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Equilibrium of Bayesian fuzzy economies and quasi-variational inequalities with random fuzzy mappings

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Abstract

In this paper, we introduce a Bayesian abstract fuzzy economy model, and we prove the existence of Bayesian fuzzy equilibrium. As applications, we prove the existence of the solutions for two types of random quasi-variational inequalities with random fuzzy mappings, and we also obtain random fixed point theorems.

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1 Introduction

The study of fuzzy games began with the paper written by Kim and Lee in 1998 [1]. This type of games is a generalization of classical abstract economies. For an overview of results concerning this topic, the reader is referred to [2]. However, the existence of random fuzzy equilibrium has not been studied so far. We introduce the new model of Bayesian abstract fuzzy economy, and we explore the existence of the Bayesian fuzzy equilibrium. Our model is characterized by a private information set, an action (strategy) fuzzy mapping, a random fuzzy constraint one and a random fuzzy preference mapping. The Bayesian fuzzy equilibrium concept is an extension of the deterministic equilibrium. We generalize the former deterministic models introduced by Debreu [3], Shafer and Sonnenschein [4], Yannelis and Prabhakar [5] or Patriche [2], and we search for applications.

Since Fichera and Stampacchia introduced the variational inequalities (in 1960s), this domain has been extensively studied. For recent results we refer the reader to [6–11] and the bibliography therein. Noor and Elsanousi [12] introduced the notion of a random variational inequality. The existence of solutions of the random variational inequality and random quasi-variational inequality problems has been proved, for instance, in [13–21].

In this paper, we first define the model of the Bayesian abstract fuzzy economy and we prove a theorem of Bayesian fuzzy equilibrium existence. Then, we apply it in order to prove the existence of solutions for the two types of random quasi-variational inequalities with random fuzzy mappings. We generalize some results obtained by Yuan in [22]. As a consequence, we obtain random fixed point theorems.

The paper is organized as follows. In the next section, some notational and terminological conventions are given. We also present, for the reader's convenience, some results on Bochner integration. In Section 3, the model of differential information abstract fuzzy economy is introduced, and the main result is also stated. Section 4 contains existence results for solutions of random quasi-variational inequalities with random fuzzy mappings.

2 Notation and definition

Throughout this paper, we shall use the following notation.

$\mathbb{R}_{++}$ denotes the set of strictly positive reals. $\operatorname{co} D$ denotes the convex hull of the set D. $\overline{\operatorname{co}}D$ denotes the closed convex hull of the set D. 2^D denotes the set of all nonempty subsets of the set D. If $D \subset Y$, where Y is a topological space, $\operatorname{cl} D$ denotes the closure of D.

For the reader's convenience, we review a few basic definitions and results from continuity and measurability of correspondences and Bochner integrable functions.

Let Z and Y be sets. Let Z, Y be topological spaces and $P : Z \to 2^Y$ be a correspondence. P is said to be *upper semicontinuous* if for each $z \in Z$ and each open set V in Y with $P(z) \subset V$, there exists an open neighborhood U of z in Z such that $P(y) \subset V$ for each $y \in U$. P is said to be *lower semicontinuous* if for each $z \in Z$ and each open set V in Y with $P(z) \cap V \neq \emptyset$, there exists an open neighborhood U of z in Z such that $P(y) \cap V \neq \emptyset$ for each $y \in U$.

Lemma 1 [22] *Let Z and Y be two topological spaces, and let D be an open subset of Z. Suppose $P_1 : Z \to 2^Y$, $P_2 : Z \to 2^Y$ are upper semicontinuous correspondences such that $P_2(z) \subset P_1(z)$ for all $z \in D$. Then the correspondence $P : Z \to 2^Y$ defined by*

$$P(z) = \begin{cases} P_1(z) & \text{if } z \notin D; \\ P_2(z) & \text{if } z \in D \end{cases}$$

is also upper semicontinuous.

Let E be a topological vector space, and let E' be the dual space of E, which consists of all continuous linear functionals on E. The real part of pairing between E' and E is denoted by $\operatorname{Re}\langle w, x\rangle$ for each $w \in E'$ and $x \in E$. The operator $P : E \to 2^{E'}$ is called *monotone* if $\operatorname{Re}\langle u - v, y - x\rangle \geq 0$ for all $u \in P(y)$ and $v \in P(x)$ and $x, y \in E$.

Let now $(\Omega, \mathcal{F}, \mu)$ be a complete, finite measure space, and Y be a topological space. The correspondence $P : \Omega \to 2^Y$ is said to have a *measurable graph* if $G_P \in \mathcal{F} \otimes \beta(Y)$, where $\beta(Y)$ denotes the Borel σ-algebra on Y and $\otimes$ denotes the product σ-algebra. The correspondence $T : \Omega \to 2^Y$ is said to be *lower measurable* if for every open subset V of Y, the set $\{\omega \in \Omega : T(\omega) \cap V \neq \emptyset\}$ is an element of $\mathcal{F}$. This notion of measurability is also called in the literature *weak measurability* or just *measurability*, in comparison with the strong measurability: the correspondence $T : \Omega \to 2^Y$ is said to be *strong measurable* if for every closed subset V of Y, the set $\{\omega \in \Omega : T(\omega) \cap V \neq \emptyset\}$ is an element of $\mathcal{F}$. In the framework we shall deal with (complete finite measure spaces), the two notions coincide (see [23]).

Recall (see Debreu [24], p.359) that if $T : \Omega \to 2^Y$ has a measurable graph, then T is lower measurable. Furthermore, if $T(\cdot)$ is closed-valued and lower measurable, then $T : \Omega \to 2^Y$ has a measurable graph.

Lemma 2 [25] *Let $P_n : \Omega \to 2^Y$, $n = 1, 2 \ldots$ be a sequence of correspondences with measurable graphs. Then the correspondences $\bigcup_n P_n$, $\bigcap_n P_n$ and $Y \setminus P_n$ have measurable graphs.*

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let Y be a Banach space.

It is known (see [25], Theorem 2, p.45) that if $x : \Omega \to Y$ is a μ-measurable function, then x is the Bochner integrable if and only if $\int_\Omega \|x(\omega)\| \, d\mu(\omega) < \infty$.

It is denoted by $L_1(\mu, Y)$, the space of equivalence classes of Y-valued Bochner integrable functions $x : \Omega \to Y$ normed by $\|x\| = \int_\Omega \|x(\omega)\| \, d\mu(\omega)$. Also, it is known (see [24], p.50) that $L_1(\mu, Y)$ is a Banach space.

The correspondence $P : \Omega \to 2^Y$ is said to be *integrably bounded* if there exists a map $h \in L_1(\mu, R)$ such that $\sup\{\|x\| : x \in P(\omega)\} \le h(\omega)$ μ*-a.e.*

We denote by S_P the set of all selections of the correspondence $P : \Omega \to 2^Y$ that belong to the space $L_1(\mu, Y)$, *i.e.*,

$$S_P = \{x \in L_1(\mu, Y) : x(\omega) \in P(\omega) \ \mu\text{-a.e.}\}.$$

We will find the conditions under which S_P is nonempty and weakly compact in $L_1(\mu, Y)$ by applying Aumann measurable selection theorem (see Appendix) and Diestel's theorem (see Appendix).

Zadeh initiated the theory of fuzzy sets [26] as a framework for phenomena, which can not be characterized precisely. We present below several notions concerning the fuzzy sets and the fuzzy mappings.

Definition 1 (Chang [27]) If Y is a topological space, then a function A from Y into $[0;1]$ is called a fuzzy set on Y. The family of all fuzzy sets on Y is denoted by $\mathcal{F}(Y)$.

(2) If X and Y are topological spaces, then a mapping $P : X \to \mathcal{F}(Y)$ is called a *fuzzy mapping*.

(3) If P is a fuzzy mapping, then, for each $x \in X$, $P(x)$ is a fuzzy set in Y and $P(x)(y) \in [0,1]$, $y \in Y$ is called the *degree of membership of* y in $P(x)$.

(4) Let $A \in \mathcal{F}(Y)$, $a \in [0,1]$, then the set $(A)_a = \{y \in Y : A(y) > a\}$ is called a *strong a-cut set* of the fuzzy set A.

The random fuzzy mappings have been defined in order to model random mechanisms generating imprecisely-valued data which can be properly described by using fuzzy sets.

Let Y be a topological space, let $\mathcal{F}(Y)$ be a collection of all fuzzy sets over Y, and let $(\Omega, \mathcal{F})$ be a measurable space.

Definition 2 (See [28]) A fuzzy mapping $P : \Omega \to \mathcal{F}(Y)$ is said to be measurable if for any given $a \in [0,1]$, $(P(\cdot))_a : \Omega \to 2^Y$ is a measurable set-valued mapping.

(2) We say that a fuzzy mapping $P : \Omega \to \mathcal{F}(Y)$ is said *to have a measurable graph* if for any given $a \in [0,1]$, the set-valued mapping $(P(\cdot))_a : \Omega \to 2^Y$ has a measurable graph.

(3) A fuzzy mapping $P : \Omega \times X \to \mathcal{F}(Y)$ is called a *random fuzzy mapping* if, for any given $x \in X$, $P(\cdot, x) : \Omega \to \mathcal{F}(Y)$ is a measurable fuzzy mapping.

3 Bayesian fuzzy equilibrium existence for Bayesian abstract fuzzy economies

3.1 The model of a Bayesian abstract fuzzy economy

The framework of fuzziness became part of the language of applied mathematics. The uncertainties characterize the individual feature of the decisions of the agents involved in different economic activities, and they can be described by using random fuzzy mappings.

In the fuzzy model of the abstract economy, which we will define below, for each agent i, the action choice is modelled by the measurable fuzzy mapping X_i, and the constraints and the preferences are modelled by the random fuzzy mappings A_i and, respectively, P_i. In the state of the world $\omega \in \Omega$, the interpretation of the number $P_i(\omega, \widetilde{x})(y) \in [0,1]$, associated with $(\widetilde{x}_i(\omega), y)$, can be the degree of intensity, with which y is preferred to $\widetilde{x}_i(\omega)$, or the degree of truth that y is preferred to $\widetilde{x}_i(\omega)$. We can also see the value $A_i(\omega, \widetilde{x})(y) \in [0,1]$ associated with $(\widetilde{x}_i(\omega), y)$, as the belief of the player i that in the state ω, if the other players choose $(\widetilde{x}_j(\omega))_{j \neq i}$, he can choose $y \in Y$. The element z_i is the action level in each state of the world, $a_i(\widetilde{x})$ expresses the perceived degree of feasibility of the strategy $\widetilde{x}$, and $p_i(\widetilde{x})$ represents the preference level of the strategy $\widetilde{x}$.

We now define the next model of the Bayesian abstract fuzzy economy, which generalizes the model in [29].

Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite measure space, where Ω denotes the set of states of nature of the world, and the σ-algebra $\mathcal{F}$ denotes the set of events. Let Y denote the strategy or commodity space, where Y is a separable Banach space.

Let I be a countable or uncountable set (the set of agents). For each $i \in I$, let $X_i : \Omega \to \mathcal{F}(Y)$ be a fuzzy mapping, and let $z_i \in (0,1]$.

Let $L_{X_i} = \{x_i \in S_{(X_i(\cdot))_{z_i}} : x_i \text{ is } \mathcal{F}_i\text{-measurable}\}$. Denote by $L_X = \prod_{i \in I} L_{X_i}$ and by $L_{X_{-i}}$ the set $\prod_{j \neq i} L_{X_j}$. An element x_i of L_{X_i} is called a strategy for agent i. The typical element of L_{X_i} is denoted by $\widetilde{x}_i$ and that of $(X_i(\omega))_{z_i}$ by $x_i(\omega)$ (or x_i).

Definition 3 A *general Bayesian abstract fuzzy economy* is a family $G = \{(\Omega, \mathcal{F}, \mu), (X_i, \mathcal{F}_i, A_i, P_i, a_i, b_i, z_i)_{i \in I}\}$, where

(a) $X_i : \Omega \to \mathcal{F}(Y)$ is the *action* (*strategy*) *fuzzy mapping* of agent i;
(b) $\mathcal{F}_i$ is a sub σ-algebra of $\mathcal{F}$, which denotes the *private information of agent i*;
(c) for each $\omega \in \Omega$, $A_i(\omega, \cdot) : L_X \to \mathcal{F}(Y)$ is the *random fuzzy constraint mapping of agent i*;
(d) for each $\omega \in \Omega$, $P_i(\omega, \cdot) : L_X \to \mathcal{F}(Y)$ is the *random fuzzy preference mapping of agent i*;
(e) $a_i : L_X \to (0,1]$ is a *random fuzzy constraint function,* and $p_i : L_X \to (0,1]$ is a *random fuzzy preference function of agent i*;
(f) $z_i \in (0,1]$ is such that for all $(\omega, x) \in \Omega \times L_X$, $(A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})} \subset (X_i(\omega))_{z_i}$ and $(P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})} \subset (X_i(\omega))_{z_i}$.

Definition 4 A *Bayesian fuzzy equilibrium for G* is a strategy profile $\widetilde{x}^* \in L_X$ such that for all $i \in I$,

(i) $\widetilde{x}_i^*(\omega) \in \mathrm{cl}(A_i(\omega, \widetilde{x}^*))_{a_i(\widetilde{x}^*)}$ μ-a.e.;
(ii) $(A_i(\omega, \widetilde{x}^*))_{a_i(\widetilde{x}^*)} \cap (P_i(\omega, \widetilde{x}^*))_{p_i(\widetilde{x}^*)} = \emptyset$ μ-a.e.

Remark 1 If the correspondences from the model above are constant with respect to Ω, we obtain the abstract fuzzy economy model.

3.2 Existence of the Bayesian fuzzy equilibrium

This is our first theorem. The constraint and preference correspondences, derived from the constraint and preference fuzzy mappings, verify the assumptions of measurable graph and weakly open lower sections. Our result is a generalization of Theorem 3 in [29].

Theorem 1 *Let I be a countable or uncountable set. Let the family $G = \{(\Omega, \mathcal{F}, \mu), (X_i, \mathcal{F}_i, A_i, P_i, a_i, b_i, z_i)_{i\in I}\}$ be a general Bayesian abstract economy satisfying* (A.1)-(A.4). *Then there exists a Bayesian fuzzy equilibrium for G.*

For each $i \in I$:

(A.1) (a) *$X_i : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to X_i(\omega)_{z_i} : \Omega \to 2^Y$ is a nonempty convex weakly compact-valued and integrably bounded correspondence*;

(b) *$X_i : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to (X_i(\omega))_{z_i} : \Omega \to 2^Y$ is $\mathcal{F}_i$-lower measurable*;

(A.2) (a) *For each $(\omega, \widetilde{x}) \in \Omega \times L_X$, $(A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})}$ is convex and has a nonempty interior in the relative norm topology of $(X_i(\omega))_{z_i}$*;

(b) *the correspondence $(\omega, \widetilde{x}) \to (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})} : \Omega \times L_X \to 2^Y$ has a measurable graph, i.e., $\{(\omega, \widetilde{x}, y) \in \Omega \times L_X \times Y : y \in (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})}\} \in \mathcal{F} \otimes ß_w(L_X) \otimes ß(Y)$, where $ß_w(L_X)$ is the Borel σ-algebra for the weak topology on L_X and $ß(Y)$ is the Borel σ-algebra for the norm topology on Y*;

(c) *the correspondence $(\omega, \widetilde{x}) \to (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})}$ has weakly open lower sections, i.e., for each $\omega \in \Omega$ and for each $y \in Y$, the set $((A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})})^{-1}(\omega, y) = \{\widetilde{x} \in L_X : y \in (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})}\}$ is weakly open in L_X*;

(d) *For each $\omega \in \Omega$, $\widetilde{x} \to \mathrm{cl}(A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})} : L_X \to 2^Y$ is upper semicontinuous in the sense that the set $\{\widetilde{x} \in L_X : \mathrm{cl}(A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})} \subset V\}$ is weakly open in L_X for every norm open subset V of Y*;

(A.3) (a) *the correspondence $(\omega, \widetilde{x}) \to (P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})} : \Omega \times L_X \to 2^Y$ has open convex values such that $(P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})} \subset (X(\omega))_{z_i}$ for each $(\omega, \widetilde{x}) \in \Omega \times L_X$*;

(b) *the correspondence $(\omega, \widetilde{x}) \to (P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})} : \Omega \times L_X \to 2^Y$ has a measurable graph*;

(c) *the correspondence $(\omega, \widetilde{x}) \to (P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})} : \Omega \times L_X \to 2^Y$ has weakly open lower sections, i.e., for each $\omega \in \Omega$ and for each $y \in Y$, the set $((P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})})^{-1}(\omega, y) = \{\widetilde{x} \in L_X : y \in (P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})}\}$ is weakly open in L_X*;

(A.4) (a) *For each $\widetilde{x}_i \in L_{X_i}$, for each $\omega \in \Omega$, $\widetilde{x}_i(\omega) \notin (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})} \cap (P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})}$.*

Proof For each $i \in I$, let us define $\Phi_i : \Omega \times L_X \to 2^Y$ by $\Phi_i(\omega, \widetilde{x}) = (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})} \cap (P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})}$. We prove first that L_X is a nonempty convex weakly compact subset in $L_1(\mu, Y)$.

Since $(\Omega, \mathcal{F}, \mu)$ is a complete finite measure space, Y is a separable Banach space, and $X_i : \Omega \to 2^Y$ has a measurable graph, by Aumann's selection theorem (see Appendix), it follows that there exists a $\mathcal{F}_i$-measurable function $f_i : \Omega \to Y$ such that $f_i(\omega) \in X_i(\omega)$ μ-a.e. Since X_i is integrably bounded, we have that $f_i \in L_1(\mu, Y)$, hence L_{X_i} is nonempty and $L_X = \prod_{i\in I} L_{X_i}$ is nonempty. Obviously, L_{X_i} is convex and L_X is also convex. Since $X_i : \Omega \to 2^Y$ is integrably bounded and it has convex weakly compact values, by Diestel's theorem (see Appendix), it follows that L_{X_i} is a weakly compact subset of $L_1(\mu, Y)$. More over, L_X is weakly compact. $L_1(\mu, Y)$ equipped with the weak topology is a locally convex topological vector space.

The correspondence Φ_i is convex valued, by Lemma 2, it has a measurable graph, and for each $\omega \in \Omega$, $\Phi_i(\omega, \cdot)$ has weakly open lower sections. Let $U_i = \{(\omega, \widetilde{x}) \in \Omega \times L_X : \Phi_i(\omega, \widetilde{x}) \neq \emptyset\}$. For each $\widetilde{x} \in L_X$, let $U_i^{\widetilde{x}} = \{\omega \in \Omega : \Phi_i(\omega, \widetilde{x}) \neq \emptyset\}$ and for each $\omega \in \Omega$, let $U_i^{\omega} = \{\widetilde{x} \in L_X : \Phi_i(\omega, \widetilde{x}) \neq \emptyset\}$. The values of Φ_{i/U_i} have nonempty interiors in the relative norm topology of $X_i(\omega)$. By the Caratheodory-type selection theorem (see Appendix), there exists a function $f_i : U_i \to Y$ such that $f_i(\omega, \widetilde{x}) \in \Phi_i(\omega, \widetilde{x})$ for all $(\omega, \widetilde{x}) \in U_i$, for each

$\widetilde{x} \in L_X, f_i(\cdot,\widetilde{x})$ is measurable on $U_i^{\widetilde{x}}$, for each $\omega \in \Omega, f_i(\omega,\cdot)$ is continuous on U_i^{ω} and, moreover $f_i(\cdot,\cdot)$ is jointly measurable.

Define $G_i : \Omega \times L_X \to 2^Y$ by $G_i(\omega,\widetilde{x}) = \begin{cases} \{f_i(\omega,\widetilde{x})\} & \text{if } (\omega,\widetilde{x}) \in U_i; \\ \mathrm{cl}(A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})} & \text{if } (\omega,\widetilde{x}) \notin U_i. \end{cases}$

Define $G_i' : L_X \to 2^{L_{X_i}}$, by $G_i'(\widetilde{x}) = \{y_i \in L_{X_i} : y_i(\omega) \in G_i(\omega,\widetilde{x})\ \mu\text{-a.e.}\}$ and $G' : L_X \to 2^{L_X}$ by $G'(\widetilde{x}) := \prod_{i\in I} G_i'(\widetilde{x})$ for each $\widetilde{x} \in L_X$. We shall prove that G' is an upper semicontinuous correspondence with respect to the weakly topology of L_X and has nonempty convex closed values. By applying Fan-Glicksberg's fixed-point Theorem [30] to G', we obtain a fixed point, which is the equilibrium point for the abstract economy.

It follows by Theorem III.40 in [31] and the projection theorem that for each $\widetilde{x} \in L_X$, the correspondence $\widetilde{x} \to \mathrm{cl}(A_i(\cdot,\widetilde{x}))_{a_i(\widetilde{x})} : \Omega \to 2^Y$ has a measurable graph. For each $\widetilde{x} \in L_X$, the correspondence $G_i(\cdot,\widetilde{x})$ has a measurable graph. Since $\Phi_i(\omega,\cdot)$ has weakly open lower sections for each $\omega \in \Omega$, it follows that U_i^{ω} is weakly open in L_X. By Lemma 1, for each $\omega \in \Omega$, $G_i(\omega,\cdot) : L_X \to 2^Y$ is upper semi-continuous in the sense that the set $\{\widetilde{x} \in L_X : G_i(\omega,\widetilde{x})\} \subset V$ is weakly open in L_X for every norm open subset V of Y. Moreover, G_i is convex and nonempty-valued.

G_i is nonempty-valued, and for each $\widetilde{x} \in L_X$, $G_i(\cdot,\widetilde{x})$ has a measurable graph. Hence, according to the Aumann measurable selection theorem for each fixed $\widetilde{x} \in L_X$, there exists an $\mathcal{F}_i$-measurable function $y_i : \Omega \to Y$ such that $y_i(\omega) \in G_i(\omega,\widetilde{x})$ μ-a.e. Since for each $(\omega,\widetilde{x}) \in \Omega \times L_X$, $G_i(\omega,\widetilde{x})$ is contained in the values of the integrably bounded correspondence $X_i(\cdot)$, then $y_i \in L_{X_i}$, and we conclude that $y_i \in G_i'(\widetilde{x})$ for each $\widetilde{x} \in L_X$. Thus, G_i' is nonempty-valued.

Since for each $\widetilde{x} \in L_X$, $G_i(\cdot,\widetilde{x})$ has a measurable graph and for each $\omega \in \Omega$, $G_i(\omega,\cdot) : L_X \to 2^Y$ is upper semicontinuous and $G_i(\omega,\widetilde{x}) \subset (X_i(\omega))_{z_i}$ for each $(\omega,\widetilde{x}) \in \Omega \times L_X$, by u.s.c. lifting theorem (see Appendix), it follows that G_i' is weakly upper semicontinuous. G_i' is convex-valued since G_i is such.

G' is a weakly upper semicontinuous correspondence, and it also has nonempty convex closed values.

The set L_X is weakly compact and convex, and then, by Fan-Glicksberg's fixed-point theorem in [30], there exists $\widetilde{x}^* \in L_X$ such that $\widetilde{x}^* \in G'(\widetilde{x}^*)$, *i.e.*, for each $i \in I$, $\widetilde{x}_i^* \in G_i'(\widetilde{x}^*)$.

Then, $\widetilde{x}_i^* \in L_{X_i}$ and $\widetilde{x}_i^*(\omega) \in G_i(\omega,\widetilde{x}^*)$ μ-a.e. Since $\widetilde{x}_i^*(\omega) \notin (A_i(\omega,\widetilde{x}^*))_{a_i(\widetilde{x}^*)} \cap (P_i(\omega,\widetilde{x}^*))_{p_i(\widetilde{x}^*)}$ μ-a.e., it follows that $(\omega,\widetilde{x}^*) \notin U_i$ for each $i \in I$ and $\widetilde{x}_i^* \in \mathrm{cl}(A_i(\omega,\widetilde{x}^*))_{a_i(\widetilde{x}^*)}$ μ-a.e. We also have that $(A_i(\omega,\widetilde{x}^*))_{a_i(\widetilde{x}^*)} \cap (P_i(\omega,\widetilde{x}^*))_{p_i(\widetilde{x}^*)} = \emptyset$. □

Example 1 Let $(\Omega,\mathcal{F},\mu)$ be the measure space, where $\Omega = [0,1]$, $\mathcal{F} = \beta([0,1])$ is the σ-algebra of the Borel measurable subsets in $[0,1]$ and μ is the Lebesgue measure.

Let $Y = \mathbb{R}$ and $I = \{1,2,\dots,n\}$.

For each $i \in I$, let us define the following.

$$\mathcal{F}_i = \mathcal{F}.$$

The random fuzzy constraint function $z_i : [0,1] \to (0,1]$ is defined by

$$z_i(\omega) = \frac{1}{i+2} \quad \text{if } \omega \in [0,1].$$

The random fuzzy mapping $X_i(\cdot):[0,1]\to\mathcal{F}(\mathbb{R})$ is defined by

$$X_i(\omega)(y)=\begin{cases}0 & \text{if } \omega\in[0,1] \text{ and } y\in(-\infty,0)\cup(1,\infty);\\ \frac{2}{i+5}y+\frac{2}{i+2} & \text{if } \omega\in[0,1] \text{ and } y\in[0,1].\end{cases}$$

Then, the correspondence $X_i:[0,1]\to 2^{\mathbb{R}}$ is defined by

$$X_i(\omega)=\left\{y\in\mathbb{R}:\frac{2}{i+5}y+\frac{2}{i+2}>\frac{1}{i+2} \text{ and } y\in[0,1]\right\}=[0,1]\quad\text{for each } \omega\in[0,1].$$

It is a nonempty convex compact valued and integrably bounded correspondence. It is also $\mathcal{F}_i$-lower measurable.

Let $L_{X_i}=\{x_i\in S_{(X_i(\cdot))_{z_i}}:x_i \text{ is } \mathcal{F}_i\text{-measurable}\}$ and $L_X=\prod_{i\in I}L_{X_i}$.

The random fuzzy constraint function $a_i:L_X\to(0,1]$ is defined by

$$a_i(\widetilde{x})=\frac{1}{2}\quad\text{for each } \widetilde{x}\in L_X.$$

For each $\omega\in[0,1]$, the random fuzzy constraint mapping of agent i, $A_i(\omega,\cdot):L_X\to\mathcal{F}(\mathbb{R})$ is defined by

$$A_i(\omega,\widetilde{x})(y)=\begin{cases}\frac{5}{10y+2} & \text{if } (\omega,\widetilde{x})\in[0,1]\times L_X \text{ and } y\in(0,1];\\ 0 & \text{if } (\omega,\widetilde{x})\in[0,1]\times L_X \text{ and } y\in(-\infty,0]\cup(1,\infty).\end{cases}$$

Then, the correspondence $(\omega,\widetilde{x})\to(A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})}:[0,1]\times L_X\to 2^{[0,1]}$ is defined by

$$\begin{aligned}\big(A_i(\omega,\widetilde{x})\big)_{a_i(\widetilde{x})}&=\{y\in[0,1]:A_i(\omega,\widetilde{x})(y)>a_i(\widetilde{x})\}\\&=\left\{y\in(0,1]:\frac{5}{10y+1}>\frac{1}{2}\right\}=\left\{y\in(0,1]:y<\frac{9}{10}\right\}=\left(0,\frac{9}{10}\right).\end{aligned}$$

For each $\omega\in[0,1]$, it has weakly open lower sections in L_X, and it has a measurable graph.

For each $(\omega,\widetilde{x})\in[0,1]\times L_X$, $(A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})}$ is convex and with nonempty interior in $[0,1]$.

For each $\omega\in[0,1]$, the correspondence $\widetilde{x}\to\operatorname{cl}(A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})}:L_X\to 2^{[0,1]}$, defined by $\operatorname{cl}(A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})}=[0,\frac{9}{10}]$ for each $\widetilde{x}\in L_X$ is upper semicontinuous and nonempty-valued.

The random fuzzy preference mapping $p_i:L_X\to(0,1]$ is defined by

$$p_i(\widetilde{x})=\frac{1}{5}\quad\text{for each } \widetilde{x}\in L_X.$$

Let us define $D_i=\prod_{j\neq i}L_{X_j}\times\{\widetilde{x}_i:[0,1]\to[0,1],\widetilde{x}_i(\omega)=k_{\widetilde{x}_i}\omega^i,\omega\in[0,1],k_{\widetilde{x}_i}\in[0,1]\}$. D_i is weakly closed in L_X.

For each $\omega\in[0,1]$, the random fuzzy preference mapping of agent i, $P_i(\omega,\cdot):L_X\to\mathcal{F}(\mathbb{R})$ is defined by

$$P_i(\omega,\widetilde{x})(y)=\begin{cases}\frac{5y+2}{5(\widetilde{x}_i(\omega)+6)} & \text{if } (\omega,\widetilde{x})\in[0,1]\times D_i \text{ and } y\in(0,1];\\ 0 & \text{otherwise.}\end{cases}$$

Then, for each $\omega \in [0,1]$, the correspondence $\widetilde{x} \to (P_i(\omega,\widetilde{x}))_{p_i(\widetilde{x})} : L_X \to 2^{[0,1]}$ is defined by

$$\big(P_i(\omega,\widetilde{x})\big)_{p_i(\widetilde{x})} = \begin{cases} \{y \in [0,1) : P_i(\omega,\widetilde{x})(y) > p_i(\widetilde{x})\} & \text{if } \widetilde{x} \in D_i; \\ \emptyset & \text{if } \widetilde{x} \notin D_i \end{cases}$$

$$= \begin{cases} (\frac{\widetilde{x}_i(\omega)+4}{5}, 1) & \text{if } \widetilde{x} \in D_i; \\ \emptyset & \text{if } \widetilde{x} \notin D_i. \end{cases}$$

For each $\omega \in \Omega$ and for each $y \in Y$, the set $((P_i(\omega,\widetilde{x}))_{p_i(\widetilde{x})})^{-1}(\omega,y) = \{\widetilde{x} \in D_i : 0 \leq \widetilde{x}_i(\omega) < 5y - 4\}$ is weakly open in D_i, then it is weakly open in L_X. Therefore, the correspondence $(\omega,\widetilde{x}) \to (P_i(\omega,\widetilde{x}))_{p_i(\widetilde{x})}$ has weakly open lower sections. It also has open convex values and a measurable graph.

For each $i \in I$, $\widetilde{x}_i(\omega) \notin (P_i(\omega,\widetilde{x}))_{p_i(\widetilde{x})}$, for each $\omega \in [0,1]$ and $\widetilde{x} \in L_X$.

All the assumptions of Theorem 1 are fulfilled, then an equilibrium exists.

For example, $\widetilde{x^*} \in L_X$ such that for each $i \in I$, $\widetilde{x}^*_i(\omega) = \frac{3}{4}\omega^i$, $\omega \in [0,1]$ is an equilibrium for the abstract fuzzy economy, that is, for each $i \in I$ and μ-a.e.:

$$\widetilde{x^*_i}(\omega) \in \mathrm{cl}\big(A_i(\omega,\widetilde{x})\big)_{a_i(\widetilde{x})} \quad \text{and} \quad \big(A_i(\omega,\widetilde{x})\big)_{a_i(\widetilde{x})} \cap \big(P_i(\omega,\widetilde{x})\big)_{p_i(\widetilde{x})} = \emptyset.$$

4 Random quasi-variational inequalities

In this section, we are establishing new random quasi-variational inequalities with random fuzzy mappings and random fixed point theorems. The proofs rely on the theorem of Bayesian fuzzy equilibrium existence for the Bayesian abstract fuzzy economy.

This is our first theorem.

Theorem 2 *Let I be a countable or uncountable set. Let $(\Omega,\mathcal{F},\mu)$ be a complete finite separable measure space, and let Y be a separable Banach space. Suppose that the following conditions are satisfied.*

For each $i \in I$:

(A.1) (a) *$X_i : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to (X_i(\omega))_{z_i} : \Omega \to 2^Y$ is a nonempty convex weakly compact-valued and integrably bounded correspondence*;

(b) *$X_i : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to (X_i(\omega))_{z_i} : \Omega \to 2^Y$ is $\mathcal{F}_i$-lower measurable*;

(A.2) (a) *For each $(\omega,\widetilde{x}) \in \Omega \times L_X$, $(A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})}$ is convex and has a nonempty interior in the relative norm topology of $(X_i(\omega))_{z_i}$*;

(b) *the correspondence $(\omega,\widetilde{x}) \to (A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})} : \Omega \times L_X \to 2^Y$ has a measurable graph, i.e., $\{(\omega,\widetilde{x},y) \in \Omega \times L_X \times Y : y \in (A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})}\} \in \mathcal{F} \otimes \beta_w(L_X) \otimes \beta(Y)$, where $\beta_w(L_X)$ is the Borel σ-algebra for the weak topology on L_X and $\beta(Y)$ is the Borel σ-algebra for the norm topology on Y*;

(c) *the correspondence $(\omega,\widetilde{x}) \to (A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})}$ has weakly open lower sections, i.e., for each $\omega \in \Omega$ and for each $y \in Y$, the set $((A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})})^{-1}(\omega,y) = \{\widetilde{x} \in L_X : y \in (A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})}\}$ is weakly open in L_X*;

(d) *For each $\omega \in \Omega$, $\widetilde{x} \to \mathrm{cl}(A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})} : L_X \to 2^Y$ is upper semicontinuous in the sense that the set $\{\widetilde{x} \in L_X : \mathrm{cl}(A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})} \subset V\}$ is weakly open in L_X for every norm open subset V of Y*;

(A.3) *$\psi_i : \Omega \times L_X \times Y \to \mathbb{R} \cup \{-\infty,+\infty\}$ is such that*:

(a) $\widetilde{x} \to \psi_i(\omega, \widetilde{x}, y)$ *is lower semicontinuous on* L_X *for each fixed* $(\omega, y) \in \Omega \times Y$;
(b) $\widetilde{x}_i(\omega) \notin \{y \in Y : \psi_i(\omega, \widetilde{x}, y) > 0\}$ *for each fixed* $(\omega, \widetilde{x}) \in \Omega \times L_X$;
(c) *for each* $(\omega, \widetilde{x}) \in \Omega \times L_X$, $\psi_i(\omega, \widetilde{x}, \cdot)$ *is quasiconcave*;
(d) *for each* $\omega \in \Omega$, $\{\widetilde{x} \in L_X : \alpha_i(\omega, \widetilde{x}) > 0\}$ *is weakly open in* L_X, *where* $\alpha_i : \Omega \times L_X \to R$ *is defined by* $\alpha_i(\omega, \widetilde{x}) = \sup_{y \in (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})}} \psi_i(\omega, \widetilde{x}, y)$ *for each* $(\omega, \widetilde{x}) \in \Omega \times L_X$;
(e) $\{(\omega, \widetilde{x}) : \alpha_i(\omega, \widetilde{x}) > 0\} \in \mathcal{F}_i \otimes B(L_X)$.

Then, there exists $\widetilde{x}^* \in L_X$ *such that for every* $i \in I$ *and* μ*-a.e.*:
(i) $\widetilde{x}_i{}^*(\omega) \in \mathrm{cl}(A_i(\omega, \widetilde{x}^*))_{a_i(\widetilde{x}^*)}$;
(ii) $\sup_{y \in (A_i(\omega, \widetilde{x}^*))_{a_i(\widetilde{x}^*)}} \psi_i(\omega, \widetilde{x}^*, y) \leq 0$.

Proof For every $i \in I$, let $P_i : \Omega \times S_X^1 \to \mathcal{F}(Y)$, and let $p_i : L_X \to (0,1]$ such that $(P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})} = \{y \in Y : \psi_i(\omega, \widetilde{x}, y) > 0\}$ for each $(\omega, \widetilde{x}) \in \Omega \times L_X$.

We shall show that the abstract economy $G = \{(\Omega, \mathcal{F}, \mu), (X_i, \mathcal{F}_i, A_i, P_i a_i, p_i, z_z)_{i \in I}\}$ satisfies all hypotheses of Theorem 1.

Suppose $\omega \in \Omega$.

According to (A.3)(a), we have that $\widetilde{x} \to (P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})} : \Omega \to 2^Y$ has open lower sections, nonempty compact values and according to (A.3)(b), $\widetilde{x}_i(\omega) \notin (P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})}$ for each $\widetilde{x} \in L_X$. Assumption (A.3)(c) implies that $\widetilde{x} \to (P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})} : \Omega \to 2^Y$ has convex values.

By the definition of α_i, we note that $\{\widetilde{x} \in L_X : (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})} \cap (P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})} \neq \emptyset\} = \{\widetilde{x} \in L_X : \alpha_i(\omega, \widetilde{x}) > 0\}$ so that $\{\widetilde{x} \in L_X : (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})} \cap (P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})} \neq \emptyset\}$ is weakly open in L_X by (A.3)(d).

According to (A.2)(b) and (A.3)(e), it follows that the correspondences $(\omega, \widetilde{x}) \to (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})} : \Omega \times L_X \to 2^Y$ and $(\omega, \widetilde{x}) \to (P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})} : \Omega \times L_X \to 2^Y$ have measurable graphs.

Thus, the Bayesian abstract fuzzy economy $G = \{(\Omega, \mathcal{F}, \mu), (X_i, \mathcal{F}_i, A_i, P_i, a_i, b_i, z_i)_{i \in I}\}$ satisfies all hypotheses of Theorem 1. Therefore, there exists $\widetilde{x}^* \in L_X$ such that for every $i \in I$:

$$\widetilde{x}_i^*(\omega) \in \mathrm{cl}\big(A_i\big(\omega, \widetilde{x}^*\big)\big)_{a_i(\widetilde{x}^*)} \ \mu\text{-a.e.} \quad \text{and} \quad \big(A_i\big(\omega, \widetilde{x}^*\big)\big)_{a_i(\widetilde{x}^*)} \cap \big(P_i\big(\omega, \widetilde{x}^*\big)\big)_{p_i(\widetilde{x}^*)} = \phi \ \mu\text{-a.e.};$$

that is, there exists $\widetilde{x}^* \in L_X$ such that for every $i \in I$ and μ-a.e.:
(i) $\widetilde{x}_i{}^*(\omega) \in \mathrm{cl}(A_i(\omega, \widetilde{x}^*))_{a_i(\widetilde{x}^*)}$;
(ii) $\sup_{y \in (A_i(\omega, \widetilde{x}^*))_{a_i(\widetilde{x}^*)}} \psi_i(\omega, \widetilde{x}^*, y) \leq 0$. □

Example 2 Let $\Omega = [0,1]$, $\mathcal{F} = \beta([0,1], \mu)$, $Y = \mathbb{R}$, $I = \{1, 2, \ldots, n\}$, and for each $i \in I$, let $\mathcal{F}_i$, X_i, a_i and D_i be defined as in Example 1.

Let us define $\psi_i : [0,1] \times L_X \times \mathbb{R} \to \mathbb{R}$ as follows: if $\widetilde{x}_i(\omega) \in [0,1)$, then,

$$\psi_i(\omega, \widetilde{x}, y) = \begin{cases} 1 & \text{if } y \in (\frac{\widetilde{x}_i(\omega)+4}{5}, 1) \text{ and } (\omega, \widetilde{x}) \in [0,1] \times D_i; \\ 0 & \text{otherwise,} \end{cases}$$

and if $\widetilde{x}_i(\omega) = 1$, $\psi_i(\omega, \widetilde{x}, y) = 0$ for each $(\omega, \widetilde{x}, y) \in [0,1] \times L_X \times \mathbb{R}$.

For each $i \in I$, let $P_i : \Omega \times L_X \to \mathcal{F}(Y)$, and let $p_i : L_X \to (0,1]$ as in Example 1, and then, $(P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})} = \{y \in Y : \psi_i(\omega, \widetilde{x}, y) > 0\} = \begin{cases} (\frac{\widetilde{x}_i(\omega)+4}{5}, 1) & \text{if } \widetilde{x} \in D_i; \\ \emptyset & \text{if } \widetilde{x} \notin D_i. \end{cases}$

From the Example 1, we have that for every $i \in I$ and for each $\omega \in [0,1]$, the correspondence $\widetilde{x} \to (P_i(\omega, \widetilde{x}))_{p_i(\widetilde{x})} : L_X \to 2^{[0,1]}$ has weakly open lower sections, open convex values and $\widetilde{x}_i(\omega) \notin (P_i(\omega,\widetilde{x}))_{p_i(\widetilde{x})}$.

$$\begin{aligned}\alpha_i(\omega,\widetilde{x}) &= \sup_{y\in(A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})}} \psi_i(\omega,\widetilde{x},y)\\ &= \sup_{y\in(0,\frac{9}{10})} \psi_i(\omega,\widetilde{x},y) = \begin{cases}1 & \text{if } \widetilde{x}_i(\omega) \in [0,\frac{1}{2});\\ 0 & \text{if } \widetilde{x}_i(\omega) \in [\frac{1}{2},1].\end{cases}\end{aligned}$$

By the definition of α_i, we note that for each $\omega \in \Omega$, $N_i(\omega) = \{\widetilde{x} \in L_X : \alpha_i(\omega,\widetilde{x}) > 0\} = \{\widetilde{x} \in D_i : \widetilde{x}_i(\omega) \in [0,\frac{1}{2})\}$ is weakly open in L_X and $N_i \in \mathcal{F}_i \otimes B(L_X)$.

In Example 1, we proved that the correspondence $(\omega,\widetilde{x}) \to (A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})} : \Omega \times L_X \to 2^{[0,1]}$ has a measurable graph.

Therefore, there exists $\widetilde{x}^* \in L_X$ such that, for each $i \in I$ and μ-a.e.:

(i) $\widetilde{x}_i^*(\omega) \in \mathrm{cl}(A_i(\omega,\widetilde{x}^*))_{a_i(\widetilde{x}^*)}$;

(ii) $\sup_{y\in(A_i(\omega,\widetilde{x}^*))_{a_i(\widetilde{x}^*)}} \psi_i(\omega,\widetilde{x}^*,y) \le 0$.

For instance, $\widetilde{x}^*$ is a solution for the variational inequality, where $\widetilde{x}^*$ is defined by $\widetilde{x}_i^*(\omega) = \frac{3}{4}\omega^i$ for each $i \in \{1,2,\ldots,n\}$ and $\omega \in \Omega$.

If |I| = 1, we obtain the following corollary of Theorem 2.

Corollary 1 *Let $(\Omega,\mathcal{F},\mu)$ be a complete finite separable measure space, and let Y be a separable Banach space. Suppose that the following conditions are satisfied*:

(A.1) (a) *$X : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to (X(\omega))_z : \Omega \to 2^Y$ is a nonempty convex weakly compact-valued and integrably bounded correspondence*;

(b) *$X : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to (X(\omega))_z : \Omega \to 2^Y$ is $\mathcal{F}$-lower measurable*;

(A.2) (a) *For each $(\omega,\widetilde{x}) \in \Omega \times L_X$, $(A(\omega,\widetilde{x}))_{a(\widetilde{x})}$ is convex and has a nonempty interior in the relative norm topology of $(X(\omega))_z$*;

(b) *the correspondence $(\omega,\widetilde{x}) \to (A(\omega,\widetilde{x}))_{a_i(\widetilde{x})} : \Omega \times L_X \to 2^Y$ has a measurable graph, i.e., $\{(\omega,\widetilde{x},y) \in \Omega \times L_X \times Y : y \in (A(\omega,\widetilde{x}))_{a(\widetilde{x})}\} \in \mathcal{F} \otimes ß_w(L_X) \otimes ß(Y)$ where $ß_w(L_X)$ is the Borel σ-algebra for the weak topology on L_X and $ß(Y)$ is the Borel σ-algebra for the norm topology on Y*;

(c) *the correspondence $(\omega,\widetilde{x}) \to (A(\omega,\widetilde{x}))_{a(\widetilde{x})}$ has weakly open lower sections, i.e., for each $\omega \in \Omega$ and for each $y \in Y$, the set $((A(\omega,\widetilde{x}))_{a(\widetilde{x})})^{-1}(\omega,y) = \{\widetilde{x} \in L_X : y \in (A(\omega,\widetilde{x}))_{a(\widetilde{x})}\}$ is weakly open in L_X*;

(d) *For each $\omega \in \Omega$, $\widetilde{x} \to \mathrm{cl}(A(\omega,\widetilde{x}))_{a(\widetilde{x})} : L_X \to 2^Y$ is upper semicontinuous in the sense that the set $\{\widetilde{x} \in L_X : \mathrm{cl}(A(\omega,\widetilde{x}))_{a(\widetilde{x})} \subset V\}$ is weakly open in L_X for every norm open subset V of Y*;

(A.3) *$\psi : \Omega \times L_X \times Y \to R \cup \{-\infty,+\infty\}$ is such that*:

(a) *$\widetilde{x} \to \psi(\omega,\widetilde{x},y)$ is lower semicontinuous on L_X for each fixed $(\omega,y) \in \Omega \times Y$*;

(b) *$\widetilde{x}(\omega) \notin \{y \in Y : \psi(\omega,\widetilde{x},y) > 0\}$ for each fixed $(\omega,\widetilde{x}) \in \Omega \times L_X$*;

(c) *for each $(\omega,\widetilde{x}) \in \Omega \times L_X$, $\psi(\omega,\widetilde{x},\cdot)$ is quasiconcave*;

(d) *for each $\omega \in \Omega$, $\{\widetilde{x} \in L_X : \alpha(\omega,\widetilde{x}) > 0\}$ is weakly open in L_X, where $\alpha : \Omega \times L_X \to R$ is defined by $\alpha(\omega,\widetilde{x}) = \sup_{y\in(A(\omega,\widetilde{x}))_{a(\widetilde{x})}} \psi(\omega,\widetilde{x},y)$ for each $(\omega,\widetilde{x}) \in \Omega \times L_X$*;

(e) *$\{(\omega,\widetilde{x}) : \alpha(\omega,\widetilde{x}) > 0\} \in \mathcal{F} \otimes B(L_X)$*;

Then, there exists $\widetilde{x}^* \in L_X$ such that μ-a.e.:

(i) $\widetilde{x}^*(\omega) \in \mathrm{cl}(A(\omega,\widetilde{x}^*))_{a(\widetilde{x}^*)}$;

(ii) $\sup_{y\in(A(\omega,\widetilde{x}^*))_{a(\widetilde{x}^*)}} \psi(\omega,\widetilde{x}^*,y) \le 0$.

As a consequence of Theorem 2, we prove the following Tan and Yuan-type [22] random quasi-variational inequality with random fuzzy mappings.

Theorem 3 *Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite separable measure space, and let Y be a separable Banach space. Suppose that the following conditions are satisfied*:

For each $i \in I$:

(A.1) (a) *$X_i : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to (X_i(\omega))_{z_i} : \Omega \to 2^Y$ is a nonempty convex weakly compact-valued and integrably bounded correspondence*;

(b) *$X_i : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to (X_i(\omega))_{z_i} : \Omega \to 2^Y$ is $\mathcal{F}_i$-lower measurable*;

A.2) (a) *For each $(\omega, \widetilde{x}) \in \Omega \times L_X$, $(A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})}$ is convex and has a nonempty interior in the relative norm topology of $(X_i(\omega))_{z_i}$*;

(b) *the correspondence $(\omega, \widetilde{x}) \to (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})} : \Omega \times L_X \to 2^Y$ has a measurable graph, i.e., $\{(\omega, \widetilde{x}, y) \in \Omega \times L_X \times Y : y \in (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})}\} \in \mathcal{F} \otimes \beta_w(L_X) \otimes \beta(Y)$ where $\beta_w(L_X)$ is the Borel σ-algebra for the weak topology on $L_X$and $\beta(Y)$ is the Borel σ-algebra for the norm topology on Y*;

(c) *the correspondence $(\omega, \widetilde{x}) \to (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})}$ has weakly open lower sections, i.e., for each $\omega \in \Omega$ and for each $y \in Y$, the set $((A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})})^{-1}(\omega, y) = \{\widetilde{x} \in L_X : y \in (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})}\}$ is weakly open in L_X*;

(d) *For each $\omega \in \Omega$, $\widetilde{x} \to \mathrm{cl}(A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})} : L_X \to 2^Y$ is upper semicontinuous in the sense that the set $\{\widetilde{x} \in L_X : \mathrm{cl}(A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})} \subset V\}$ is weakly open in L_X for every norm open subset V of Y*;

(A.3) *$G_i : \Omega \times Y \to \mathcal{F}(Y')$ and $g_i : Y \to (0,1]$ are such that*

(a) *for each $\omega \in \Omega$, $y \to (G_i(\omega, y))_{g_i(y)} : Y \to 2^{Y'}$ is monotone (that is $\mathrm{Re}\langle u - v, y - x\rangle \geq 0$ for all $u \in (G_i(\omega, y))_{g_i(y)}$ and $v \in (G_i(\omega, x))_{g_i(x)}$ and $x, y \in Y$) with nonempty values*;

(b) *for each $\omega \in \Omega$, $y \to (G_i(\omega, y))_{g_i(y)} : L \cap Y \to 2^{Y'}$ is lower semicontinuous from the relative topology of Y into the weak*-topology $\sigma(Y', Y)$ of Y' for each one-dimensional flat $L \subset Y$*;

(A.4) (a) *for each fixed $\omega \in \Omega$, the set $\{\widetilde{x} \in S_X^1 : \sup_{y \in (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})}} [\sup_{u \in (G_i(\omega, y))_{g_i(y)}} \mathrm{Re}\langle u, \widetilde{x} - y\rangle] > 0\}$ is weakly open in L_X*;

(b) $\{(\omega, \widetilde{x}) : \sup_{u \in (G_i(\omega, y))_{g_i(y)}} \mathrm{Re}\langle u, \widetilde{x}_i(\omega) - y\rangle > 0\} \in \mathcal{F} \otimes B(L_X)$.

Then, there exists $\widetilde{x}^* \in L_X$ such that for every $i \in I$ and μ-a.e.:

(i) $\widetilde{x}_i^*(\omega) \in \mathrm{cl}(A_i(\omega, \widetilde{x}^*))_{a_i(\widetilde{x}^*)}$;

(ii) $\sup_{u \in (G_i(\omega, \widetilde{x}^*(\omega)))_{g_i(\widetilde{x}^*(\omega))}} \mathrm{Re}\langle u, \widetilde{x}_i^*(\omega) - y\rangle \leq 0$ *for all $y \in ((A_i(\omega, \widetilde{x}^*))_{a_i(\widetilde{x}^*)})$*.

Proof Let us define $\psi_i : \Omega \times L_X \times Y \to R \cup \{-\infty, +\infty\}$ by

$$\psi_i(\omega, \widetilde{x}, y) = \sup_{u \in (G_i(\omega, y))_{g_i(y)}} \mathrm{Re}\langle u, \widetilde{x}_i(\omega) - y\rangle \quad \text{for each } (\omega, \widetilde{x}, y) \in \Omega \times L_X \times Y.$$

We have that $\widetilde{x} \to \psi_i(\omega, \widetilde{x}, y)$ is lower semicontinuous on L_X for each fixed $(\omega, y) \in \Omega \times Y$ and $\widetilde{x}_i(\omega) \notin \{y \in Y : \psi_i(\omega, \widetilde{x}, y) > 0\}$ for each fixed $(\omega, \widetilde{x}) \in \Omega \times L_X$.

We also know that for each $(\omega, \widetilde{x}) \in \Omega \times L_X$, $\psi_i(\omega, \widetilde{x}, \cdot)$ is concave. This fact is a consequence of assumption (A.3)(a).

All the hypotheses of Theorem 2 are satisfied. According to Theorem 2, there exists $\widetilde{x}^* \in L_X$ such that $\widetilde{x}_i^*(\omega) \in \mathrm{cl}(A_i(\omega, \widetilde{x}^*))_{a_i(\widetilde{x}^*)}$ for every $i \in I$ and

(1) $\sup_{y \in (A_i(\omega, \widetilde{x}^*))_{a_i(\widetilde{x}^*)}} \sup_{u \in (G_i(\omega, y))_{g_i(y)}} [\mathrm{Re}\langle u, \widetilde{x}_i^*(\omega) - y\rangle] \leq 0$ for every $i \in I$.

Finally, we will prove that $\sup_{y\in(A_i(\omega,\widetilde{x}^*))_{a_i(\widetilde{x}^*)}}\sup_{u\in(G_i(\omega,\widetilde{x}^*(\omega)))_{g_i(\widetilde{x}^*(\omega))}}[\mathrm{Re}\langle u,\widetilde{x}_i^*(\omega)-y\rangle]\leq 0$ for every $i\in I$.

In order to do that, let us consider $i\in I$ and the fixed point $\omega\in\Omega$.

Let $y\in(A_i(\omega,\widetilde{x}^*))_{a_i(\widetilde{x}^*)}$, $\lambda\in[0,1]$ and $z_\lambda^i(\omega):=\lambda y+(1-\lambda)\widetilde{x}_i^*(\omega)$. According to assumption (A.2)(a), $z_\lambda^i(\omega)\in A_i(\omega,\widetilde{x}^*)$.

According to (1), we have $\sup_{u\in(G_i(\omega,z_\lambda^i(\omega)))_{g_i(z_\lambda^i(\omega))}}[\mathrm{Re}\langle u,\widetilde{x}_i^*(\omega)-z_\lambda(\omega)\rangle]\leq 0$ for each $\lambda\in[0,1]$.

Therefore, for each $\lambda\in[0,1]$, we have that

$$\begin{aligned}
&t\Bigl\{\sup_{u\in(G_i(\omega,z_\lambda^i(\omega)))_{g_i(z_\lambda^i(\omega))}}\bigl[\mathrm{Re}\bigl\langle u,\widetilde{x}_i^*(\omega)-y\bigr\rangle\bigr]\Bigr\}\\
&\quad=\sup_{u\in(G_i(\omega,z_\lambda^i(\omega)))_{g_i(z_\lambda^i(\omega))}}t\bigl[\mathrm{Re}\bigl\langle u,\widetilde{x}_i^*(\omega_i)-y\bigr\rangle\bigr]\\
&\quad=\sup_{u\in(G_i(\omega,z_\lambda^i(\omega)))_{g_i(z_\lambda^i(\omega))}}\bigl[\mathrm{Re}\bigl\langle u,\widetilde{x}_i^*(\omega)-z_\lambda^i(\omega)\bigr\rangle\bigr]\leq 0.
\end{aligned}$$

It follows that for each $\lambda\in[0,1]$,

(2) $\sup_{u\in(G_i(\omega,z_\lambda^i(\omega)))_{g_i(z_\lambda^i(\omega))}}[\mathrm{Re}\langle u,\widetilde{x}_i^*(\omega)-y\rangle]\leq 0$.

Now, we are using the lower semicontinuity of $y\to(G_i(\omega,y))_{g_i(y)}:L\cap Y\to 2^{Y'}$ in order to show the conclusion. For each $z_0\in(G_i(\omega,\widetilde{x}_i^*(\omega)))_{g_i(\widetilde{x}_i^*(\omega))}$ and $e>0$ let us consider $U_{z_0}^i$, the neighborhood of z_0 in the topology $\sigma(Y',Y)$, defined by $U_{z_0}^i:=\{z\in Y':|\mathrm{Re}\langle z_0-z,\widetilde{x}_i^*(\omega)-y\rangle|<e\}$. As $y\to(G_i(\omega,y))_{g_i(y)}:L\cap Y\to 2^{Y'}$ is lower semicontinuous, where $L=\{z_\lambda^i(\omega):\lambda\in[0,1]\}$ and $U_{z_0}^i\cap(G_i(\omega,\widetilde{x}_i^*(\omega)))_{g_i(\widetilde{x}_i^*(\omega))}\neq\emptyset$, there exists a nonempty neighborhood $N(\widetilde{x}_i^*(\omega))$ of $\widetilde{x}_i^*(\omega)$ in L such that for each $z\in N(\widetilde{x}_i^*(\omega))$, we have that $U_{z_0}^i\cap(G_i(\omega,z))_{g_i(z)}\neq\emptyset$. Then there exists $\delta\in(0,1]$, $t\in(0,\delta)$ and $u\in(G_i(\omega,z_\lambda^i(\omega)))_{g_i(z_\lambda^i(\omega))}\cap U_{z_0}^i\neq\emptyset$ such that $\mathrm{Re}\langle z_0-u,\widetilde{x}_i^*(\omega)-y\rangle<e$. Therefore, $\mathrm{Re}\langle z_0,\widetilde{x}_i^*(\omega)-y\rangle<\mathrm{Re}\langle u_i,\widetilde{x}_i^*(\omega)-y\rangle+e$.

It follows that

$$\mathrm{Re}\bigl\langle z_0,\widetilde{x}_i^*(\omega)-y\bigr\rangle<\mathrm{Re}\bigl\langle u,\widetilde{x}_i^*(\omega)-y\bigr\rangle+e<e.$$

The last inequality comes from (2). Since $e>0$ and $z_0\in(G_i(\omega,\widetilde{x}_i^*(\omega)))_{g_i(\widetilde{x}_i^*(\omega))}$ have been chosen arbitrarily, the next relation holds:

$$\mathrm{Re}\bigl\langle z_0,\widetilde{x}_i^*(\omega)-y\bigr\rangle<0.$$

Hence, for each $i\in I$, we have that $\sup_{u\in(G_i(\omega,\widetilde{x}^*(\omega)))_{g_i(\widetilde{x}^*(\omega))}}[\mathrm{Re}\langle z_0,\widetilde{x}_i^*(\omega)-y\rangle]\leq 0$ for every $y\in\mathrm{cl}(A_i(\omega,\widetilde{x}^*))_{a_i(\widetilde{x}^*)}$. □

Example 3 Let $(\Omega,\mathcal{F},\mu)$ be the measure space, where $\Omega=[0,1]$, $\mathcal{F}=\beta([0,1])$ is the σ-algebra of the Borel measurable subsets in $[0,1]$, and μ is the Lebesgue measure.

Let $Y=\mathbb{R}$ and $I=\{1,2,\dots,n\}$.

For each $i\in I$, let us define the following

$$\mathcal{F}_i=\mathcal{F}.$$

The correspondence X_i is as in Example 1, that is, $(X_i(\omega))_{z_i} = [0,1]$ for each $\omega \in [0,1]$.

For each $\omega \in [0,1]$, $(X_i(\omega))_{z_i}$ is a nonempty convex weakly compactly valued and integrably bounded correspondence. It is also $\mathcal{F}_i$-lower measurable.

Let $L_{X_i} = \{x_i \in S_{(X_i(\cdot))_{z_i}} : x_i \text{ is } \mathcal{F}_i\text{-measurable}\}$, and let $L_X = \prod_{i \in I} L_{X_i}$.

Let us define $D_i = \prod_{j \neq i} L_{X_j} \times \{\widetilde{x}_i : [0,1] \to [0,1], \widetilde{x}_i(\omega) = k_{\widetilde{x}_i}, \omega \in [0,1], k_{\widetilde{x}_i} \in [0,1]\}$. D_i is weakly closed in L_X.

The random fuzzy constraint function $a_i : L_X \to (0,1]$ is defined by

$$a_i(\widetilde{x}) = \frac{1}{2} \quad \text{for each } \widetilde{x} \in L_X.$$

For each $\omega \in [0,1]$, the random fuzzy constraint mapping of agent i, $A_i(\omega,\cdot) : L_X \to \mathcal{F}(\mathbb{R})$ is defined by

$$A_i(\omega,\widetilde{x})(y) = \begin{cases} \frac{19}{20(y+1)} & \text{if } (\omega,\widetilde{x}) \in [0,1] \times D_i \text{ and } y \in (0,1]; \\ \frac{1}{4y} & \text{if } (\omega,\widetilde{x}) \in [0,1] \times (L_X \backslash D_i) \text{ and } y \in (0,1]; \\ 0 & \text{if } (\omega,\widetilde{x}) \in [0,1] \times L_X \text{ and } y = 1. \end{cases}$$

Then, the correspondence $(\omega,\widetilde{x}) \to (A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})} : [0,1] \times L_X \to 2^{[0,1]}$ is defined by

$$\big(A_i(\omega,\widetilde{x})\big)_{a_i(\widetilde{x})} = \begin{cases} (0, \frac{9}{10}) & \text{if } (\omega,\widetilde{x}) \in [0,1] \times D_i; \\ (0, \frac{1}{2}] & \text{otherwise.} \end{cases}$$

For each $\omega \in [0,1]$, it has weakly open lower sections in L_X, and it has a measurable graph.

For each $(\omega,\widetilde{x}) \in [0,1] \times L_X$, $(A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})}$ is convex and with nonempty interior in $[0,1]$.

For each $\omega \in [0,1]$, the correspondence $\widetilde{x} \to \mathrm{cl}(A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})} : L_X \to 2^{[0,1]}$ is upper semicontinuous and nonempty-valued.

For each $\omega \in [0,1]$, let $G_i(\omega,\cdot) : \mathbb{R} \to \mathcal{F}(\mathbb{R})$, and let $g_i : \mathbb{R} \to (0,1]$ be such that

$$g_i(y) = \begin{cases} \frac{1}{4} & \text{if } y \in (-\infty, \frac{1}{2}]; \\ \frac{3}{4} & \text{if } y \in (\frac{1}{2}, \infty) \end{cases} \quad \text{and}$$

$$G_i(\omega,y)(z) = \begin{cases} \frac{1}{2} & \text{if } y \in (-\infty, \frac{1}{2}] \text{ and } z = 0; \\ 1 & \text{if } y \in (\frac{1}{2}, \infty) \text{ and } z \in \{y, y+1\}; \\ 0 & \text{otherwise.} \end{cases}$$

Then, for each $\omega \in [0,1]$, $(G_i(\omega,\cdot))_{g_i(\cdot)} : \mathbb{R} \to 2^{\mathbb{R}}$ is defined by

$$\big(G_i(\omega,y)\big)_{g_i(y)} = \begin{cases} \{0\} & \text{if } y \leq \frac{1}{2}; \\ \{y, y+1\} & \text{if } y > \frac{1}{2} \end{cases} \quad \text{for each } (\omega,y) \in [0,1] \times \mathbb{R}.$$

For each $\omega \in [0,1]$, $(G_i(\omega,\cdot))_{g_i(\cdot)} : \mathbb{R} \to 2^{\mathbb{R}}$ is monotone with nonempty values and lower semicontinuous.

For each fixed $\omega \in \Omega$, let $m_i(\omega) = \sup_{y \in (A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})}} [\sup_{u \in (G_i(\omega,y))_{g_i(y)}} u(\widetilde{x}_i(\omega) - y)]$.

If $\widetilde{x}_i(\omega) \geq \frac{9}{10}$, $m_i(\omega) > 0$.

If $\widetilde{x}_i(\omega) < \frac{9}{10}$, $m_i(\omega) = \begin{cases} \sup_{y\in(\frac{1}{2},\frac{9}{10})}(y+1)(\widetilde{x}_i(\omega)-y) & \text{if } \frac{1}{2} < y < \widetilde{x}_i(\omega); \\ 0 & \text{if } y \in [\widetilde{x}_i(\omega), \frac{9}{10}) \end{cases}$

Therefore, if $\frac{1}{2} < \widetilde{x}_i(\omega) < \frac{9}{10}$, $m_i(\omega) = \sup_{y\in(\frac{1}{2},\widetilde{x}_i(\omega))}(y+1)(\widetilde{x}_i(\omega)-y) > 0$ and if $0 < \widetilde{x}_i(\omega) \leq \frac{1}{2}$, $m_i(\omega) = 0$.

Consequently, $m_i(\omega) > 0$ for each $\widetilde{x}_i(\omega) \in (\frac{1}{2}, 1]$.

Then, for each $\omega \in [0,1]$, the set

$$\begin{aligned} M_i(\omega) &= \Big\{\widetilde{x} \in L_X : \sup_{y\in(A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})}}\Big[\sup_{u\in(G_i(\omega,y))_{g_i(y)}} u\big(\widetilde{x}_i(\omega)-y\big)\Big] > 0\Big\} \\ &= \Big\{\widetilde{x} \in D_i : \widetilde{x}_i(\omega) \in \Big(\frac{1}{2}, 1\Big]\Big\} \end{aligned}$$

is weakly open in L_X and $M_i \in \mathcal{F} \otimes B(L_X)$.

There exists $\widetilde{x}^* \in L_X$ such that, for each $i \in I$, $\widetilde{x}_i^*(\omega) = 0$ for each $\omega \in [0,1]$ and μ-a.e.

(i) $\widetilde{x}_i^*(\omega) \in \mathrm{cl}((A_i(\omega,\widetilde{x}^*))_{a_i(\widetilde{x}^*)})$;

(ii) $\sup_{u\in(G_i(\omega,\widetilde{x}^*(\omega)))_{g_i(\widetilde{x}^*(\omega))}} \mathrm{Re}\langle u, \widetilde{x}_i^*(\omega)-y\rangle \leq 0$ for all $y \in (A_i(\omega,\widetilde{x}^*))_{a_i(\widetilde{x}^*)}$.

If |I| = 1, we obtain the following corollary of Theorem 3.

Corollary 2 *Let $(\Omega,\mathcal{F},\mu)$ be a complete finite separable measure space, and let Y be a separable Banach space. Suppose that the following conditions are satisfied.*

(A.1) (a) *$X : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to (X(\omega))_z : \Omega \to 2^Y$ is a nonempty convex weakly compact-valued and integrably bounded correspondence*;

(b) *$X : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to (X(\omega))_z : \Omega \to 2^Y$ is $\mathcal{F}$-lower measurable*;

(A.2) (a) *For each $(\omega,\widetilde{x}) \in \Omega \times L_X$, $(A(\omega,\widetilde{x}))_{a(\widetilde{x})}$ is convex and has a nonempty interior in the relative norm topology of $(X(\omega))_z$*;

(b) *the correspondence $(\omega,\widetilde{x}) \to (A(\omega,\widetilde{x}))_{a(\widetilde{x})} : \Omega \times L_X \to 2^Y$ has a measurable graph, i.e., $\{(\omega,\widetilde{x},y) \in \Omega \times L_X \times Y : y \in (A(\omega,\widetilde{x}))_{a(\widetilde{x})}\} \in \mathcal{F} \otimes ß_w(L_X) \otimes ß(Y)$, where $ß_w(L_X)$ is the Borel σ-algebra for the weak topology on L_X and $ß(Y)$ is the Borel σ-algebra for the norm topology on Y*;

(c) *the correspondence $(\omega,\widetilde{x}) \to (A(\omega,\widetilde{x}))_{a(\widetilde{x})}$ has weakly open lower sections, i.e., for each $\omega \in \Omega$ and for each $y \in Y$, the set $((A(\omega,\widetilde{x}))_{a(\widetilde{x})})^{-1}(\omega,y) = \{\widetilde{x} \in L_X : y \in (A(\omega,\widetilde{x}))_{a(\widetilde{x})}\}$ is weakly open in L_X*;

(d) *For each $\omega \in \Omega$, $\widetilde{x} \to \mathrm{cl}(A(\omega,\widetilde{x}))_{a(\widetilde{x})} : L_X \to 2^Y$ is upper semicontinuous in the sense that the set $\{\widetilde{x} \in L_X : \mathrm{cl}(A(\omega,\widetilde{x}))_{a(\widetilde{x})} \subset V\}$ is weakly open in L_X for every norm open subset V of Y*;

(A.3) *$G : \Omega \times Y \to \mathcal{F}(Y')$ and $g : Y \to (0,1]$ are such that*:

(a) *for each $\omega \in \Omega$, $y \to (G(\omega,y))_{g(y)} : Y \to 2^{Y'}$ is monotone with nonempty values*;

(b) *for each $\omega \in \Omega$, $y \to (G(\omega,y))_{g(y)} : L \cap Y \to 2^{Y'}$ is lower semicontinuous from the relative topology of Y into the weak *-topology $\sigma(Y',Y)$ of Y' for each one-dimensional flat $L \subset Y$*;

(A.4) (a) *for each fixed $\omega \in \Omega$, the set $\{\widetilde{x} \in S_X^1 : \sup_{y\in(A(\omega,\widetilde{x}))_{a(\widetilde{x})}}[\sup_{u\in(G(\omega,y))_{g(y)}} \mathrm{Re}\langle u,\widetilde{x}-y\rangle] > 0\}$ is weakly open in L_X*;

(b) $\{(\omega,\widetilde{x}) : \sup_{u\in(G(\omega,y))_{g(y)}} \mathrm{Re}\langle u,\widetilde{x}-y\rangle > 0\} \in \mathcal{F} \otimes B(L_X)$.

Then, there exists $\widetilde{x}^* \in L_X$ such that μ-a.e.:

(i) $\widetilde{x}^*(\omega) \in \mathrm{cl}(A(\omega,\widetilde{x}^*))_{a(\widetilde{x}^*)}$;

(ii) $\sup_{u\in(G(\omega,\widetilde{x}^*(\omega)))_{g(\widetilde{x}^*(\omega))}} \mathrm{Re}\langle u,\widetilde{x}^*(\omega)-y\rangle \leq 0$ *for all* $y \in ((A(\omega,\widetilde{x}^*))_{a(\widetilde{x}^*)})$.

We obtain the following random fixed point theorem by using a similar kind of proof as in the case of Theorem 1. This result is a generalization of Browder fixed point theorem [32].

Theorem 4 *Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite separable measure space, and let Y be a separable Banach space. Suppose that the following conditions are satisfied.*

For each $i \in I$:

(A.1) (a) *$X_i : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to (X_i(\omega))_{z_i} : \Omega \to 2^Y$ is a nonempty convex weakly compact-valued and integrably bounded correspondence*;

(b) *$X_i : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to (X_i(\omega))_{z_i} : \Omega \to 2^Y$ is $\mathcal{F}_i$-lower measurable*;

(A.2) (a) *For each $(\omega, \widetilde{x}) \in \Omega \times L_X$, $(A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})}$ is convex and has a nonempty interior in the relative norm topology of $(X_i(\omega))_z$*;

(b) *the correspondence $(\omega, \widetilde{x}) \to (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})} : \Omega \times L_X \to 2^Y$ has a measurable graph, i.e., $\{(\omega, \widetilde{x}, y) \in \Omega \times L_X \times Y : y \in (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})}\} \in \mathcal{F} \otimes ß_w(L_X) \otimes ß(Y)$, where $ß_w(L_X)$ is the Borel σ-algebra for the weak topology on L_X and $ß(Y)$ is the Borel σ-algebra for the norm topology on Y*;

(c) *the correspondence $(\omega, \widetilde{x}) \to (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})}$ has weakly open lower sections, i.e., for each $\omega \in \Omega$ and for each $y \in Y$, the set $((A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})})^{-1}(\omega, y) = \{\widetilde{x} \in L_X : y \in (A_i(\omega, \widetilde{x}))_{a_i(\widetilde{x})}\}$ is weakly open in L_X*;

Then, there exists $\widetilde{x}^* \in L_X$ such that for every $i \in I$ and μ-a.e., $\widetilde{x}_i{}^*(\omega) \in (A_i(\omega, \widetilde{x}^*))_{a_i(\widetilde{x}^*)}$.

If $|\mathrm{I}| = 1$, we obtain the following result.

Theorem 5 *Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite separable measure space, and let Y be a separable Banach space. Suppose that the following conditions are satisfied.*

(A.1) (a) *$X : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to (X(\omega))_z : \Omega \to 2^Y$ is a nonempty convex weakly compact-valued and integrably bounded correspondence*;

(b) *$X : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to (X(\omega))_z : \Omega \to 2^Y$ is $\mathcal{F}$-lower measurable*;

(A.2) (a) *For each $(\omega, \widetilde{x}) \in \Omega \times L_X$, $(A(\omega, \widetilde{x}))_{a(\widetilde{x})}$ is convex and has a nonempty interior in the relative norm topology of $(X(\omega))_z$*;

(b) *the correspondence $(\omega, \widetilde{x}) \to (A(\omega, \widetilde{x}))_{a(\widetilde{x})} : \Omega \times L_X \to 2^Y$ has a measurable graph, i.e., $\{(\omega, \widetilde{x}, y) \in \Omega \times L_X \times Y : y \in (A(\omega, \widetilde{x}))_{a_i(\widetilde{x})}\} \in \mathcal{F} \otimes ß_w(L_X) \otimes ß(Y)$, where $ß_w(L_X)$ is the Borel σ-algebra for the weak topology on L_X and $ß(Y)$ is the Borel σ-algebra for the norm topology on Y*;

(c) *the correspondence $(\omega, \widetilde{x}) \to (A(\omega, \widetilde{x}))_{a(\widetilde{x})}$ has weakly open lower sections, i.e., for each $\omega \in \Omega$ and for each $y \in Y$, the set $((A(\omega, \widetilde{x}))_{a(\widetilde{x})})^{-1}(\omega, y) = \{\widetilde{x} \in L_X : y \in (A(\omega, \widetilde{x}))_{a_i(\widetilde{x})}\}$ is weakly open in L_X*;

Then, there exists $\widetilde{x}^* \in L_X$ such that $\widetilde{x}^*(\omega) \in (A(\omega, \widetilde{x}^*))_{a(\widetilde{x}^*)}$ μ-a.e.

Example 4 Let $\Omega = [0,1]$, $\mathcal{F} = ß([0,1], \mu)$, $Y = \mathbb{R}$, $I = \{1, 2, \dots, n\}$.

For each $i \in \{1, 2, \dots, n\}$ let us define the following mathematical objects.

Let X_i, L_{X_i} and L_X be defined as in Example 1. $\mathcal{M}_i = \{[0, \frac{1}{2}), [\frac{1}{2}, \frac{2}{3}), \dots, [\frac{i-1}{i}, 1]\}$ and $\mathcal{F}_i = \sigma(\mathcal{M}_i)$.

$C_i = \{\widetilde{x}_i : [0,1] \to [0,1] : \widetilde{x}_i(\omega) = \begin{cases} c_{\widetilde{x}_i} & \text{if } \omega \in [\frac{i-1}{i}, 1]; \\ 0 & \text{otherwise,} \end{cases}$ where $c_{\widetilde{x}_i} \in [0,1]$ is constant$\}$ and $D_i = \prod_{j \neq i} L_{X_j} \times C_i$.

We notice that if $x_i \in C_i$, then it is $\mathcal{F}_i$-measurable and μ-integrable, then $C_i \subset L_{X_i}$ and C_i is weakly closed in L_{X_i}.

The random fuzzy constraint function $a_i : L_X \to (0,1]$ is defined by

$$a_i(\widetilde{x}) = \frac{1}{3} \quad \text{for each } \widetilde{x} \in L_X.$$

For each $\omega \in [0,1]$, the random fuzzy constraint mapping of agent i, $A_i(\omega,\cdot) : L_X \to \mathcal{F}(\mathbb{R})$ is defined by

$$A_i(\omega,\widetilde{x})(y) = \begin{cases} \frac{\widetilde{x}_i(\omega)+2}{5(y+1)} & \text{if } (\omega,\widetilde{x}) \in [0,1] \times D_i \text{ and } y \in (0,1]; \\ 0 & \text{otherwise.} \end{cases}$$

Then, the correspondence $(\omega,\widetilde{x}) \to (A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})} : [0,1] \times L_X \to 2^{[0,1]}$ is defined by

$$\begin{aligned} \big(A_i(\omega,\widetilde{x})\big)_{a_i(\widetilde{x})} &= \big\{y \in [0,1] : A_i(\omega,\widetilde{x})(y) > a_i(\widetilde{x})\big\} \\ &= \begin{cases} [0, \frac{1+3\widetilde{x}_i(\omega)}{5}) & \text{if } (\omega,\widetilde{x}) \in [0,1] \times D_i; \\ [0,1] & \text{otherwise.} \end{cases} \end{aligned}$$

It has weakly open lower sections in L_X, and it has a measurable graph.

For each $(\omega,\widetilde{x}) \in [0,1] \times L_X$, $(A_i(\omega,\widetilde{x}))_{a_i(\widetilde{x})}$ is convex and with a nonempty interior in $[0,1]$.

There exists $\widetilde{x}^* \in L_X$ such that for every $i \in I$, $\widetilde{x}_i{}^*(\omega) \in (A_i(\omega,\widetilde{x}^*))_{a_i(\widetilde{x}^*)}$. For instance, let $\widetilde{x}^*$ such that $\widetilde{x}_1{}^*(\omega) = \frac{1}{3}$ if $\omega \in [0,1]$ and $\widetilde{x}_i{}^*(\omega) = \begin{cases} \frac{1}{i+1} & \text{if } \omega \in [\frac{i-1}{i},1]; \\ 0 & \text{otherwise.} \end{cases}$ for each $i \in \{2,\dots,n\}$.

Appendix

The results below have been used in the proof of our theorems. For more details and further references see the paper quoted.

Theorem 6 (Projection theorem) *Let $(\Omega,\mathcal{F},\mu)$ be a complete, finite measure space, and let Y be a complete separable metric space. If H belongs to $\mathcal{F} \otimes ß(Y)$, its projection $Proj_\Omega(H)$ belongs to $\mathcal{F}$.*

Theorem 7 (Aumann measurable selection theorem [33]) *Let $(\Omega,\mathcal{F},\mu)$ be a complete finite measure space, let Y be a complete, separable metric space, and let $T : \Omega \to 2^Y$ be a nonempty valued correspondence with a measurable graph, i.e., $G_T \in \mathcal{F} \otimes \beta(Y)$. Then there is a measurable function $f : \Omega \to Y$ such that $f(\omega) \in T(\omega)$ μ-a.e.*

Theorem 8 (Diestel's theorem [34], Theorem 3.1) *Let $(\Omega,\mathcal{F},\mu)$ be a complete finite measure space, let Y be a separable Banach space, and let $T : \Omega \to 2^Y$ be an integrably bounded, convex, weakly compact and a nonempty valued correspondence. Then $S_T = \{x \in L_1(\mu,Y) : x(\omega) \in T(\omega)$ μ-a.e.$\}$ is weakly compact in $L_1(\mu,Y)$.*

Theorem 9 (Carathéodory-type selection theorem [25]) *Let $(\Omega,\mathcal{F},\mu)$ be a complete measure space, let Z be a complete separable metric space, and let Y be a separable Banach space. Let $X : \Omega \to 2^Y$ be a correspondence with a measurable graph, i.e., $G_X \in \mathcal{F} \otimes ß(Y)$ and let $T : \Omega \times Z \to 2^Y$ be a convex-valued correspondence (possibly empty) with a measurable graph, i.e., $G_T \in \mathcal{F} \otimes ß(Z) \otimes ß(Y)$, where $ß(Y)$ and $ß(Z)$ are the Borel σ-algebras of Y and Z , respectively.*

Suppose that

(a) *for each $\omega \in \Omega$, $T(\omega,x) \subset X(\omega)$ for all $x \in Z$.*

(b) *for each* $\omega \in \Omega$, $T(\omega, \cdot)$ *has open lower sections in* Z, *i.e., for each* $\omega \in \Omega$ *and* $y \in Y$, $T^{-1}(\omega, y) = \{x \in Z : y \in T(\omega, x)\}$ *is open in* Z.

(c) *for each* $(\omega, x) \in \Omega \times Z$, *if* $T(\omega, x) \neq \emptyset$, *then* $T(\omega, x)$ *has a nonempty interior in* $X(\omega)$.

Let $U = \{(\omega, x) \in \Omega \times Z : T(\omega, x) \neq \emptyset\}$ *and for each* $x \in Z$, $U^x = \{\omega \in \Omega : (\omega, x) \in U\}$ *and for each* $\omega \in \Omega$, $U^\omega = \{x \in Z : (\omega, x) \in U\}$. *Then for each* $x \in Z$, U^x *is a measurable set in* Ω, *and there exists a Caratheodory-type selection from* $T_{|U}$, *i.e., there exists a function* $f : U \to Y$ *such that* $f(\omega, x) \in T(\omega, x)$ *for all* $(\omega, x) \in U$, *for each* $x \in Z$, $f(\cdot, x)$ *is measurable on* U^x *and for each* $\omega \in \Omega$, $f(\omega, \cdot)$ *is continuous on* U^ω. *Moreover,* $f(\cdot, \cdot)$ *is jointly measurable.*

Theorem 10 (U.s.c. lifting theorem [33]) *Let* Y *be a separable space, let* $(\Omega, \mathcal{F}, \mu)$ *be a complete finite measure space, and let* $X : \Omega \to 2^Y$ *be an integrably bounded, nonempty, convex valued correspondence such that for all* $\omega \in \Omega$, $X(\omega)$ *is a weakly compact, convex subset of* Y. *Denote by* S_X *the set* $\{x \in L_1(\mu, Y) : x(\omega) \in X(\omega)\ \mu\text{-a.e.}\}$. *Let* $T : \Omega \times S_X \to 2^Y$ *be a nonempty, closed, convex-valued correspondence such that* $T(\omega, x) \subset X(\omega)$ *for all* $(\omega, x) \in \Omega \times S_X^1$. *Assume that for each fixed* $x \in S_X$, $T(\cdot, x)$ *has a measurable graph, and that for each fixed* $\omega \in \Omega$, $T(\omega, \cdot) : S_X \to 2^Y$ *is u.s.c. in the sense that the set* $\{x \in S_X : T(\omega, x) \subset V\}$ *is weakly open in* S_X *for every norm open subset* V *of* Y. *Define the correspondence* $\Phi : S_X \to 2^{S_X}$ *by*

$$\Phi(x) = \{y \in S_X : y(\omega) \in T(\omega, x)\ \mu\text{-a.e.}\}.$$

Then Φ *is weakly u.s.c., i.e., the set* $\{x \in S_X : \Phi(x) \subset V\}$ *is weakly open in* S_X *for every weakly open subset* V *of* S_X.

Competing interests
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