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# A note to the convergence rates in precise asymptotics

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**Abstract**

Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with zero mean. Set $S_n = \sum_{k=1}^{n} X_k$, $EX^2 = \sigma^2 > 0$, and $\lambda_{r,p}(\epsilon) = \sum_{n=1}^{\infty} n^{r/p-2} P(|S_n| \geq n^{1/p}\epsilon)$. In this paper, the author discusses the rate of approximation of $\frac{p}{r-p} E|N|^{2(r-p)/(2-p)}$ by $\epsilon^{2(r-p)/(2-p)} \lambda_{r,p}(\epsilon)$ under suitable moment conditions, where $N$ is normal with zero mean and variance $\sigma^2 > 0$, which improves the results of Gut and Steinebach (J. Math. Anal. Appl. 390:1-14, 2012) and extends the work He and Xie (Acta Math. Appl. Sin. 29:179-186, 2013). Specially, for the case $r = 2$ and $p = \frac{1}{\beta+1}$, $\beta > -\frac{1}{2}$, the author discusses the rate of approximation of $\frac{\sigma^2}{2\beta+1}$ by $\epsilon^2 \lambda_{2,1/(\beta+1)}(\epsilon)$ under the condition $EX^2 I(|X| > t) = O(t^{-\delta} l(t))$ for some $\delta > 0$, where $l(t)$ is a slowly varying function at infinity.

**MSC:** 60F15; 60G50

**Keywords:** convergence rate; precise asymptotics; slowly varying function

## 1 Introduction

Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables. Set $S_n = \sum_{k=1}^{n} X_k$ and $\lambda_{r,p}(\epsilon) = \sum_{n=1}^{\infty} n^{r/p-2} P(|S_n| \geq n^{1/p}\epsilon)$. Heyde [1] proved that

$$\lim_{\epsilon \to 0} \epsilon^2 \lambda_{2,1}(\epsilon) = \sigma^2,$$

whenever $EX = 0$ and $EX^2 = \sigma^2 < \infty$. Klesov [2] studied the rate of the approximation of $\sigma^2$ by $\epsilon^2 \lambda_{2,1}(\epsilon)$ under the condition $E|X|^3 < \infty$. He and Xie [3] improved the results of Klesov [2]. Gut and Steinebach [4] extended the results of Klesov [2] and obtained the following Theorem $A$. Gut and Steinebach [5] studied the general idea of proving precise asymptotics.

**Theorem A** *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d.random variables with zero mean and $0 < p < 2$, $r \geq 2$.*

(1) *If $EX^2 = \sigma^2 > 0$ and $E|X|^q < \infty$ for some $r < q \leq 3$, then*

$$\epsilon^{2(r-p)/(2-p)} \lambda_{r,p}(\epsilon) - \frac{p}{r-p} E|N|^{2(r-p)/(2-p)} = o\left(\epsilon^{\frac{p(q-2)(r-p)}{(q-p)(2-p)}}\right).$$

(2) *If $EX^2 = \sigma^2 > 0$ and $E|X|^q < \infty$ for some $q \geq 3$ with $q > \frac{2r-3p}{2-p}$, then*

$$\epsilon^{2(r-p)/(2-p)} \lambda_{r,p}(\epsilon) - \frac{p}{r-p} E|N|^{2(r-p)/(2-p)} = o\left(\epsilon^{\frac{2p(r-p)}{(2-p)(p+2q-pq)}}\right),$$

*where $N$ is normal with mean $0$ and variance $\sigma^2 > 0$.*

The purpose of this paper is to strengthen Theorem $A$ and extend the theorem of He and Xie [3] under suitable moment conditions. In addition, we shall discuss the rate at which $\epsilon^2\lambda_{2,1/(\beta+1)}(\epsilon)$ converges to $\frac{\sigma^2}{2\beta+1}$ under the condition $T(t) = O(t^{-\delta}l(t))$ for some $\delta > 0$, where $T(t) = EX^2I(|X| > t)$, $l(t)$ is a slowly varying function at infinity. Throughout this paper, C represents a positive constant, though its value may change from one appearance to the next, and $[x]$ denotes the integer part of $x$. $\Phi(x)$ is the standard normal distribution function, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$, $\varphi(x) = \Phi'(x)$.

## 2 Main results

From Gut and Steinebach [6], it is easy to obtain the following lemma.

**Lemma 2.1** *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. normal distribution random variables with zero mean and variance $\sigma^2 > 0$. Set $0 < p < 2$ and $r \geq 2$, then*

$$\epsilon^{2(r-p)/(2-p)}\sum_{n=1}^{\infty} n^{(r/p)-2}P\big(|S_n| \geq n^{1/p}\epsilon\big) - \frac{p}{r-p}E|N|^{2(r-p)/(2-p)} = \begin{cases} O(\epsilon^{2(r-p)/(2-p)}), & r < 3p, \\ O(\epsilon^{4p/(2-p)}), & r \geq 3p. \end{cases} \tag{2.1}$$

**Lemma 2.2** (Bingham *et al.* [7]) *Let $l(t)$ be a slowly varying function. We have*

(1) *for any $\eta > 0$,*

$$\lim_{t\to\infty} t^{\eta}l(t) = \infty, \qquad \lim_{t\to\infty} t^{-\eta}l(t) = 0;$$

(2) *if $0 < \delta < 1$, then*

$$\int_a^t s^{-\delta}l(s)\,ds \sim \frac{1}{1-\delta}t^{1-\delta}l(t), \quad t \to \infty;$$

(3) *if $\delta > 1$, then*

$$\int_t^{\infty} s^{-\delta}l(s)\,ds \sim -\frac{1}{1-\delta}t^{1-\delta}l(t), \quad t \to \infty;$$

(4) *if $\delta = 1$, then $L(t) = \int_t^{\infty}\frac{l(s)}{s}\,ds$, $m(t) = \int_a^t \frac{l(s)}{s}\,ds$ are slowly varying functions*; *and*

$$\lim_{t\to\infty}\frac{l(t)}{L(t)} = 0, \qquad \lim_{t\to\infty}\frac{l(t)}{m(t)} = 0.$$

**Theorem 2.1** *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d.random variables with zero mean and $0 < p < 2$, $r \geq 2$.*

(1) *If $EX^2 = \sigma^2 > 0$ and $E|X|^3 < \infty$ for some $r < 3$, then*

$$\epsilon^{2(r-p)/(2-p)}\lambda_{r,p}(\epsilon) - \frac{p}{r-p}E|N|^{2(r-p)/(2-p)} = \begin{cases} O(\epsilon^{2(r-p)/(2-p)}), & 2 \leq r < \frac{3p}{2}, \\ O(\epsilon^{p/(2-p)}\log\frac{1}{\epsilon}), & r = \frac{3p}{2}, \\ O(\epsilon^{p/(2-p)}), & \frac{3p}{2} < r < 3. \end{cases} \tag{2.2}$$

(2) *If* $EX^2 = \sigma^2 > 0$ *and* $E|X|^{2+\delta} < \infty$ *for some* $0 < \delta < 1$, $r < 2 + \delta$, *then*

$$\epsilon^{2(r-p)/(2-p)}\lambda_{r,p}(\epsilon) - \frac{p}{r-p}E|N|^{2(r-p)/(2-p)} = \begin{cases} O(\epsilon^{2(r-p)/(2-p)}), & 2 \le r < (1+\delta/2)p, \\ O(\epsilon^{p\delta/(2-p)}\log\frac{1}{\epsilon}), & r = (1+\delta/2)p, \\ o(\epsilon^{p\delta/(2-p)}), & (1+\delta/2)p < r < 2+\delta. \end{cases} \tag{2.3}$$

(3) *If* $EX^2 = \sigma^2 > 0$ *and* $E|X|^q < \infty$ *for some* $q \ge 3$ *with* $q > \frac{2r-3p}{2-p}$, *then*

$$\epsilon^{2(r-p)/(2-p)}\lambda_{r,p}(\epsilon) - \frac{p}{r-p}E|N|^{2(r-p)/(2-p)} = \begin{cases} O(\epsilon^{2(r-p)/(2-p)}), & 2 \le r < \frac{3p}{2}, \\ O(\epsilon^{p/(2-p)}\log\frac{1}{\epsilon}), & r = 3p/2, \\ O(\epsilon^{p/(2-p)}), & r > 3p/2, \end{cases} \tag{2.4}$$

*where* $N$ *is normal with mean* 0 *and variance* $\sigma^2 > 0$.

**Remark 2.1** Clearly, Theorem 1 and Theorem 2 in He and Xie [3] are special cases of Theorem 2.1, by taking $r = 2$ and $p = 1$.

**Remark 2.2** If $0 < p < 2$, $r \ge 2$, we have $\min(\frac{2(r-p)}{2-p}, \frac{p\delta}{2-p}) > \frac{p\delta(r-p)}{(2+\delta-p)(2-p)}$ for $r < 2 + \delta = q \le 3$ and $\min(\frac{2(r-p)}{2-p}, \frac{p}{2-p}) > \frac{2(r-p)p}{(2-p)(p+2q-pq)}$ for some $q \ge 3$ with $q > \frac{2r-3p}{2-p}$. So, the results of Theorem 2.1 are stronger than those of Theorem A.

**Theorem 2.2** *Let* $\{X, X_n; n \ge 1\}$ *be a sequence of i.i.d random variables with zero mean, and let* $T(t) = O(t^{-\delta}l(t))$ *for some* $\delta > 0$, *where* $l(t)$ *is a slowly varying function at infinity. Set* $EX^2 = \sigma^2 > 0$ *and* $\beta > -\frac{1}{2}$.

(1) *If* $\delta > 1$, *then*

$$\epsilon^2\lambda_{2,1/(\beta+1)}(\epsilon) - \frac{\sigma^2}{2\beta+1} = \begin{cases} O(\epsilon^2), & -\frac{1}{2} < \beta < -\frac{1}{4}, \\ O(\epsilon^2\log\frac{1}{\epsilon}), & \beta = -\frac{1}{4}, \\ O(\epsilon^{1/(2\beta+1)}), & \beta > -\frac{1}{4}. \end{cases} \tag{2.5}$$

(2) *If* $0 < \delta < 1$, *then*

$$\epsilon^2\lambda_{2,1/(\beta+1)}(\epsilon) - \frac{\sigma^2}{2\beta+1} = \begin{cases} O(\epsilon^2), & -\frac{1}{2} < \beta < -\frac{1}{2} + \frac{\delta}{4}, \\ O(\epsilon^{\delta/(2\beta+1)}l(\epsilon^{-1/(2\beta+1)})), & \beta \ge -\frac{1}{2} + \frac{\delta}{4}. \end{cases} \tag{2.6}$$

(3) *If* $\delta = 1$, *then*

$$\epsilon^2\lambda_{2,1/(\beta+1)}(\epsilon) - \frac{\sigma^2}{2\beta+1} = \begin{cases} O(\epsilon^2 + \epsilon^2\int_1^{\epsilon^{-1/(2\beta+1)}}\frac{l(t)}{t}\,dt), & -\frac{1}{2} < \beta < -\frac{1}{4}, \\ O(\epsilon^2(1 + \int_1^{\epsilon^{-5}}\frac{l(t)}{t}\,dt)\log\frac{1}{\epsilon}), & \beta = -\frac{1}{4}, \\ O(\epsilon^{1/(2\beta+1)}(1 + \int_1^{\epsilon^{-(2\beta+3)/(2\beta+1)}}\frac{l(t)}{t}\,dt)), & \beta > -\frac{1}{4}. \end{cases} \tag{2.7}$$

**Remark 2.3** For $r = 2$, $p = \frac{1}{\beta+1}$. If $l(t) = 1$, then the result of Theorem 2.2 is weaker than that of Theorem 2.1 for $0 < \delta < 1$, $\beta \ge -\frac{1}{2} + \frac{\delta}{4}$, and weaker than that of Theorem 2.1 for

$\delta = 1$. But the condition $T(t) = O(t^{-\delta})$ is weaker than the condition $E|X|^{2+\delta} < \infty$. If $l(t) \to 0$ as $t \to \infty$, then the result of Theorem 2.2 is the same as that of Theorem 2.1 for $0 < \delta < 1$.

**Remark 2.4** For $\delta > 0$, the condition $E|X|^{2+\delta} < \infty$ is neither sufficient nor necessary for the condition $T(t) = O(t^{-\delta} l(t))$. Here are some suitable examples.

**Example 1** Let $X$ be a random variable with density $f(x) = \frac{C(1+\delta \ln|x|)}{|x|^{3+\delta} \ln^2|x|} I(|x| > e)$, where $C$ is a normalizing constant, and $0 < \delta < 1$, then $EX = 0$ and $T(t) = \frac{C}{t^\delta \ln t} I(t > e)$, $l(t) = \frac{1}{\ln t}$ is a slowly varying function at infinity. But $E|X|^{2+\delta} = C \int_{|x|>e} \frac{1+\delta \ln|x|}{|x| \ln^2|x|}\,dx = \infty$.

**Example 2** Let $X$ be a random variable with density $f(x) = \frac{C(\delta \ln^2|x| + |x|(\ln|x| - 1))}{|x|^{\delta+3} \ln^2|x| e^{|x|/\ln|x|}} I(|x| > e)$, where $0 < \delta < 1$, then $EX = 0$ and $T(t) = \frac{C}{t^\delta e^{t/\ln(t)}} I(t > e)$, $h(t) = \frac{1}{e^{t/\ln t}}$, $E|X|^{2+\delta} < \infty$. But $h(t) = \frac{1}{e^{t/\ln t}}$ is not a slowly varying function at infinity.

In fact, we have the following result.

**Theorem 2.3** *Suppose $X$ is a real random variable and $\delta > 0$. Then $E|X|^{2+\delta} < \infty$ if and only if $t^\delta T(t) \to 0$ and $\int_t^\infty s^{\delta-1} T(s)\,ds \to 0$ as $t \to \infty$.*

**Remark 2.5** If $t^\delta T(t)$ is bounded as $t \to \infty$ for some $\delta > 0$, then we have $E|X|^{2+\alpha} < \infty$ for every $\alpha \in (0, \delta)$ from Theorem 2.3.

**Remark 2.6** Let $X$ be a random variable with zero mean. If there exist positive constants $C_1$ and $C_2$ such that $C_1 l(t) \le t^\delta T(t) \le C_2 l(t)$ for sufficiently large $t$ and some $\delta > 0$, where $l(t)$ is a slowly varying function at infinity, then from Lemma 2.2(4) and Theorem 2.3, we have

$$E|X|^{2+\delta} < \infty \quad \Leftrightarrow \quad \int_t^\infty \frac{l(s)}{s}\,ds \to 0 \quad \text{as } t \to \infty.$$

## 3 Proofs of the main results

*Proof of Theorem* 2.1 Without loss of generality, we suppose that $\sigma^2 = 1$, $0 < \epsilon < 1$. Since

$$P\big(|S_n| \ge n^{1/p}\epsilon\big) = 2\big(1 - \Phi\big(n^{(2-p)/2p}\epsilon\big)\big) + R_n, \tag{3.1}$$

where

$$R_n = P\big(S_n \le -n^{1/p}\epsilon\big) - \Phi\big(-n^{1/p-1/2}\epsilon\big) + \Phi\big(n^{1/p-1/2}\epsilon\big) - P\big(S_n \le n^{1/p}\epsilon\big).$$

From (3.1), we have

$$\begin{aligned} &\epsilon^{2(r-p)/(2-p)}\lambda_{r,p}(\epsilon) - \frac{p}{r-p} E|N|^{2(r-p)/(2-p)} \\ &\quad = 2\epsilon^{2(r-p)/(2-p)} \sum_{n=1}^\infty n^{r/p-2}\big(1 - \Phi\big(n^{(2-p)/2p}\epsilon\big)\big) - \frac{p}{r-p} E|N|^{2(r-p)/(2-p)} \\ &\qquad + \epsilon^{2(r-p)/(2-p)} \sum_{n=1}^\infty n^{r/p-2} R_n. \end{aligned} \tag{3.2}$$

By Lemma 2.1, in order to prove Theorem 2.1, we only need to estimate $\epsilon^{2(r-p)/(2-p)} \times \sum_{n=1}^{\infty} n^{r/p-2} R_n$.

(1) On account of a non-uniform estimate of the central limit theorem by Nagaev [8], for every $x \in R$,

$$\left| P\left(\frac{S_n}{\sqrt{n}} < x\right) - \Phi(x) \right| \leq \frac{CE|X|^3}{\sqrt{n}(1+|x|)^3}. \tag{3.3}$$

By (3.3), $|R_n| \leq \frac{CE|X|^3}{\sqrt{n}(1+\epsilon n^{(2-p)/2p})^3}$.

(a) If $r < 3p/2$, then

$$\epsilon^{2(r-p)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2} R_n \leq C\epsilon^{2(r-p)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-5/2} = O\big(\epsilon^{2(r-p)/(2-p)}\big). \tag{3.4}$$

(b) If $3p/2 < r < 3$, then

$$\begin{aligned} &\epsilon^{2(r-p)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2} R_n \\ &\quad \leq C\epsilon^{2(r-p)/(2-p)} \sum_{n=1}^{\infty} \frac{n^{r/p-2}}{\sqrt{n}(1+\epsilon n^{(2-p)/2p})^3} \\ &\quad \leq C\epsilon^{2(r-p)/(2-p)} \left( \sum_{n=1}^{[\epsilon^{-2p/(2-p)}]} \frac{n^{r/p-2}}{\sqrt{n}} + \epsilon^{-3} \sum_{n=[\epsilon^{-2p/(2-p)}]+1}^{\infty} n^{r/p-5/2-(6-3p)/2p} \right) \\ &\quad = O\big(\epsilon^{p/(2-p)}\big). \end{aligned} \tag{3.5}$$

(c) If $r = 3p/2$, then

$$\begin{aligned} \epsilon^{2(r-p)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2} R_n &\leq C\epsilon^{p/(2-p)} \left( \sum_{n=1}^{[\epsilon^{-2p/(2-p)}]} \frac{1}{n} + \epsilon^{-3} \sum_{n=[\epsilon^{-2p/(2-p)}]+1}^{\infty} n^{-1-(6-3p)/2p} \right) \\ &= O\left(\epsilon^{p/(2-p)} \log \frac{1}{\epsilon}\right). \end{aligned} \tag{3.6}$$

From (2.1), (3.2), (3.4), (3.5) and (3.6), we obtain (2.2). This completes the proof of part (1).

(2) By the inequality in Osipov and Petrov [9], there exists a bounded and decreasing function $\psi(u)$ on the interval $(0,\infty)$ such that $\lim_{u\to\infty} \psi(u) = 0$ and

$$\left| P\left(\frac{1}{\sqrt{n}\sigma} S_n < x\right) - \Phi(x) \right| \leq \frac{\psi(\sqrt{n}(1+|x|))}{n^{\delta/2}(1+|x|)^{2+\delta}}.$$

Let $x = n^{(2-p)/2p}\epsilon$, we have $|R_n| \leq \frac{2\psi(\sqrt{n}(1+n^{(2-p)/2p}\epsilon))}{n^{\delta/2}(1+n^{(2-p)/2p}\epsilon)^{2+\delta}}$, so that:

(a) If $2 < r < (1+\delta/2)p$, then

$$\epsilon^{2(r-p)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2} R_n \leq \epsilon^{2(r-p)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2-\delta/2} = O\big(\epsilon^{2(r-p)/(2-p)}\big). \tag{3.7}$$

(b) If $(1+\delta/2)p < r < 2+\delta$, then by noticing that $\lim_{u\to\infty}\psi(u) = 0$ for any $\eta > 0$, there exists a natural number $N_0$ such that $\psi(\sqrt{n}) < \eta$ whenever $n > N_0$. We conclude that

$$\begin{aligned}
&\epsilon^{2(r-p)/(2-p)}\sum_{n=1}^{\infty} n^{r/p-2}R_n \\
&\quad\le C\epsilon^{2(r-p)/(2-p)}\sum_{n=1}^{\infty}\frac{2n^{r/p-2}\psi(\sqrt{n}(1+n^{(2-p)/2p}\epsilon))}{n^{\delta/2}(1+\epsilon n^{(2-p)/2p})^{2+\delta}} \\
&\quad\le C\epsilon^{2(r-p)/(2-p)}\left(\sum_{n=1}^{N_0} n^{r/p-2-\delta/2}\psi(\sqrt{n}) + \eta\sum_{n=N_0+1}^{[\epsilon^{-2p/(2-p)}]} n^{r/p-2-\delta/2}\right) \\
&\qquad + C\epsilon^{2(r-p)/(2-p)-2-\delta}\psi\left(\epsilon^{-p/(2-p)}\right)\sum_{n=[\epsilon^{-2p/(2-p)}]+1}^{\infty} n^{r/p-2-\delta/2-(1/p-1/2)(2+\delta)} \\
&\quad\le \epsilon^{2(r-p)/(2-p)}N_0^{r/p-1-\delta/2} + C\eta\epsilon^{p\delta/(2-p)} + C\psi\left(\epsilon^{-p/(2-p)}\right)\epsilon^{p\delta/(2-p)} \\
&\quad= o\left(\epsilon^{p\delta/(2-p)}\right).
\end{aligned} \tag{3.8}$$

(c) If $r = (1+\delta/2)p$, then

$$\epsilon^{2(r-p)/(2-p)}\sum_{n=1}^{\infty} n^{r/p-2}R_n = O\left(\epsilon^{p\delta/(2-p)}\log\frac{1}{\epsilon}\right). \tag{3.9}$$

By (2.1) and combining with (3.2), (3.7), (3.8) and (3.9), we obtain (2.3), which completes the proof of part (2).

(3) We make use of the following large deviation estimate in Petrov [10]:

$$\left|P\left(\frac{1}{\sqrt{n}\sigma}S_n < x\right) - \Phi(x)\right| \le \frac{C}{\sqrt{n}(1+|x|)^q}, \quad x > 0.$$

So, $|R_n| \le \frac{C}{\sqrt{n}(1+\epsilon n^{(2-p)/2p})^q}$. Hence we have the following.

(a) If $r < 3p/2$, then

$$\epsilon^{2(r-p)/(2-p)}\sum_{n=1}^{\infty} n^{r/p-2}R_n \le \epsilon^{2(r-p)/(2-p)}\sum_{n=1}^{\infty} n^{r/p-5/2} = O\left(\epsilon^{2(r-p)/(2-p)}\right). \tag{3.10}$$

(b) If $r > 3p/2$, then $\frac{r}{p} - \frac{5}{2} - \frac{2q-pq}{2p} < -1$. By noting that $q > \frac{2r-3p}{2-p}$, we obtain

$$\begin{aligned}
\epsilon^{2(r-p)/(2-p)}\sum_{n=1}^{\infty} n^{r/p-2}R_n &\le C\epsilon^{2(r-p)/(2-p)}\sum_{n=1}^{\infty}\frac{n^{r/p-2}}{(1+\epsilon n^{(2-p)/2p})^q\sqrt{n}} \\
&\le C\epsilon^{2(r-p)/(2-p)}\sum_{n=1}^{[\epsilon^{-2p/(2-p)}]} n^{r/p-2-1/2} \\
&\quad + C\epsilon^{2(r-p)/(2-p)-q}\sum_{n=[\epsilon^{-2p/(2-p)}]+1}^{\infty} n^{r/p-2-1/2-(2-p)q/2p} \\
&= O\left(\epsilon^{p/(2-p)}\right).
\end{aligned} \tag{3.11}$$

(c) If $r = 3p/2$, then

$$\begin{aligned}\epsilon^{2(r-p)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2} R_n &\le C\epsilon^{p/(2-p)} \sum_{n=1}^{[\epsilon^{-2p/(2-p)}]} \frac{1}{n} + C\epsilon^{p/(2-p)-q} \sum_{n=[\epsilon^{-2p/(2-p)}]+1}^{\infty} n^{-1-(2-p)q/2p} \\ &= O\left(\epsilon^{p/(2-p)} \log \frac{1}{\epsilon}\right). \qquad (3.12)\end{aligned}$$

By (2.1), from (3.2), (3.10), (3.11) and (3.12), we have (2.4), which completes the proof of part (3). □

*Proof of Theorem* 2.2 We write

$$\begin{aligned}&\epsilon^2 \lambda_{2,1/(\beta+1)}(\epsilon) - \frac{1}{2\beta+1} \\ &\quad = \left(\frac{2\epsilon^2}{\sqrt{2\pi}} \sum_{n=1}^{\infty} n^{2\beta} \int_{\epsilon n^{\beta+1/2}}^{\infty} e^{-t^2/2}\,dt - \frac{1}{2\beta+1}\right) \\ &\qquad + \epsilon^2 \left(\sum_{n=1}^{[\epsilon^{-4/(2\beta+1)}]} + \sum_{n=[\epsilon^{-4/(2\beta+1)}]+1}^{\infty}\right) n^{2\beta} \left(P(|S_n| \ge \epsilon n^{\beta+1}) - \frac{2}{\sqrt{2\pi}} \int_{\epsilon n^{\beta+1/2}}^{\infty} e^{-t^2/2}\,dt\right) \\ &\quad =: I_1 + I_2 + I_3. \qquad (3.13)\end{aligned}$$

First, according to Lemma 2.1, we have

$$I_1 = \begin{cases} O(\epsilon^2), & -\frac{1}{2} < \beta < \frac{1}{2}, \\ O(\epsilon^{4/(2\beta+1)}), & \beta \ge \frac{1}{2}. \end{cases} \qquad (3.14)$$

For $I_3$, applying Lemma 2.3 of Xie and He [11], and letting $x = 2y = n^{\beta+1}\epsilon$, we obtain

$$P(|S_n| \ge n^{\beta+1}\epsilon) \le nP\left(|X| \ge \frac{1}{2} n^{\beta+1}\epsilon\right) + 8e^2\epsilon^{-4} n^{-4\beta-2}. \qquad (3.15)$$

Observing the following fact

$$\frac{2}{\sqrt{2\pi}} \int_{\epsilon n^{\beta+1/2}}^{\infty} e^{-t^2/2}\,dt = 2\left(1 - \Phi\left(n^{\beta+\frac{1}{2}}\epsilon\right)\right) \le \frac{2\varphi(n^{\beta+1/2}\epsilon)}{n^{\beta+1/2}\epsilon} = O\left(\epsilon^{-5} n^{-5\beta-5/2}\right), \qquad (3.16)$$

from (3.15) and (3.16), we have

$$\begin{aligned}I_3 &\le \epsilon^2 \sum_{n=[\epsilon^{-4/(2\beta+1)}]+1}^{\infty} n^{2\beta} P(|S_n| \ge \epsilon n^{\beta+1}) + \epsilon^2 \sum_{n=[\epsilon^{-4/(2\beta+1)}]+1}^{\infty} \frac{2n^{2\beta}}{\sqrt{2\pi}} \int_{\epsilon n^{\beta+1/2}}^{\infty} e^{-t^2/2}\,dt \\ &\le \epsilon^2 \sum_{n=[\epsilon^{-4/(2\beta+1)}]+1}^{\infty} n^{2\beta+1} P\left(|X| > \frac{\epsilon n^{\beta+1}}{2}\right) + C\epsilon^{-2} \sum_{n=[\epsilon^{-4/(2\beta+1)}]+1}^{\infty} n^{-2\beta-2} \\ &\quad + C\epsilon^{-3} \sum_{n=[\epsilon^{-4/(2\beta+1)}]+1}^{\infty} n^{-3\beta-5/2}\end{aligned}$$

$$\begin{aligned}
&\le \epsilon^2 \sum_{n=[\epsilon^{-4/(2\beta+1)}]+1}^{\infty} n^{2\beta+1} \int_{|x|\ge\frac{1}{2}n^{\beta+1}\epsilon} dF(x) + O(\epsilon^2) + O(\epsilon^3) \\
&\le \epsilon^2 \sum_{n=[\epsilon^{-4/(2\beta+1)}]+1}^{\infty} n^{2\beta+1} \sum_{k=n}^{\infty} \int_{\frac{1}{2}k^{\beta+1}\epsilon\le x<\frac{1}{2}(k+1)^{\beta+1}\epsilon} dF(x) + O(\epsilon^2) \\
&\le \epsilon^2 \sum_{k=[\epsilon^{-4/(2\beta+1)}]+1}^{\infty} \sum_{n=1}^{k} n^{2\beta+1} \int_{\frac{1}{2}k^{\beta+1}\epsilon\le x<\frac{1}{2}(k+1)^{\beta+1}\epsilon} dF(x) + O(\epsilon^2) \\
&\le C\epsilon^2 \sum_{k=[\epsilon^{-4/(2\beta+1)}]+1}^{\infty} k^{2\beta+2} \int_{\frac{1}{2}k^{\beta+1}\epsilon\le x<\frac{1}{2}(k+1)^{\beta+1}\epsilon} dF(x) + O(\epsilon^2) \\
&\le C \sum_{k=[\epsilon^{-4/(2\beta+1)}]+1}^{\infty} \int_{\frac{1}{2}k^{\beta+1}\epsilon\le x<\frac{1}{2}(k+1)^{\beta+1}\epsilon} x^2\, dF(x) + O(\epsilon^2) \\
&\le C \int_{x\ge\frac{1}{2}\epsilon^{-(2\beta+3)/(2\beta+1)}} x^2\, dF(x) + O(\epsilon^2) \\
&= CT(\epsilon^{-(2\beta+3)/(2\beta+1)}) + O(\epsilon^2).
\end{aligned}$$

Using the assumption on $T(t)$ and Lemma 2.2(1), we can obtain

$$I_3 = \begin{cases} O(\epsilon^2), & -\frac{1}{2} < \beta \le \frac{\min(\delta,1)}{4} - \frac{1}{2}, \\ O(\epsilon^{1/(2\beta+1)}), & \beta \ge -\frac{1}{4}, \delta \ge 1, \\ O(\epsilon^{\delta/(2\beta+1)} l(\epsilon^{-1/(2\beta+1)})), & \beta \ge -\frac{1}{2} + \frac{\delta}{4}, 0 < \delta < 1. \end{cases} \tag{3.17}$$

For $I_2$, by Bikelis's inequality (see [12]), we have

$$\begin{aligned}
I_2 &\le \epsilon^2 \sum_{n=1}^{[\epsilon^{-4/(2\beta+1)}]} \frac{Cn^{2\beta}}{(1+\epsilon n^{\beta+1/2})^3\sqrt{n}} \int_0^{(1+\epsilon n^{\beta+1/2})\sqrt{n}} T(v)\, dv \\
&\le \epsilon^2 \sum_{n=1}^{[\epsilon^{-2/(2\beta+1)}]} n^{2\beta-1/2} \int_0^{(1+\epsilon n^{\beta+1/2})\sqrt{n}} T(v)\, dv \\
&\quad + \epsilon^{-1} \sum_{n=[\epsilon^{-2/(2\beta+1)}]+1}^{[\epsilon^{-4/(2\beta+1)}]} n^{-\beta-2} \int_0^{(1+\epsilon n^{\beta+1/2})\sqrt{n}} T(v)\, dv.
\end{aligned}$$

Now, the proof of Theorem 2.2 will be divided into the following cases.

Case 1 of $\delta > 1$.

Noting that $T(t) \le EX^2 = 1$, let $\delta_1$ be a real number such that $1 < \delta_1 < \delta$, by Lemma 2.2(1), $\lim_{t\to\infty} t^{\delta_1-\delta} l(t) = 0$. Therefore, there is a real number $T_0 > 0$ such that $|\frac{l(t)}{t^{\delta-\delta_1}}| < 1$ whenever $t > T_0$. Then

$$\int_0^\infty T(t)\, dt \le \int_0^1 T(t)\, dt + \int_1^\infty T(t)\, dt \le C + \int_{T_0}^\infty \frac{1}{t^{\delta_1}}\, dt < \infty.$$

We have

$$I_2 \le C\epsilon^2 \sum_{n=1}^{[\epsilon^{-2/(2\beta+1)}]} n^{2\beta-1/2} + C\epsilon^{-1} \sum_{n=[\epsilon^{-2/(2\beta+1)}]+1}^{[\epsilon^{-4/(2\beta+1)}]+1} n^{-\beta-2}$$
$$= \begin{cases} O(\epsilon^2), & -\frac{1}{2} < \beta \le -\frac{1}{4}, \\ O(\epsilon^2 \log \frac{1}{\epsilon}), & \beta = -\frac{1}{4}, \\ O(\epsilon^{1/(2\beta+1)}), & \beta > -\frac{1}{4}. \end{cases} \tag{3.18}$$

From (3.13), (3.14), (3.17) and (3.18), we obtain (2.5).

Case 2 of $0 < \delta < 1$.

(a) If $-\frac{1}{2} < \beta < -\frac{1}{2} + \frac{\delta}{4}$, then $\sum_{n=1}^{\infty} n^{2\beta-1/2} < \infty$ and $\int_1^{\infty} t^{4\beta+1-\delta} l(t)\,dt < \infty$. Making use of Lemma 2.2(2)-(3), we have

$$\begin{aligned}
I_2 &\le C\epsilon^2 \sum_{n=1}^{[\epsilon^{-2/(2\beta+1)}]} n^{2\beta-1/2}\left(1 + \int_1^{2\sqrt{n}} T(t)\,dt\right) \\
&\quad + C\epsilon^{-1} \sum_{n=[\epsilon^{-2/(2\beta+1)}]+1}^{[\epsilon^{-4/(2\beta+1)}]} n^{-\beta-2}\left(1 + \int_1^{2\epsilon n^{\beta+1}} T(t)\,dt\right) \\
&\le C\epsilon^2 \sum_{n=1}^{[\epsilon^{-2/(2\beta+1)}]} n^{2\beta-1/2}(\sqrt{n})^{1-\delta} l(\sqrt{n}) + O(\epsilon^2) \\
&\quad + C\epsilon^{1/(2\beta+1)} + C\epsilon^{-1} \sum_{n=[\epsilon^{-2/(2\beta+1)}]+1}^{[\epsilon^{-4/(2\beta+1)}]} n^{-\beta-2}\bigl(2n^{\beta+1}\epsilon\bigr)^{1-\delta} l\bigl(2n^{\beta+1}\epsilon\bigr) \\
&\le C\epsilon^2 \int_1^{\epsilon^{-2/(2\beta+1)}} x^{2\beta-1/2}(\sqrt{x})^{1-\delta} l(\sqrt{x})\,dx \\
&\quad + C\epsilon^{-\delta} \int_{\epsilon^{-2/(2\beta+1)}}^{\infty} x^{-\beta-2} l\bigl(2x^{\beta+1}\epsilon\bigr) x^{(\beta+1)(1-\delta)}\,dx + O\bigl(\epsilon^2\bigr) \\
&\le C\epsilon^2 \int_1^{\epsilon^{-1/(2\beta+1)}} t^{4\beta+1-\delta} l(t)\,dt + C\int_{\epsilon^{-1/(2\beta+1)}}^{\infty} \frac{l(t)}{t^{1+\delta}}\,dt + O\bigl(\epsilon^2\bigr) \\
&\le C\epsilon^2 + C\epsilon^{\delta/(2\beta+1)} l\bigl(\epsilon^{-1/(2\beta+1)}\bigr) \\
&= O\bigl(\epsilon^2\bigr).
\end{aligned} \tag{3.19}$$

(b) If $\beta \ge -\frac{1}{2} + \frac{\delta}{4}$, then we have

$$\begin{aligned}
I_2 &\le C\epsilon^2 \epsilon^{-(4\beta+1)/(2\beta+1)}\left(1 + \int_1^{2\epsilon^{-1/(2\beta+1)}} T(t)\,dt\right) + C\epsilon^{1/(2\beta+1)} + C\epsilon^{\delta/(2\beta+1)} l\bigl(\epsilon^{-1/(2\beta+1)}\bigr) \\
&\le C\epsilon^{1/(2\beta+1)}\bigl(1 + \bigl(2\epsilon^{-(1-\delta)/(2\beta+1)} l\bigl(\epsilon^{-1/(2\beta+1)}\bigr)\bigr)\bigr) + C\epsilon^{\delta/(2\beta+1)} l\bigl(\epsilon^{-1/(2\beta+1)}\bigr) \\
&\le C\epsilon^{\delta/(2\beta+1)} l\bigl(\epsilon^{-1/(2\beta+1)}\bigr).
\end{aligned} \tag{3.20}$$

Therefore

$$I_2 = \begin{cases} O(\epsilon^2), & -\frac{1}{2} < \beta \le -\frac{1}{2} + \frac{\delta}{4}, \\ O(\epsilon^{\delta/(2\beta+1)} l(\epsilon^{1/(2\beta+1)})), & \beta \ge -\frac{1}{2} + \frac{\delta}{4}. \end{cases} \tag{3.21}$$

Combining the estimate with (3.11) and (3.14), by (3.10), this implies that (2.6) follows.

Case 3 of $\delta = 1$.

(a) If $-\frac{1}{2} < \beta < -\frac{1}{4}$, then $\sum_{n=1}^{\infty} n^{2\beta-\frac{1}{2}} < \infty$. We have

$$\begin{aligned} I_2 &\le C\epsilon^2 \left(1 + \int_1^{\epsilon^{-1/(2\beta+1)}} T(t)\,dt\right) \sum_{n=1}^{[\epsilon^{-2/(2\beta+1)}]} n^{2\beta-1/2} \\ &\quad + C\epsilon^{-1}\left(1 + \int_1^{\epsilon^{-\frac{2\beta+3}{2\beta+1}}} T(t)\,dt\right) \sum_{n=[\epsilon^{-2/(2\beta+1)}]+1}^{[\epsilon^{-4/(2\beta+1)}]} n^{-\beta-2} \\ &\le C\epsilon^2 \left(1 + \int_1^{-1/(2\beta+1)} \frac{l(t)}{t}\,dt\right) + \epsilon^{1/(2\beta+1)}\left(1 + \int_1^{\epsilon^{-\frac{2\beta+3}{2\beta+1}}} \frac{l(t)}{t}\,dt\right) \\ &\le C\epsilon^2 \left(1 + \int_1^{\epsilon^{-1/(2\beta+1)}} \frac{l(t)}{t}\,dt\right). \end{aligned} \tag{3.22}$$

(b) If $\beta > -\frac{1}{4}$, then we have

$$\begin{aligned} I_2 &\le C\epsilon^2 \epsilon^{\frac{-2}{2\beta+1}(2\beta+1/2)} \left(1 + \int_1^{\epsilon^{-1/(2\beta+1)}} \frac{l(t)}{t}\,dt\right) + C\epsilon^{1/(2\beta+1)}\left(1 + \int_1^{\epsilon^{-(2\beta+3)/(2\beta+1)}} \frac{l(t)}{t}\,dt\right) \\ &\le C\epsilon^{1/(2\beta+1)}\left(1 + \int_1^{-1/(2\beta+1)} \frac{l(t)}{t}\,dt\right) + \epsilon^{1/(2\beta+1)}\left(1 + \int_1^{\epsilon^{-(2\beta+3)/(2\beta+1)}} \frac{l(t)}{t}\,dt\right) \\ &\le C\epsilon^{1/(2\beta+1)}\left(1 + \int_1^{\epsilon^{-(2\beta+3)/(2\beta+1)}} \frac{l(t)}{t}\,dt\right). \end{aligned} \tag{3.23}$$

(c) If $\beta = -\frac{1}{4}$, then we have

$$\begin{aligned} I_2 &\le C\epsilon^2 \log\frac{1}{\epsilon}\left(1 + \int_1^{\epsilon^{-2}} \frac{l(t)}{t}\,dt\right) + C\epsilon^2\left(1 + \int_1^{\epsilon^{-5}} \frac{l(t)}{t}\,dt\right) \\ &\le C\epsilon^2 \log\frac{1}{\epsilon}\left(1 + \int_1^{\epsilon^{-5}} \frac{l(t)}{t}\,dt\right) \end{aligned}$$

so that

$$I_2 = \begin{cases} O(\epsilon^2(1 + \int_1^{\epsilon^{-1/(2\beta+1)}} \frac{l(t)}{t}\,dt)), & -\frac{1}{2} < \beta \le -\frac{1}{4}, \\ O(\epsilon^2(1 + \int_1^{\epsilon^{-5}} \frac{l(t)}{t}\,dt)\log\frac{1}{\epsilon}), & \beta = -\frac{1}{4}, \\ O(\epsilon^{1/(2\beta+1)}(1 + \int_1^{\epsilon^{-(2\beta+3)/(2\beta+1)}} \frac{l(t)}{t}\,dt)), & \beta > -\frac{1}{4}. \end{cases} \tag{3.24}$$

Combining the estimate with (3.14) and (3.17), by (3.13), this implies that (2.7) follows, and hence Theorem 2.2 is proved. □

*Proof of Theorem* 2.3 Set $T_1(t) = E|X|^{2+\delta}I(|X| > t)$. First, note that

$$\begin{aligned} E|X|^{2+\delta}I\bigl(|X| > t\bigr) &= \int_{|x|>t} |x|^{2+\delta}\,dF(x) \\ &= \int_{|x|>t} x^2\left(\int_t^{|x|} \delta y^{\delta-1}\,dy\right)dF(x) + t^\delta \int_{|x|>t} x^2\,dF(x) \\ &= \int_t^\infty \delta y^{\delta-1}\left(\int_{|x|>y} x^2\,dF(x)\right)dy + t^\delta T(t) \\ &= \delta\int_t^\infty s^{\delta-1}T(s)\,ds + t^\delta T(t). \end{aligned}$$

We have

$$T_1(t) = \delta\int_t^\infty s^{\delta-1}T(s)\,ds + t^\delta T(t).$$

Since $\int_t^\infty s^{\delta-1}T(s)\,ds \ge 0$, $t^\delta T(t) \ge 0$, we have

$$T_1(t) \to 0 \quad \Leftrightarrow \quad t^\delta T(t) \to 0 \quad \text{and} \quad \int_t^\infty s^{\delta-1}T(s)\,ds \to 0 \quad \text{as } t \to \infty.$$

Next, it is easy to get

$$E|X|^{2+\delta} < \infty \quad \Leftrightarrow \quad T_1(t) \to 0 \quad \text{as } t \to \infty.$$

From the above facts, the proof of Theorem 2.3 is complete. □

**Competing interests**
The author declares that they have no competing interests.

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