}\int_B\left(\left|\mathcal{T}_\alpha f_1(x)\right|^q + \left|\mathcal{T}_\alpha f_2(x)\right|^q\right)w^q(x)\,dx\\
&\quad=: L + LL.
\end{aligned}$$

To estimate the term L, using the fact that $\mathcal{T}_\alpha$ is bounded from $L^p(w^p)$ to $L^q(w^q)$ with $w \in A_{(p,q)}$, we can get

$$\int_B\left|\mathcal{T}_\alpha f_1(x)\right|^q w^q(x)\,dx \le C\|f\|^q_{M_{p,\lambda}(w^p,w^q)}w^q(B)^{q\lambda/p},$$

which implies that $L \le C\|f\|_{M_{p,\lambda}(w^p,w^q)}$.

For the term LL, by similar arguments to those of Theorem 1.1, we obtain

$$\begin{aligned}
LL &\le C\sum_k\left(2^{-k(n-\alpha)}\int_{A_k}\left|f(y)\right|dy\right)^q w^q(B)^{1-q\lambda/p}\\
&\le C\sum_k\left(2^{-k(n-\alpha)}\|f\|_{M_{p,\lambda}(w^p,w^q)}\left|2^{k+1}B\right|^{1-\alpha/n}\frac{1}{w^q(2^{k+1}B)^{1/q-\lambda/p}}\right)^q w^q(B)^{1-q\lambda/p}\\
&\le C\|f\|^q_{M_{p,\lambda}(w^p,w^q)}\left(\sum_{k=1}^{\infty}\frac{w^q(B)^{(1/q-\lambda/p)}}{w^q(2^{k+1}B)^{(1/q-\lambda/p)}}\right)^q\\
&\le C\|f\|^q_{M_{p,\lambda}(w^p,w^q)}.
\end{aligned}$$

We have completed the proof of (a).

Using an argument quite similar to the one in the proof of (a), we can prove (b). We omit the proof here. □

Proof of Theorem 1.5 The proof of Theorem 1.5 is similar to that of Theorem 1.3, except using $w \in A_{(p,q)}$. □

3 Applications to nondivergence elliptic equations

In this section, we shall give some applications of our main results to nondivergence elliptic equations. The Dirichlet problem on the second-order elliptic equation in nondivergence form is

$$\begin{cases} Lu = \sum_{i,j}^{n} a_{ij}(x) u_{x_i} u_{x_j} = f & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Here $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, Ω is a bounded domain of $\mathbb{R}^n$. The coefficients $(a_{ij})_{i,j=1}^n$ of L are symmetric and uniformly elliptic, *i.e.*, for some $\nu \geq 1$ and every $\xi \in \mathbb{R}^n$, $a_{ij}(x) = a_{ji}(x)$ and $\nu^{-1}|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \nu|\xi|^2$ with a.e. $x \in \Omega$. In [12], Fan, Lu and Yang investigated the regularity in $M_{p,\lambda}(\Omega)$ of the strong solution to (3.1) with $a_{ij} \in VMO(\Omega)$, the space of the functions of vanishing mean oscillation introduced by Sarason in [33]. The main methods of [12] are based on integral representation formulas established in [34] for the second derivatives of the solution u to (3.1), *a priori* estimate of the solution to (3.1) and on the theories of singular integrals and sublinear commutators in corresponding Morrey spaces.

By extending some theorems of [12] to weighted versions, we can also establish the regularity in weighted Morrey spaces $M_{p,\lambda}$ of strong solutions to problem (3.1).

Theorem 3.1 *Let* $w \in A_p$ $(1 < p < \infty)$, $f \in M_{p,\lambda}(w)$ *with* $0 < \lambda < 1$. *Then* (3.1) *has a unique solution* $u \in W^2M_{p,\lambda}(w)$ *satisfying*

$$\|u\|_{W^2M_{p,\lambda}(w)} \leq C\|f\|_{M_{p,\lambda}(w)},$$

where $W^2M_{p,\lambda}(w)$ *is the Sobolev-Morrey space.* $u \in W^2M_{p,\lambda}(w)$ *means* u *and its distributional derivatives,* u_{x_i}, $u_{x_ix_j}$ $(i,j = 1, \dots, n)$ *are in* $M_{p,\lambda}(w)$.

The proof of Theorem 3.1 is very similar to that of [12], we omit the details. Here, we only take two main results to explain this similarity. All other proofs of the corresponding theorems are straightforward. Firstly, Theorem 1.3 in Section 1 is just the weighted version of important Theorem 2.1 in [12]. Next, we give the proof of another important result (the weighted version of Theorem 2.3 of [12]).

Let $\mathbb{R}_+^n = \{x = (x', x_n) : x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0\}$, $L_+^p(w) = L^p(w, \mathbb{R}_+^n)$ and $M_{p,\lambda}^+ = M_{p,\lambda}(w, \mathbb{R}_+^n)$. To establish the boundary estimates of the solutions to (3.1), we need the following general theorem for sublinear operators.

Theorem 3.2 *Let* $1 < p < \infty$, $0 < \lambda < 1$, $w \in A_p$, $\tilde{x} = (x', -x_n)$ *for* $x = (x', x_n) \in \mathbb{R}_+^n$. *If a sublinear operator* $\mathfrak{T}$ *is bounded on* $L_+^p(w)$ *for any* $f \in L_+^1(w)$ *with compact support and satisfies*

$$\left|\mathfrak{T}f(x)\right| \leq C \int_{\mathbb{R}_+^n} \frac{|f(y)|}{|\tilde{x} - y|^n}\, dy, \tag{3.2}$$

then $\mathfrak{T}$ *is bounded on* $M_{p,\lambda}^+(w)$.

Proof Let $z \in \mathbb{R}^n_+$ and $\delta > 0$. Set $B^+_\delta(z) = B_\delta(z) \cap \mathbb{R}^n_+$, where $B_\delta(z) = \{y \in \mathbb{R}^n : |z - y| < \delta\}$. We consider the following two cases.

Case 1. $0 \le z_n < 2\delta$. In this case, we write

$$f(y) = f(y)\chi_{B^+_{2^4\delta}(z)}(y) + \sum_{i=4}^{\infty} f(y)\chi_{B^+_{2^{i+1}\delta}(z)/B^+_{2^i\delta}(z)}(y) =: \sum_{i=3}^{\infty} f_i(y).$$

Therefore, by the $L^p_+(w)$ boundedness of $\mathfrak{T}$ and (3.2), we obtain

$$\begin{aligned}
&\frac{1}{w(B^+_\delta)^{\lambda/p}} \left(\int_{B^+_\delta} \left|\mathfrak{T}f(x)\right|^p w(x)\,dx \right)^{1/p} \\
&\quad\le \frac{1}{w(B^+_\delta)^{\lambda/p}} \sum_{i=3}^{\infty} \left(\int_{B^+_\delta} \left|\mathfrak{T}f_i(x)\right|^p w(x)\,dx \right)^{1/p} \\
&\quad\le \frac{C}{w(B^+_\delta)^{\lambda/p}} \|f_3\|_{L^p_+(w)} + \frac{C}{w(B^+_\delta)^{\lambda/p}} \sum_{i=4}^{\infty} \left(\int_{B^+_\delta} \left(\int_{B^+_{2^{i+1}\delta}(z)/B^+_{2^i\delta}(z)} \frac{|f(y)|}{|\tilde{x} - y|^n}\,dy \right)^p w(x)\,dx \right)^{1/p} \\
&\quad\le C\|f\|_{M^+_{p,\lambda}(w)} + C\sum_{i=4}^{\infty} \frac{1}{(2^i\delta)^n} \left(\int_{B^+_{2^{i+1}\delta}} \left|f(y)\right|\,dy \right) w\left(B^+_\delta\right)^{(1-\lambda)/p} \\
&\quad\le C\|f\|_{M^+_{p,\lambda}(w)} \left(1 + \sum_{i=4}^{\infty} \frac{w(B^+_\delta)^{(1-\lambda)/p}}{w(B^+_{2^{i+1}\delta})^{(1-\lambda)/p}} \right) \\
&\quad\le C\|f\|_{M^+_{p,\lambda}(w)}.
\end{aligned}$$

In the last inequality, we have used Lemma 2.1.

Case 2. There exists $i \in \mathbb{N}$ such that $2^i\delta \le z_n < 2^{i+1}\delta$. In this case, we write

$$f(y) = f(y)\chi_{B^+_{2^{i+1}\delta}(z)}(y) + \sum_{j=1}^{\infty} f(y)\chi_{B^+_{2^{i+j+4}\delta}(z)}(y) =: \sum_{j=0}^{\infty} f_j(y).$$

By (3.2) and Lemma 2.1, we have

$$\begin{aligned}
&\frac{1}{w(B^+_\delta)^{\lambda/p}} \left(\int_{B^+_\delta} \left|\mathfrak{T}f(x)\right|^p w(x)\,dx \right)^{1/p} \\
&\quad\le \frac{C}{w(B^+_\delta)^{\lambda/p}} \left(\int_{B^+_\delta} \left(\int_{B^+_{2^{i+4}\delta}(z)} \frac{|f(y)|}{|\tilde{x} - y|^n}\,dy \right)^p w(x)\,dx \right)^{1/p} \\
&\qquad + \frac{C}{w(B^+_\delta)^{\lambda/p}} \sum_{j=1}^{\infty} \left(\int_{B^+_\delta} \left(\int_{B^+_{2^{i+j+4}\delta}(z)/B^+_{2^{i+j+3}\delta}(z)} \frac{|f(y)|}{|\tilde{x} - y|^n}\,dy \right)^p w(x)\,dx \right)^{1/p} \\
&\quad\le C\|f\|_{M^+_{p,\lambda}(w)} \left(\frac{w(B^+_\delta)^{(1-\lambda)/p}}{w(B^+_{2^{i+4}\delta})^{(1-\lambda)/p}} + \sum_{j=1}^{\infty} \frac{w(B^+_\delta)^{(1-\lambda)/p}}{w(B^+_{2^{i+j}\delta})^{(1-\lambda)/p}} \right) \\
&\quad\le C\|f\|_{M^+_{p,\lambda}(w)}.
\end{aligned}$$

□

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

SS carried out the study for norm inequalities for sublinear operators and drafted the manuscript. ZF carried out the study for norm inequalities for the commutators of certain sublinear operators. FZ participated in the study for the regularity of strong solutions to nondivergence elliptic equations. All authors read and approved the final manuscript.

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