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BMO-Lorentz martingale spaces

Xueying Zhang[1*], Xuming Yi[1] and Chuanzhou Zhang[2]

*Correspondence:
zhxying315@163.com
[1]School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, China
Full list of author information is available at the end of the article

Abstract
In this paper BMO-Lorentz martingale spaces are investigated. We give the characterization of BMO-Lorentz martingale spaces. Moreover, we discuss the relationship between the Carleson measure and BMO-Lorentz martingales. As a consequence, we find a new way to characterize the geometrical properties of a Banach space.

1 Introduction and preliminaries

Since 1951 when they were first introduced by Lorentz in [1], Lorentz spaces have attracted more and more attention. A lot of results were obtained such as normability, duality, interpolation, and so on [2–7].

We know that martingale theory is intimately related to harmonic analysis. In martingale case, Weisz [8] and Long [9] considered the spaces $H_{p,q}$ and the interpolations between them, respectively. It is well known that the validity of a classical (scalar-valued) result in the vector-valued setting, *i.e.*, for functions or martingales with values in a Banach space X, depends on the geometrical or topological properties of X.

It was exactly with this in mind that Xu [10] developed the vector-valued Littlewood-Paley theory, which was inspired by Pisier's celebrated work [8] on martingale inequalities in uniformly convex spaces. Very recently, Ouyang and Xu [11] studied the endpoint case of the main results of [10] by means of the classical relationship between BMO functions and Carleson measures. Jiao [12] discussed the relationship between Carleson measures and vector-valued martingales.

Let (Ω,μ) be a nonatomic σ-finite measure space. Suppose that f is a measurable function on a measure space (Ω,μ). We define its distribution function

$$\lambda_f(t)=\mu\bigl\{\omega:\bigl\|f(\omega)\bigr\|>t\bigr\},\quad t\geq 0,$$

and its decreasing rearrangement function

$$f^*(t)=\inf\bigl\{s>0:\lambda_f(s)\leq t\bigr\}.$$

Given a measurable function f on a measure space (Ω,μ) and $0<p,q\leq\infty$, define

$$\|f\|_{L^{p,q}}=\begin{cases}(\int_0^\infty(t^{\frac{1}{p}}f^*(t))^q\frac{dt}{t})^{\frac{1}{q}} & \text{if } q<\infty,\\ \sup_{t>0}t^{\frac{1}{p}}f^*(t) & \text{if } q=\infty.\end{cases}$$

Remark 1.1 Observe that for all $0 < p, r < \infty$ and $0 < q \le \infty$ we have

$$\big\| |g|^r \big\|_{L^{p,q}} = \|g\|^r_{L^{pr,qr}}. \tag{1.1}$$

Unfortunately, the functions $\|\cdot\|_{L^{p,q}}$ do not satisfy the triangle inequality. However, since for all $t > 0$, $(f+g)^*(t) \le f^*(t/2) + g^*(t/2)$, we have

$$\|f+g\|_{L^{p,q}} \le c_{p,q}\big(\|f\|_{L^{p,q}} + \|g\|_{L^{p,q}}\big), \tag{1.2}$$

where $c_{p,q} = 2^{1/p}\max\{1, 2^{(1-q)/q}\}$.

The set of all f with $\|f\|_{L^{p,q}} < \infty$ is denoted by $L^{p,q}(X,\mu)$ and is called the Lorentz space with indices p and q. Observe that the definition implies that $L^{\infty,\infty} = L^\infty$.

Let (Ω, Σ, P) be a complete probability space, and let $(\Sigma_n)_{n\ge 0}$ be a nondecreasing sequence of sub-σ-algebras of Σ with $\Sigma = \sigma(\bigcup_{n\ge0}\Sigma_n)$. We denote by E and E_n the expectation and conditional expectation with respect to Σ and Σ_n, respectively. For a martingale $f = (f_n)_{n\ge0}$ with martingale difference $df_n = f_n - f_{n-1}$, $n \ge 0$, $f_{-1} \equiv 0$, we define its maximal function, p-square function, respectively, as usual:

$$Mf = \sup_n \|f_n\|, \qquad S^{(p)}(f) = \left(\sum_{n=1}^{\infty} \|df_n\|^p\right)^{1/p}.$$

We say that an X-valued martingale $f = (f_n)_{n\ge0} \in L^{p,q}(X)$ if $\sup_n \|f_n\|_{L^{p,q}} < \infty$.

The space $BMO^a_{L^{p,q}(X)}$ ($a \ge 1, 1 < p, q \le \infty$) consists of all martingale $f \in L^{p,q}(X)$ such that

$$\|f\|_{BMO^a_{L^{p,q}(X)}} = \sup_n \big\| \big(E\big(\|f - f_{n-1}\|^a | \Sigma_n\big)\big)^{1/a} \big\|_{L^{p,q}(X)} < \infty.$$

Remark 1.2 The spaces $BMO^a_{L^{p,q}(X)}$ are independent of a and all corresponding norms are equivalent. This allows us to denote any of them by $BMO_{L^{p,q}(X)}$.

Proof If $a \ge 1$, $\varphi(x) = x^a$ is a convex function, by Jensen's inequality, we have $E(\|f - f_{n-1}\||\Sigma_n) \le (E(\|f - f_{n-1}\|^a|\Sigma_n))^{1/a}$, which implies $E(\|f - f_{n-1}\||\Sigma_n)^*(t) \le ((E(\|f - f_{n-1}\|^a| \Sigma_n))^{1/a})^*(t)$, *i.e.*, $BMO^a_{L^{p,q}(X)} \subset BMO^1_{L^{p,q}(X)}$.

On the contrary, let $g_n = E(\|f - f_{n-1}\||\Sigma_n)$, $Mg = \sup_n \|g_n\|$, $h_n = E(\|f - f_{n-1}\|^a|\Sigma_n)^{1/a}$, $Mh = \sup_n \|h_n\|$. Now we set $A_t = \{\omega : Mh > t\}$. Then $Mg > t$ a.e. on A_t. (Factually, if there is a $B \subset A_t$ with $\mu(B) > 0$ such that $Mg \le t$ on B, then $E(\|f - f_{n-1}\||\Sigma_n) \le t = E(t|\Sigma_n)$ a.e. on B, which implies $\|f - f_{n-1}\| \le t$ a.e. on B. This is a contradiction for A_t.) So, $P\{\omega : Mh > t\} \le cP\{\omega : Mg > t\}$. Then

$$\begin{aligned}
\|f\|_{BMO^a_{L^{p,q}(X)}} &\le \left(q\int_0^\infty \big[tP\big(Mh(\omega) > t\big)^{1/p}\big]^q \frac{dt}{t}\right)^{1/q} \\
&\le c\left(q\int_0^\infty \big[tP\big(Mg(\omega) > t\big)^{1/p}\big]^q \frac{dt}{t}\right)^{1/q} \le c\|f\|_{BMO^1_{L^{p,q}(X)}},
\end{aligned}$$

i.e., $BMO^1_{L^{p,q}(X)} \subset BMO^a_{L^{p,q}(X)}$. Thus we complete the proof. □

Remark 1.3 If $p = q = \infty$, $BMO_{L^{p,q}(X)}$ is the classical BMO space.

Remark 1.4 For the classical BMO space, we have the following statement (see [12]):

$$\|f\|_{BMO} \sim \sup_{\tau} \mu(\tau < \infty)^{-1/p} \|f - f_{\tau-1}\|_{L_p}, \quad 1 \le p < \infty. \tag{1.3}$$

It is well known that $L^{p,q}$ is a subspace of $L^{p,r}$ for $0 < p \le \infty$, $0 < q < r \le \infty$ and L^{p_2,q_2} is a subspace of L^{p_1,q_1} for $1 < p_1 \le p_2 \le \infty$, $1 < q_1, q_2 \le \infty$. Thus we have the following proposition.

Proposition 1.5 (1) *If* $0 < p \le \infty$, $0 < q < r \le \infty$, $BMO_{L^{p,q}} \subseteq BMO_{L^{p,r}}$.
(2) *If* $1 < p_1 \le p_2 \le \infty$, $1 < q_1, q_2 \le \infty$, $BMO_{L^{p_2,q_2}} \subseteq BMO_{L^{p_1,q_1}}$.

Theorem 1.6 *Let* $f = (f_n)_{n\ge 0}$ *be an X-valued martingale in* $L^{p,q}(X)$, $1 < p, q \le \infty$. *Then* $f \in BMO_{L^{p,q}(X)}$ *if and only if there exists an adapted process* $\theta = (\theta)_{n\ge 0}$ *such that*

$$C_\theta = \sup_n \bigl\| E\bigl(\|f - \theta_{n-1}\| \,|\, \Sigma_n\bigr) \bigr\|_{L^{p,q}(X)} < \infty.$$

And, in any case, we have

$$|||f|||_{BMO_{L^{p,q}(X)}} := \inf_\theta C_\theta \le \|f\|_{BMO_{L^{p,q}(X)}} \le c|||f|||_{BMO_{L^{p,q}(X)}}.$$

Proof Assume $f \in BMO_{L^{p,q}(X)}$. Then, obviously,

$$|||f|||_{BMO_{L^{p,q}(X)}} \le \|f\|_{BMO_{L^{p,q}(X)}}.$$

Now let $|||f|||_{BMO_{L^{p,q}(X)}} < \infty$ and $\theta = (\theta)_{n\ge 0}$ be any one such that $C_\theta < \infty$. Then we have

$$\begin{aligned} E\bigl(\|f - f_{n-1}\| \,|\, \Sigma_n\bigr) &\le E\bigl(\|f - \theta_{n-1}\| \,|\, \Sigma_n\bigr) + \|\theta_{n-1} - f_{n-1}\| \\ &= E\bigl(\|f - \theta_{n-1}\| \,|\, \Sigma_n\bigr) + \bigl\|E\bigl((f - \theta_{n-1}) | \Sigma_{n-1}\bigr)\bigr\| \\ &\le E\bigl(\|f - \theta_{n-1}\| \,|\, \Sigma_n\bigr) + E\bigl(\|f - \theta_{n-1}\| \,|\, \Sigma_{n-1}\bigr) \\ &= E\bigl(\|f - \theta_{n-1}\| \,|\, \Sigma_n\bigr) + E\bigl(E\bigl(\|f - \theta_{n-1}\| \,|\, \Sigma_n\bigr) | \Sigma_{n-1}\bigr). \end{aligned}$$

Taking 'inf' over all possible θ and (1.2), we get the desired inequality. □

Theorem 1.7 *Let* $f = (f_n)_{n\ge 0}$ *be an X-valued martingale in* $BMO_{L^{p,q}(X)}$, *where* $\frac{1}{p} + \frac{1}{s_1} = \frac{1}{r}$ *and* $\frac{1}{q} + \frac{1}{s_2} = \frac{1}{s}$, $1 < p, q, r, s_1, s_2 \le \infty$. *Then*

$$\|f\|_{BMO_{L^{p,q}(X)}} \sim \sup_\tau \mu\bigl(\{\tau < \infty\}\bigr)^{-\frac{1}{s_1}} \|f - f_{\tau-1}\|_{L^{r,s}(X)},$$

where 'sup' *is taken over all stopping times* τ.

Proof Assume that $\|f\|_{BMO_{L^{p,q}(X)}} < \infty$, τ is any stopping time. Then, by Hölder's inequality, we have

$$\begin{aligned} \|f - f_{\tau-1}\|_{L^{r,s}(X)} &= \bigl\|(f - f_{\tau-1})\chi_{\{\tau<\infty\}}\bigr\|_{L^{r,s}(X)} \\ &\le c\|f - f_{\tau-1}\|_{L^{p,q}(X)} \|\chi_{\{\tau<\infty\}}\|_{L^{s_1,s_2}(X)} \end{aligned}$$

$$\begin{aligned}
&\leq c \sup_{\|g\|_{(L^{p,q})^*}\leq 1}\left|\int_{\{\tau<\infty\}}\big(\|f-f_{\tau-1}\|\big)g\,d\mu\right| \cdot \|\chi_{\{\tau<\infty\}}\|_{L^{s_1,s_2}(X)} \\
&= c\left(\frac{q}{p}\right)^{1/q} \sup_{\|g\|_{(L^{p,q})^*}\leq 1}\left|\int_{\{\tau<\infty\}} E\big(\|f-f_{\tau-1}\|\cdot g|\Sigma_\tau\big)\,d\mu\right| \cdot \mu(\tau<\infty)^{\frac{1}{s_1}} \\
&\leq c\left(\frac{q}{p}\right)^{1/q} \|f\|_{BMO_{L^{p,q}}} \cdot \mu(\tau<\infty)^{\frac{1}{s_1}}.
\end{aligned}$$

This proves one half of the assertion. Conversely, assume that $\beta = \sup_\tau \mu(\{\tau<\infty\})^{-\frac{1}{s_1}} \|f - f_{\tau-1}\|_{L^{r,s}(X)} < \infty$, and τ is any stopping time, $F \in \Sigma_\tau$, $F \subset \{\tau < \infty\}$. Define

$$\tau_F = \begin{cases} \tau & \text{if } \omega \in F, \\ \infty & \text{if } \omega \notin F. \end{cases}$$

Thus we get

$$\begin{aligned}
\beta &\geq \mu\big(\{\tau<\infty\}\big)^{-\frac{1}{s_1}} \|f-f_{\tau-1}\|_{L^{r,s}(X)} \\
&= \frac{1}{\mu(F)^{1/s_1}} \|f-f_{\tau_F-1}\|_{L^{r,s}(X)} \\
&\geq \frac{1}{\mu(F)^{1/s_1}} \|f-f_{\tau_F-1}\|_{L^{r\wedge 1,r\wedge 1}(X)} \\
&\geq \frac{1}{\mu(F)} \|f-f_{\tau_F-1}\|_{L^{1,1}(X)} \\
&= \frac{1}{\mu(F)} \int_F \|f-f_{\tau_F-1}\|\,du.
\end{aligned}$$

That is, $E(\|f-f_{\tau_F-1}\||\Sigma_{\tau_F}) \leq \beta$. By Remark 1.2 we have

$$\|f\|_{BMO_{L^{p,q}(X)}} \leq c\beta.$$

Thus we complete the proof of the theorem. □

Proposition 1.8 *Particularly, if* $s = r$ *and* $p = q = \infty$, *we get Remark* 1.4.

By Theorem 1.6 and Theorem 1.7, we have the following proposition.

Proposition 1.9 *Let* $f = (f_n)_{n\geq 0}$ *be an X-valued martingale in* $BMO_{L^{p,q}(X)}$, *where* $\frac{1}{p} + \frac{1}{s_1} = \frac{1}{r}$ *and* $\frac{1}{q} + \frac{1}{s_2} = \frac{1}{s}$, $1 < p, q, r, s_1, s_2 \leq \infty$. *Then*

$$\|f\|_{BMO_{L^{p,q}(X)}} \sim \inf_\theta \sup_\tau \mu\big(\{\tau<\infty\}\big)^{-\frac{1}{s_1}} \|f-\theta_{\tau-1}\|_{L^{r,s}(X)},$$

where 'sup' *is taken over all stopping times* τ *and* 'inf' *is taken over all adapted process* $\theta = (\theta)_{n\geq 0}$.

2 Carleson measure and BMO-Lorentz martingale spaces

Definition 2.1 Let ν be a nonnegative measure on $\Omega \times N$, where N is equipped with the counting measure dm. Let $\hat{\tau}$ denote the tent over τ:

$$\hat{\tau} = \{(\omega, k) : k \geq \tau(\omega), \tau(\omega) < \infty\}.$$

ν is said to be an s-Carleson measure if

$$|||\nu||| = \sup_{\tau} \frac{\nu(\hat{\tau})}{\mu(\{\tau < \infty\})^s} < \infty,$$

where τ runs through all stopping times.

Theorem 2.2 *Let $1 < p, q \leq \infty$, and $g = (g_n)_{n\geq 0}$ be a real-valued martingale and $d\nu = |\Delta_k g|^2 \delta_k \, d\mu$, where δ_k is the Dirac measure centered at k. So, if $g \in BMO^2_{L^{2p,2q}}$, ν is a $1/p'$-Carleson measure. Moreover, if $1 < p, q < \infty$, the converse is also true, where $\frac{1}{p'} + \frac{1}{p} = 1$, $\frac{1}{q'} + \frac{1}{q} = 1$.*

Proof Let $g = (g_n)_{n\geq 0}$ be a real-valued martingale, let ν be generated by g as above, and let τ be any stopping time. Then, for $1 < q < \infty$,

$$\begin{aligned}
\nu(\hat{\tau}) &= E\left(\sum_{k=0}^{\infty} |\Delta_k g|^2 \chi_{\{\tau(\omega)\leq k\}}\right) \\
&= E\left(E\left(\sum_{k=\tau(\omega)}^{\infty} |\Delta_k g|^2 \Big| \Sigma_\tau\right) \chi_{\{\tau(\omega)\leq k\}}\right) \\
&= E\left(E\left(|g - g_{\tau-1}|^2 | \Sigma_\tau\right) \chi_{\{\tau(\omega)\leq k\}}\right) \\
&\leq c\left\|E\left(|g - g_{\tau-1}|^2 | \Sigma_\tau\right)\right\|_{L^{p,q}} \cdot \|\chi_{\{\tau(\omega)\leq k\}}\|_{L^{p',q'}} \\
&= c\left\|E\left(|g - g_{\tau-1}|^2 | \Sigma_\tau\right)^{\frac{1}{2}\cdot 2}\right\|_{L^{p,q}} \cdot \|\chi_{\{\tau(\omega)\leq k\}}\|_{L^{p',q'}} \\
&\leq c\left\|E\left(|g - g_{\tau-1}|^2 | \Sigma_\tau\right)^{1/2}\right\|^2_{L^{2p,2q}} \cdot \mu\left(\{\tau(\omega) \leq k\}\right)^{1/p'} \\
&\leq c\|g\|^2_{BMO^2_{L^{2p,2q}}} \cdot \mu\left(\{\tau(\omega) \leq k\}\right)^{1/p'}.
\end{aligned}$$

So, $g \in BMO^2_{L^{2p,2q}}$ implies that ν is a $1/p'$-Carleson measure and $|||\nu||| \leq \|g\|^2_{BMO^2_{L^{2p,2q}}}$.

Conversely, for any n and any $F \in \Sigma_n$, we define

$$\tau = \begin{cases} n & \text{if } \omega \in F, \\ \infty & \text{if } \omega \notin F. \end{cases}$$

Since ν is a $1/p'$-Carleson measure, we have

$$\begin{aligned}
|||\nu||| &\geq \frac{1}{\mu(\{\tau < \infty\})^{1/p'}} \nu\left(\{(\omega, k) : k \geq \tau(\omega), \tau(\omega) < \infty\}\right) \\
&= \frac{1}{\mu(F)^{1/p'}} \int_F \sum_{k=n}^{\infty} |\Delta_k g|^2 \, d\mu
\end{aligned}$$

$$\geq \frac{1}{\mu(F)} \int_F \sum_{k=n}^{\infty} |\Delta_k g|^2 \, d\mu$$
$$= \frac{1}{\mu(F)} \int_F |g - g_{n-1}|^2 \, d\mu.$$

That is, $|||\nu||| \geq E(|g - g_{n-1}|^2 | \Sigma_n)$. Then we have $\|g\|^2_{BMO^2_{L^{2p,2q}}} \leq c|||\nu|||$. Thus, we complete the proof of the theorem. □

3 The characterization of Banach space's geometrical properties

Let $1 < q < \infty$. Then a Banach space X has an equivalent q-uniformly convex norm if and only if for one $1 < p < \infty$ (or equivalently, for every $1 < p < \infty$) there exists a positive constant c such that

$$\|S^{(q)}(f)\|_p \leq c \sup_n \|f_n\|_p$$

for all finite L_p-martingales f with values in X. Again, the validity of the converse inequality amounts to saying that X has an equivalent q-uniformly smooth norm.

Definition 3.1 Let X_1 and X_2 be two Banach spaces. Let $L(X_1, X_2)$ denote the space of all bounded linear operators from X_1 to X_2. Let $\nu = (\nu_n)_{n\geq 1}$ be an adapted sequence such that $\nu_n \in L_\infty(L(X_1, X_2))$ and $\sup_{n\geq 1} \|\nu_n\|_{L_\infty(L(X_1,X_2))} \leq 1$. Then the martingale transform T associated to ν is defined as follows. For any X_1-valued martingale $f = (f_n)_{n\geq 1}$,

$$(Tf)_n = \sum_{k=1}^{n} \nu_k df_k.$$

We get the following results from [13, 14].

Lemma 3.2 *With the assumptions above, the following statements are equivalent*:

(1) *There exists a positive constant c such that*

$$\|Tf\|_{BMO(X_2)} \leq c\|f\|_{BMO(X_1)}, \quad \forall f \in BMO(X_1).$$

(2) *There exists a positive constant c such that*

$$\|(Tf)^*\|_{BMO(X_2)} \leq c\|f\|_{BMO(X_1)}, \quad \forall f \in BMO(X_1).$$

(3) *For some* $1 \leq p < \infty$ (*or equivalently, for every* $1 \leq p < \infty$), *there exists a positive constant c such that*

$$\|Tf\|_{L^p(X)} \leq c\|f\|_{L^p(X)}, \quad \forall f \in L^p(X).$$

Theorem 3.3 *Let X be a Banach space*, $2 \leq q < \infty$, $1 < p < \infty$. *Then the following statements are equivalent*:

(1) *There exists a positive constant c such that for any finite X-valued martingale*,

$$\|S^{(q)}(f)\|_{BMO_{L^{p,q}(X)}} \leq c\|f\|_{BMO_{L^{p,q}(X)}}.$$

(2) *X has an equivalent norm which is q-uniformly convex.*

Proof (1) $\Rightarrow$ (2) Let $\frac{1}{p}+\frac{1}{s_1}=\frac{1}{r}$, $\frac{1}{q}+\frac{1}{s_2}=\frac{1}{r}$, $1<p,q,r,s_1,s_2\leq\infty$ By Theorem 1.7 we have

$$\|S^{(q)}(f)\|_{BMO_{L^{p,q}(X)}}\sim\sup_{\tau}\mu(\{\tau<\infty\})^{-\frac{1}{s_1}}\|S^{(q)}(f)-S^{(q)}_{\tau-1}(f)\|_{L^{r,r}(X)},\tag{3.1}$$

$$\|f\|_{BMO^a_{L^{p,q}(X)}}\sim\sup_{\tau}\mu(\{\tau<\infty\})^{-\frac{1}{s_1}}\|f-f_{\tau-1}\|_{L^{r,r}(X)}.\tag{3.2}$$

So, if (1) holds, we have

$$\|S^{(q)}(f)-S^{(q)}_{\tau-1}(f)\|_{L^{r,r}(X)}\leq c\|f-f_{\tau-1}\|_{L^{r,r}(X)}.$$

By Remark 1.4 we have

$$\|S^{(q)}(f)\|_{BMO}\leq c\|f\|_{BMO}.\tag{3.3}$$

We now consider a martingale transform operator Q from the family of X-valued martingales to that of $l_q(X)$-valued martingales. Let $v\in L(X,l_q(X))$ be the operator defined by $v_k(x)=\{x_j\}_{j=1}^{\infty}$ for $x\in X$, where $x_j=x$ if $j=k$ and $x_j=0$ otherwise. Q is the martingale transform associated to the sequence (v_k):

$$(Qf)_n=\sum_{k=1}^{n}v_k df_k=(df_1,df_2,\dots,df_n,0,\dots).$$

Then

$$(Qf)^*=S^{(q)}(f).\tag{3.4}$$

By (3.3) and (3.4) we have

$$\|(Qf)^*\|_{BMO}\leq c\|f\|_{BMO}.$$

By Lemma 3.2 we have

$$\|S^{(q)}(f)\|_q=\|(Qf)^*\|_q\leq c\|f\|_q.$$

Thus, by Pisier's theorem, X has an equivalent q-uniformly convex norm.

(2) $\Rightarrow$ (1) Suppose that X has an equivalent q-uniformly convex norm. By Pisier's theorem [15], we find, for any $1\leq n<\infty$,

$$E\left(\sum_{i=n}^{\infty}\|df_i\|^q\Big|\Sigma_n\right)\leq cE(\|f-f_{n-1}\|^q|\Sigma_n).$$

Since $E(|S^{(q)}(f)-S^{(q)}_{n-1}(f)|^q|\Sigma_n)\leq cE(|(S^{(q)}(f))^q-(S^{(q)}_{n-1}(f))^q||\Sigma_n)$, we have

$$E(|S^{(q)}(f)-S^{(q)}_{n-1}(f)|^q|\Sigma_n)\leq cE(\|f-f_{n-1}\|^q|\Sigma_n).$$

Thus

$$\|S^{(q)}(f)\|_{BMO_{L^{p,q}(X)}} \le c\|f\|_{BMO_{L^{p,q}(X)}}.$$

We complete the proof. □

Theorem 3.4 *Let X be a Banach space and $1 < p \le 2$, $1 < q < \infty$. If there exists a positive constant c such that for any finite X-valued martingale*

$$\|f\|_{BMO_{L^p(X)}} \le c\|S^{(p)}(f)\|_{BMO_{L^p(X)}}, \tag{3.5}$$

then X has an equivalent norm which is p-uniformly smooth.

On the contrary, if X has an equivalent norm which is p-uniformly smooth, then

$$\|f\|_{BMO^a_{L^{p,q}(X)}} \le c\|S^{(p)}(f)\|_{BMO^a_{L^{p,q}(X)}}$$

for every martingale f.

Proof Let X^* be the dual space of X. It suffices to prove that X^* has an equivalent p'-uniformly convex norm, where p' is the conjugate index of p. By Pisier's theorem, this is equivalent to showing that

$$\|S^{(p')}(g)\|_{L_{p'}} \le c\|g^*\|_{L_{p'}} = c\|g\|_{H^*_{p'}(X^*)}. \tag{3.6}$$

Recall that $H^*_{p'}(X^*)$ is defined by

$$H^*_{p'}(X^*) = \{X^*\text{-valued martingale } g = (g_n) : g^* \in L_{p'}\}.$$

It is well known that $BMO_{L^p(X)}$ can be identified as a subspace of $H^*_{p'}(X^*)$. Thus, for any finite martingale, $f = (f_n)_{n\ge 0} \in BMO_{L^p(X)}$ and $g = (g_n)_{n\ge 0} \in H^*_{p'}(X^*)$.

$$|\langle g, f\rangle| = \int_\Omega \langle g(\omega), f(\omega)\rangle\, dP \le c\|f\|_{BMO_{L^p(X)}} \cdot \|g\|_{H^*_{p'}(X^*)}.$$

On the other hand, $\|S^{(p')}(g)\|_{L_{p'}}$ is the norm of the difference sequence (dg_n) in $L_{p'}(l_{p'}(X^*))$.

Thus

$$\begin{aligned}\|S^{(p')}(g)\|_{L_{p'}} &= \sup_{(a_k)}\Big\{\Big|\sum\langle dg_k, a_k\rangle\Big| : \|(a_k)\|_{L_p(l_p(X))} \le 1\Big\}\\ &= \sup_{(a_k)}\Big\{\Big|\sum\langle dg_k, E_k(a_k) - E_{k-1}(a_k)\rangle\Big| : \|(a_k)\|_{L_p(l_p(X))} \le 1\Big\}.\end{aligned}$$

Set $df_k = E_k(a_k) - E_{k-1}(a_k)$ and $f = \sum df_k$. Then f is an X-valued martingale. We have

$$\|S^{(p')}(g)\|_{L_{p'}} = \sup_{(a_k)}\Big\{\Big|\sum\langle dg_k, df_k\rangle\Big|\Big\} \le c\|f\|_{BMO_{L^p(X)}} \cdot \|g\|_{H^*_{p'}(X^*)}. \tag{3.7}$$

It remains to estimate $\|f\|_{BMO_{L^p(X)}}$. Since $\|(a_k)\|_{L_p(l_p(X))} \le 1$, we can also get the conditional case $E((\sum_{k=n}^{\infty}\|a_k\|^p)|\Sigma_n) \le 1$. Then by (3.5)

$$\begin{aligned}\|f\|_{BMO_{L^p(X)}} &\le c\left\|S^{(p)}(f)\right\|_{BMO_{L^p(X)}}\\ &\le c\sup_n\left\|E\left(\left\|S^{(p)}(f)-S^{(p)}_{n-1}(f)\right\|^p|\Sigma_n\right)\right\|_{L_p}\\ &\le c\sup_n\left\|E\left(\left\|S^{(p)}(f)^p-S^{(p)}_{n-1}(f)^p\right\||\Sigma_n\right)\right\|_{L_p}\\ &\le c\sup_n\left\|E\left(\sum_{k=n}^{\infty}\left\|E_k(a_k)-E_{k-1}(a_k)\right\|^p\Big|\Sigma_n\right)\right\|_{L_p}\\ &\le c\left\|E\left(\left(\sum_{k=n}^{\infty}\|a_k\|^p\right)\Big|\Sigma_n\right)\right\|_{L^p}\le c.\end{aligned}$$

On the contrary, we define $\hat{f}^\tau = (\hat{f}_i^\tau,)$, where $\hat{\Sigma}_i = \Sigma_{\tau+i}, \hat{f}_i^\tau = f_{\tau+i} - f_\tau$, $i \ge 0$. So, we have

$$\left(S^{(p)}\left(\hat{f}^\tau\right)\right)^p = S^{(p)}(f)^p - S^{(p)}_\tau(f)^p.$$

Suppose that X has an equivalent p-uniformly smooth norm. Then by Pisier's theorem, we have

$$\left\|\hat{f}^\tau\right\|_p \le c\left\|(S^{(p)}\left(\hat{f}^\tau\right)\right\|_p,$$

i.e.,

$$E\left(\|f - f\tau\|^p\right) \le E\left(S^{(p)}(f)^p - S^{(p)}_\tau(f)^p\right).$$

Moreover, we conditionalize it, we will get

$$E\left(\|f - f\tau\|^p|\Sigma_{\tau+1}\right) \le E\left(S^{(p)}(f)^p - S^{(p)}_\tau(f)^p|\Sigma_{\tau+1}\right).$$

By the definition and Theorem 1.7, we get

$$\|f\|_{BMO^a_{L^{p,q}(X)}} \le c\left\|S^{(p)}(f)\right\|_{BMO^a_{L^{p,q}(X)}}.$$

Thus we complete the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, China. [2]College of Science, Wuhan University of Science and Technology, Wuhan, Hubei 430065, China.

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