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Infinite propagation speed for the two component b-family system

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Abstract
In this paper, we propose the infinite propagation speed for the two component b-family system. No matter what the profile of the compactly supported initial datum $(u_0(x), \rho_0(x))$ is, for any $t > 0$ in its lifespan, the solution $u(x, t)$ is positive at infinity and negative at negative infinity.

MSC: 37L05; 35Q58; 26A12

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1 Introduction

In this essay, we consider the following model, named two-component b-family system

$$\begin{cases} m_t = um_x + k_1 mu_x + k_2\rho\rho_x, & t > 0, x \in \mathbb{R}, \\ \rho_t = k_3(\rho u)_x, & t > 0, x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $m = u - u_{xx}$. As far as we known, it seems that system (1.1) appears initially by Guha in [1]. There are two cases about this system (i) $k_1 = b$, $k_2 = 2b$ and $k_3 = 1$; (ii) $k_1 = b + 1$, $k_2 = 2$ and $k_3 = b$ with $b \in \mathbb{R}$. In [2], they applied Kato's theory [3] to establish the local well-posedness for the Cauchy problem of (1.1). It is proved that there exists a unique solution $(u, \rho) \in C([0, T); H^s \times H^{s-1})$ for any $(u_0, \rho_0) \in H^s \times H^{s-1}$ with $s > \frac{3}{2}$. The precise blow-up scenarios and some blow-up criteria were also established in [2].

We only consider the case $k_3 = 1$,

$$\begin{cases} m_t = um_x + k_1 mu_x + k_2\rho\rho_x, & t > 0, x \in \mathbb{R}, \\ \rho_t = (\rho u)_x, & t > 0, x \in \mathbb{R}. \end{cases} \tag{1.2}$$

Obviously, under the constraint of $\rho = 0$, system (1.2) reduces to the b-family equations

$$m_t + um_x + bmu_x = 0, \tag{1.3}$$

which were derived physically by Holm and Staley in [4]. Detailed description of the corresponding strong solutions to (1.3) with u_0, being its initial data, was given by Zhou in [5]. He established a sufficient condition in profile on the initial data for blow-up in finite time. The necessary and sufficient condition for blow-up is still a challenging problem for us at present. More precisely, Theorem 3.1 in [5] means that no matter what the profile of the compactly supported initial datum $u_0(x)$ is (no matter whether it is positive or negative), for any $t > 0$ in its lifespan, the solution $u(x,t)$ is positive at infinity and negative at negative infinity, it is really a very nice property for the b-family equations. For $b = 2$ and $b = 3$, (1.3) is the famous Camassa-Holm equation [6] and Degasperis-Procesi equation [7], respectively. Many papers [8–14] are devoted to their studies.

Another related system is the two component Camassa-Holm system. Recently, Constantin and Ivanov in [15] gave a demonstration about its derivation in view of the fluid shallow water theory from the hydrodynamic point of view. This generalization, similarly to the Camassa-Holm equation, possessed the peakon, multi-kink solutions and the bi-Hamiltonian structure [16] and is integrable. Well-posedness and wave breaking mechanism were discussed in [17], and the existence of global solutions was analyzed in [18]. The infinite propagation speed was studied by Henry in [19].

In the following section, we will show our main results and give the detailed proof.

2 Main results

Motivated by Mckean's deep observation for the Camassa-Holm equation [11], we can do the similar particle trajectory as

$$\begin{cases} q_t = -u(q,t), & 0 < t < T, x \in \mathbb{R}, \\ q(x,0) = x, & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where T is the life span of the solution, then q is a diffeomorphism of the line. Differentiating the first equation in (2.1) with respect to x, one has

$$\frac{dq_t}{dx} = q_{xt} = -u_x(q,t)q_x, \quad t \in (0,T).$$

Hence

$$q_x(x,t) = \exp\left\{\int_0^t -u_x(q,s)\,ds\right\}, \qquad q_x(x,0) = 1.$$

Our first result will show that m and ρ have compact support if their initial data have this property. m and ρ are the solutions of system (1.2).

Theorem 2.1 *Assume that $m_0 = u_0 - u_{0xx}$ has compact support, contained in the interval $[\alpha_{m_0}, \beta_{m_0}]$, and that ρ_0 is also compactly supported, with support contained in $[\alpha_{\rho_0}, \beta_{\rho_0}]$. If $T = T(u_0, \rho_0) > 0$ is the maximal existence time of the unique classical solutions (u,ρ) to the system* (1.2) *with the given initial data $u_0(x)$ and $\rho_0(x)$, then for any $t \in [0,T)$, $m(x,t)$ and $\rho(x,t)$ have compact support.*

Proof By the particle trajectory defined in (2.1), we find that

$$\begin{aligned} \frac{d}{dt} m\big(q(x,t),t\big)q_x^{k_1}(x,t) &= m_t q_x^{k_1} + m_x q_t q_x^{k_1} + k_1 m q_{xt} q_x^{k_1-1} \\ &= (\rho_t - u m_x - k_1 m u_x)q_x^{k_1} = k_2 \rho\rho_x q_x^{k_1} \end{aligned}$$

and

$$\frac{d}{dt}\rho\big(q(x,t),t\big)q_x(x,t) = \rho_t q_x + \rho_x q_t q_x + \rho q_{xt} = \big(\rho_t - (u\rho)_x\big)q_x = 0.$$

Therefore,

$$\rho\big(q(x,t),t\big)q_x(x,t) = \rho_0(x_0).$$

Since $q_x \geq 0$ and ρ_0 is compactly supported, it follows from this relation that $\rho(\cdot,t)$ is compactly supported for all times $t \in [0,T)$, with support contained in the interval $[q(\alpha_{\rho_0},t), q(\beta_{\rho_0},t)]$. Setting

$$\alpha = \min\{\alpha_{m_0}, \alpha_{\rho_0}\}, \qquad \beta = \max\{\beta_{m_0}, \beta_{\rho_0}\}.$$

Due to

$$\frac{d}{dt}m\big(q(x,t),t\big)q_x^{k_1}(x,t) = 0, \quad \text{on } x \in \mathbb{R} - [\alpha_{\rho_0}, \beta_{\rho_0}],$$

it follows that $m(\cdot,t)$ is compactly supported, with its support contained in the interval $[q(\alpha,t), q(\beta,t)]$, for all $t \in [0,T)$. □

In order to prove Theorem 2.3, we will use the following result.

Lemma 2.2 [9] *Let $u \in C^2(\mathbb{R}) \cap H^2(\mathbb{R})$ be such that $m = u - u_{xx}$ has compact support. Then u has compact support if and only if*

$$\int_{\mathbb{R}} e^x m\,dx = \int_{\mathbb{R}} e^{-x} m\,dx = 0. \tag{2.2}$$

Theorem 2.3 *Let $k_1 \in [0,3]$, $k_2 \geq 0$, assume that the function u_0 has compact support. Let $T \geq 0$ be the maximal existence time of the unique solution $u(x,t)$ with initial data $u_0(x)$. If at every $t \in [0,T)$, $u(x,t)$ has compact support, then u and ρ are identically zero.*

Proof Using (1) and differentiating the left hand side of (2.2) with respect to t we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} e^x m\,dx &= \int_{\mathbb{R}} e^x m_t\,dx = \int_{\mathbb{R}} e^x u m_x\,dx + \int_{\mathbb{R}} e^x k_1 u_x m\,dx + \int_{\mathbb{R}} e^x k_2 \rho\rho_x\,dx \\ &= \int_{\mathbb{R}} e^x u m_x\,dx + \int_{\mathbb{R}} e^x k_1 u_x m\,dx + \int_{\mathbb{R}} e^x k_2 \rho\rho_x\,dx \\ &= -\int_{\mathbb{R}} e^x u m\,dx + \int_{\mathbb{R}} e^x (k_1 - 1) u_x m\,dx - \int_{\mathbb{R}} e^x \frac{k_2}{2}\rho^2\,dx \\ &= \int_{\mathbb{R}} \frac{-k_1}{2}u^2 + \frac{k_1 - 3}{2}u_x^2 - \frac{k_2}{2}\rho^2\,dx,\end{aligned}$$

where all boundary terms after integration by parts vanish as both $m(\cdot,t)$ and, by assumption, $u(\cdot,t)$ have compact support for all $t \in [0,T)$.

The expression under the integral on the right hand side of this relation must be identically zero by (2.2). This implies that all of the terms in the bracket must be identically zero, and in particular $u = \rho = 0$. This completes the proof. □

The theorem above means that if $u \not\equiv 0$ is a function with compact support, then the classical solution $u(x,t)$ of system (1.2) must instantly lose the compactness of its support. The following theorem will give a detailed description about the profile of the solution $u(x,t)$.

Theorem 2.4 *Let $0 \leq k_1 \leq 3$ and $k_2 \geq 0$, u is a nontrivial solution of* (1.2). *If $u_0(x) = u(x,0)$ has compact support $[\alpha_{u_0}, \beta_{u_0}]$, and ρ_0 is also initially compactly supported, on the interval $[\alpha_{\rho_0}, \beta_{\rho_0}]$, then for $t \in (0,T]$, we have*

$$u(x,t) = \begin{cases} f_-(t)e^{-x}, & \text{for } x > q(\beta,t), \\ f_+(t)e^{x}, & \text{for } x < q(\alpha,t), \end{cases}$$

where $f_-(t)$ and $f_+(t)$ denote continuous nonvanishing functions with $f_-(t) < 0$ and $f_+(t) > 0$ for $t \in (0,T]$. Furthermore, $f_-(t)$ is a strictly decreasing function, while $f_+(t)$ is increasing function.

Proof Since u_0 and ρ_0 are compactly supported. By Theorem 2.1, m is compactly supported with its support contained in the interval $[q(\alpha,t), q(\beta,t)]$. Hence the following functions are well-defined:

$$E(t) = \int_{\mathbb{R}} e^x m(x,t)\,dx \quad \text{and} \quad F(t) = \int_{\mathbb{R}} e^{-x} m(x,t)\,dx,$$

with

$$E_0 = \int_{\mathbb{R}} e^x m_0(x)\,dx = 0 \quad \text{and} \quad F_0 = \int_{\mathbb{R}} e^{-x} m_0(x)\,dx = 0.$$

Then for $x > q(\beta,t)$, we have

$$u(x,t) = \frac{1}{2}e^{-|x|} * m(x,t) = \frac{1}{2}e^{-x}\int_{q(\alpha,t)}^{q(\beta,t)} e^{\tau} m(\tau,t)\,d\tau = \frac{1}{2}e^{-x}E(t). \tag{2.3}$$

Similarly, when $x < q(\alpha,t)$, we get

$$u(x,t) = \frac{1}{2}e^{-|x|} * m(x,t) = \frac{1}{2}e^{x}\int_{q(\alpha,t)}^{q(\beta,t)} e^{-\tau} m(\tau,t)\,d\tau = \frac{1}{2}e^{x}F(t). \tag{2.4}$$

Hence, as consequences of (2.3) and (2.4), we have

$$u(x,t) = -u_x(x,t) = u_{xx}(x,t) = \frac{1}{2}e^{-x}E(t), \quad \text{as } x > q(\beta,t) \tag{2.5}$$

and

$$u(x,t) = u_x(x,t) = u_{xx}(x,t) = \frac{1}{2}e^{x}F(t), \quad \text{as } x < q(\alpha,t). \tag{2.6}$$

On the other hand,

$$\frac{dE(t)}{dt} = \int_{\mathbb{R}} e^x m_t(x,t)\,dx.$$

It is easy to get

$$m_t = uu_x - (uu_x)_{xx} + \partial_x\left(\frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2\right) + k_2\rho\rho_x. \tag{2.7}$$

Substituting the identity (2.7) into $dE(t)/dt$, we obtain

$$\begin{aligned}\frac{dE(t)}{dt} &= \int_{\mathbb{R}} e^x\left(uu_x - (uu_x)_{xx} + \partial_x\left(\frac{k_1}{2}u^2 + \frac{3-k_2}{2}u_x^2\right)\right)dx + \int_{\mathbb{R}} e^x(k_2\rho\rho_x)\,dx \\ &= \int_{\mathbb{R}} e^x\left(\frac{-k_1}{2}u^2 + \frac{k_2-3}{2}u_x^2 - \frac{k_2}{2}\rho^2\right)dx,\end{aligned}$$

where we use (2.5) and (2.6).

Therefore, in the lifespan of the solution, we have

$$E(t) = \int_0^t \int_{\mathbb{R}} e^x\left(\frac{-k_1}{2}u^2 + \frac{k_1-3}{2}u_x^2 - \frac{k_2}{2}\rho^2\right)(x,\tau)\,dx\,d\tau < 0.$$

By the same argument, one can check that the following identity for $F(t)$ is true

$$F(t) = \int_0^t \int_{\mathbb{R}} e^{-x}\left(\frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2\right)(x,\tau)\,dx\,d\tau > 0.$$

In order to complete the proof, it is sufficient to let $f_-(t) = \frac{1}{2}E(t)$ and $f_+(t) = \frac{1}{2}F(t)$. □

Remark 2.1 Theorem 2.3 means that no matter what the profile of the compactly supported initial datum $u_0(x)$ is (no matter whether it is positive or negative), for any $t > 0$ in its lifespan, the solution $u(x,t)$ is negative at infinity and positive at negative infinity.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZW proposed the problems and finished the whole manuscript. YJ proved Theorems 2.1 and 2.3. LJ proved Theorem 2.4. All authors read and approved the final manuscript.

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