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(λ, μ)-Fuzzy linear subspaces

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Abstract
In this paper, we first introduce the concepts of (λ, μ)-fuzzy subfields. Then we generalize the concepts of fuzzy linear spaces, we define (λ, μ)-fuzzy linear subspaces, and we obtain some of their fundamental properties. Lastly, we list some possible directions of the extending of the present work.

Keywords: (λ, μ)-fuzzy subfield; (λ, μ)-fuzzy linear subspace; homomorphism; linear transformation

1 Introduction and preliminaries

The concept of fuzzy sets was first introduced by Zadeh [1] in 1965, and then the fuzzy sets have been used in the reconsideration of classical mathematics. Recently, Yuan [2] introduced the concept of fuzzy subgroup with thresholds. A fuzzy subgroup with thresholds λ and μ is also called a (λ, μ)-fuzzy subgroup. Yao continued to research (λ, μ)-fuzzy normal subgroups, (λ, μ)-fuzzy quotient subgroups and (λ, μ)-fuzzy subrings in [3–5].

Nanda [6] introduced the concepts of fuzzy field and fuzzy linear space and gave some results. Biswas [7] pointed out that Proposition 4.1 of [6] was incorrect and redefined fuzzy field and fuzzy linear space. Gu and Lu [8] listed two examples to show that Proposition 4.5 in [6] is also incorrect and redefined the concept of fuzzy linear space.

In this paper, we first introduce the concepts of (λ, μ)-fuzzy subfields. Next, we generalize the concepts of fuzzy linear spaces over fuzzy fields to (λ, μ)-fuzzy linear subspaces over (λ, μ)-fuzzy subfields, and give some fundamental properties.

Let us recall some definitions and notions.

By a fuzzy subset of a nonempty set X we mean a mapping from X to the unit interval $[0,1]$. If A is a fuzzy subset of X, then we denote $A_\alpha = \{x \in X | A(x) \geq \alpha\}$ for all $\alpha \in [0,1]$.

Throughout this paper, we always assume that $0 \leq \lambda < \mu \leq 1$.

2 (λ, μ)-fuzzy subfields

Definition 1 A fuzzy subset F of a field $\mathbf{F}$ is said to be a (λ, μ)-fuzzy subfield of $\mathbf{F}$ if $\forall k, l \in \mathbf{F}$, we have

(1) $F(k + l) \vee \lambda \geq F(k) \wedge F(l) \wedge \mu$;
(2) $F(-k) \vee \lambda \geq F(k) \wedge \mu$;
(3) $F(kl) \vee \lambda \geq F(k) \wedge F(l) \wedge \mu$;
(4) $F(k^{-1}) \vee \lambda \geq F(k) \wedge \mu$, where $k \neq 0$.

From the previous definition, we can easily conclude that a fuzzy subfield defined by Gu [8, 9] or Biswas [7] is a $(0,1)$-fuzzy subfield.

Proposition 1 *Let F be a (λ,μ)-fuzzy subfield of a field $\mathbf{F}$, then $\forall k \in \mathbf{F}$, we have*

(1) $F(0) \vee \lambda \geq F(k) \wedge \mu$;
(2) $F(1) \vee \lambda \geq F(k) \wedge \mu$, *where $k \neq 0$*;
(3) $F(0) \vee \lambda \geq F(1) \wedge \mu$.

Proof $F(0) \vee \lambda = F(k + (-k)) \vee \lambda \vee \lambda \geq (F(k) \wedge F(-k) \wedge \mu) \vee \lambda = (F(k) \vee \lambda) \wedge (F(-k) \vee \lambda) \wedge (\mu \vee \lambda) \geq F(k) \wedge (F(k) \wedge \mu) \wedge \mu = F(k) \wedge \mu$. Thus we complete the proof of (1).

(2) can be proved similarly and (3) is a corollary of (1). □

Theorem 1 *Let F be a fuzzy subset of a field $\mathbf{F}$. Then the following are equivalent*:

(1) *F is a (λ,μ)-fuzzy subfield of $\mathbf{F}$*;
(2) *F_α is a subfield of $\mathbf{F}$, for any $\alpha \in (\lambda,\mu]$, where $F_\alpha \neq \emptyset$.*

Proof It is a corollary of Proposition 2.4 of [4]. □

We use $\mathbf{F_1}$, $\mathbf{F_2}$ to stand for two fields in the following and define $\sup \emptyset = 0$, where $\emptyset$ is the empty set.

Proposition 2 *Let $f : \mathbf{F_1} \to \mathbf{F_2}$ be a homomorphism, and let F_1 be a (λ,μ)-fuzzy subfield of $\mathbf{F_1}$. Then $f(F_1)$ is a (λ,μ)-fuzzy subfield of $\mathbf{F_2}$, where*

$$f(F_1)(y) = \sup_{x \in \mathbf{F_1}} \{F_1(x) | f(x) = y\}, \quad \forall y \in \mathbf{F_2}.$$

Proof Similar to the proof of Proposition 2.7 in [4]. □

Proposition 3 *Let $f : \mathbf{F_1} \to \mathbf{F_2}$ be a homomorphism, and let F_2 be a (λ,μ)-fuzzy sublattice of $\mathbf{F_2}$. Then $f^{-1}(F_2)$ is a (λ,μ)-fuzzy sublattice of $\mathbf{F_1}$, where*

$$f^{-1}(F_2)(x) = F_2(f(x)), \quad \forall x \in \mathbf{F_1}.$$

Proof Similar to the proof of Proposition 2.8 in [4]. □

3 (λ,μ)-fuzzy linear subspaces

Definition 2 Let F be a (λ,μ)-fuzzy subfield of a field $\mathbf{F}$, $\mathbf{V}$ be a linear space over $\mathbf{F}$ and V be a fuzzy subset of $\mathbf{V}$. V is called a (λ,μ)-fuzzy linear subspace of $\mathbf{V}$ over F if for all $x, y \in \mathbf{V}$, $k \in \mathbf{F}$, we have

(1) $V(x + y) \vee \lambda \geq V(x) \wedge V(y) \wedge \mu$;
(2) $V(-x) \vee \lambda \geq V(x) \wedge \mu$;
(3) $V(kx) \vee \lambda \geq F(k) \wedge V(x) \wedge \mu$;
(4) $F(1) \vee \lambda \geq V(x) \wedge \mu$.

Obviously, the previous definition is a generalization of fuzzy linear space defined by Gu and Tu (see Definition 3.1 in [9]).

Theorem 2 *Let F be a (λ,μ)-fuzzy subfield of a field $\mathbf{F}$, let $\mathbf{V}$ be a linear space over $\mathbf{F}$, and let V be a fuzzy subset of $\mathbf{V}$. V is a (λ,μ)-fuzzy linear subspace of $\mathbf{V}$ over F if and only if, for all $x, y \in \mathbf{V}$, $k, l \in \mathbf{F}$, we have*

(1) $V(kx + ly) \vee \lambda \geq F(k) \wedge V(x) \wedge F(l) \wedge V(y) \wedge \mu$;
(2) $F(1) \vee \lambda \geq V(x) \wedge \mu$.

Proof '⇒'
For all $x, y \in \mathbf{V}$, $k, l \in \mathbf{F}$, we have

$$\begin{aligned} V(kx + ly) \vee \lambda &= V(kx + ly) \vee \lambda \vee \lambda \geq \big(V(kx) \wedge V(ly) \wedge \mu\big) \vee \lambda \\ &= \big(V(kx) \vee \lambda\big) \wedge \big(V(ly) \vee \lambda\big) \wedge (\mu \vee \lambda) \\ &\geq F(k) \wedge V(x) \wedge F(l) \wedge V(y) \wedge \mu. \end{aligned}$$

'⇐'
From $F(1) \vee \lambda \geq V(x) \wedge \mu$, we know that $\lambda \geq V(x) \wedge \mu$ or $F(1) \geq V(x) \wedge \mu$. Two cases are possible:
Case 1. If $\lambda \geq V(x) \wedge \mu$, then
(1) $V(x + y) \vee \lambda \geq \lambda \geq V(x) \wedge \mu \geq V(x) \wedge V(y) \wedge \mu$;
(2) $V(-x) \vee \lambda \geq \lambda \geq V(x) \wedge \mu$;
(3) $V(kx) \vee \lambda \geq \lambda \geq V(x) \wedge \mu \geq F(k) \wedge V(x) \wedge \mu$.
Case 2. $F(1) \geq V(x) \wedge \mu$, then
(1) $V(x + y) \vee \lambda = V(1x + 1y) \vee \lambda \geq F(1) \wedge V(x) \wedge F(1) \wedge V(y) \wedge \mu = V(x) \wedge V(y) \wedge \mu$;
(2) $V(-x) \vee \lambda = V(-1x + 0x) \vee \lambda \vee \lambda \geq (F(-1) \wedge V(x) \wedge F(0) \wedge V(x) \wedge \mu) \vee \lambda \geq (F(-1) \vee \lambda) \wedge (V(x) \vee \lambda) \wedge (F(0) \vee \lambda) \wedge (\mu \vee \lambda) \geq (F(1) \wedge \mu) \wedge V(x) \wedge (F(1) \wedge \mu) \wedge \mu = V(x) \wedge \mu$.
(3) $V(kx) \vee \lambda = V(kx + 0x) \vee \lambda \vee \lambda \geq (F(k) \wedge V(x) \wedge F(0) \wedge V(x) \wedge \mu) \vee \lambda = (F(k) \vee \lambda) \wedge (V(x) \vee \lambda) \wedge (F(0) \vee \lambda) \wedge (\mu \vee \lambda) \geq F(k) \wedge V(x) \wedge (F(1) \wedge \mu) \wedge \mu = F(k) \wedge V(x) \wedge \mu$.
□

Theorem 3 *Let F be a (λ, μ)-fuzzy subfield of a field* $\mathbf{F}$, *let* $\mathbf{V}$ *be a linear space over* $\mathbf{F}$, *and let V be a fuzzy subset of* $\mathbf{V}$. *Then the following are equivalent*:
(1) *V is a (λ, μ)-fuzzy linear subspace over F*;
(2) *V_α is a linear subspace over F_α, for any $\alpha \in (\lambda, \mu]$, where $F_\alpha \neq \emptyset$ and $V_\alpha \neq \emptyset$.*

Proof '(1) ⇒ (2)'
Let V be a (λ, μ)-fuzzy linear subspace over F.
Take any $\alpha \in (\lambda, \mu]$ such that $F_\alpha \neq \emptyset$ and $V_\alpha \neq \emptyset$; we need to show that $kx + ly \in V_\alpha$, $\forall x, y \in V_\alpha$, $\forall k, l \in F_\alpha$.
From $F(k) \geq \alpha$, $F(l) \geq \alpha$, $V(x) \geq \alpha$, $V(y) \geq \alpha$, and $\alpha \leq \mu$, we conclude that $V(kx + ly) \vee \lambda \geq F(k) \wedge V(x) \wedge F(l) \wedge V(y) \wedge \mu \geq \alpha \wedge \mu = \alpha$. Note that $\lambda < \alpha$, we obtain that $V(kx+ly) \geq \alpha$. So $kx + ly \in V_\alpha$.
'(1) ⇐ (2)'
Conversely, let V_α be a linear subspace over F_α, for any $\alpha \in (\lambda, \mu]$. If there exist $x, y \in V$ and $k, l \in F$ such that $V(kx + ly) \vee \lambda < \alpha = F(k) \wedge V(x) \wedge F(l) \wedge V(y) \wedge \mu$, then $\alpha \in (\lambda, \mu]$, $x, y \in V_\alpha$, $k, l \in F_\alpha$. But $kx + ly \notin A_\alpha$. This is a contradiction with that V_α is a linear subspace over F_α.
Again, if there exists $x \in V$ such that $F(1) \vee \lambda < \alpha = V(x) \wedge \mu$, then $\alpha \in (\lambda, \mu]$, $x \in V_\alpha$ and $1 \notin F_\alpha$. This is a contradiction to that V_α is a linear subspace over a subfield F_α of $\mathbf{F}$. □

We use $\mathbf{V_1}$, $\mathbf{V_2}$ to stand for two linear spaces over the same field $\mathbf{F}$ in the following.

Proposition 4 *Let F be a (λ,μ)-fuzzy subfield of a field $\mathbf{F}$, let $f:\mathbf{V_1}\to\mathbf{V_2}$ be a linear transformation over $\mathbf{F}$, and V_1 be a (λ,μ)-fuzzy linear subspace of $\mathbf{V_1}$ over F. Then $f(V_1)$ is a (λ,μ)-fuzzy linear subspace of $\mathbf{V_2}$ over F, where*

$$f(V_1)(y)=\sup_{x\in\mathbf{V_1}}\bigl\{V_1(x)|f(x)=y\bigr\},\quad \forall y\in\mathbf{V_2}.$$

Proof If $f^{-1}(y_1)=\emptyset$ or $f^{-1}(y_2)=\emptyset$ for any $y_1,y_2\in V_2$, then $(f(V_1)(ky_1+ly_2))\vee\lambda\geq 0=F(k)\wedge f(V_1)(y_1)\wedge F(l)\wedge f(V_1)(y_2)\wedge\mu$.

Suppose that $f^{-1}(y_1)\neq\emptyset, f^{-1}(y_2)\neq\emptyset$ for any $y_1,y_2\in V_2$, then $f^{-1}(ky_1+ly_2)\neq\emptyset$. So for any $k,l\in F$, we have

$$\begin{aligned}
&f(V_1)(ky_1+ly_2)\vee\lambda\\
&\quad=\sup_{t\in\mathbf{V_1}}\bigl\{A(t)|f(t)=ky_1+ly_2\bigr\}\vee\lambda\\
&\quad=\sup_{t\in\mathbf{V_1}}\bigl\{V_1(t)\vee\lambda|f(t)=ky_1+ly_2\bigr\}\\
&\quad\geq\sup_{x_1,x_2\in\mathbf{V_1}}\bigl\{\bigl(V_1(kx_1+lx_2)\bigr)\vee\lambda|f(x_1)=y_1,f(x_2)=y_2\bigr\}\\
&\quad\geq\sup_{x_1,x_2\in\mathbf{V_1}}\bigl\{\bigl(F(k)\wedge V_1(x_1)\wedge F(l)\wedge V_1(x_2)\bigr)\wedge\mu|f(x_1)=y_1,f(x_2)=y_2\bigr\}\\
&\quad=\Bigl(\sup_{x_1\in\mathbf{V_1}}\bigl\{V_1(x_1)|f(x_1)=y_1\bigr\}\wedge\sup_{x_2\in V_1}\bigl\{V_1(x_2)|f(x_2)=y_2\bigr\}\Bigr)\wedge F(k)\wedge F(l)\wedge\mu\\
&\quad=f(V_1)(y_1)\wedge f(V_1)(y_2)\wedge F(k)\wedge F(l)\wedge\mu.
\end{aligned}$$

And for all $y\in\mathbf{V_2}$, we have

$$\begin{aligned}
F(1)\vee\lambda&=\sup_{x\in\mathbf{V_1}}\bigl\{F(1)\vee\lambda|V_1(x)=y\bigr\}\\
&\geq\sup_{x\in\mathbf{V_1}}\bigl\{V_1(x)\wedge\mu|V_1(x)=y\bigr\}\\
&=\sup_{x\in\mathbf{V_1}}\bigl\{V_1(x)|V_1(x)=y\bigr\}\wedge\mu\\
&=f(V_1)(y)\wedge\mu.
\end{aligned}$$

Thus $f(V_1)$ is a (λ,μ)-fuzzy linear subspace of $\mathbf{V_2}$ over F. □

Proposition 5 *Let F be a (λ,μ)-fuzzy subfield of a field $\mathbf{F}$, let $f:\mathbf{V_1}\to\mathbf{V_2}$ be a linear transformation over $\mathbf{F}$, and let V_2 be a (λ,μ)-fuzzy linear subspace of $\mathbf{V_2}$ over F. Then $f^{-1}(V_2)$ is a (λ,μ)-fuzzy linear subspace of $\mathbf{V_1}$ over F, where*

$$f^{-1}(V_2)(x)=V_2\bigl(f(x)\bigr),\quad\forall x\in\mathbf{V_1}.$$

Proof For any $x_1,x_2\in V_1$ and $k,l\in F$, we have

$$\begin{aligned}
f^{-1}(V_2)(kx_1+lx_2)\vee\lambda&=V_2\bigl(f(kx_1+lx_2)\bigr)\vee\lambda\\
&=V_2\bigl(kf(x_1)+lf(x_2)\bigr)\vee\lambda
\end{aligned}$$

$$\geq F(k) \wedge V_2\big(f(x_1)\big) \wedge F(l) \wedge V_2\big(f(x_2)\big) \wedge \mu$$
$$= F(k) \wedge f^{-1}(V_2)(x_1) \wedge F(l) \wedge f^{-1}(V_2)(x_2) \wedge \mu.$$

And for all $x \in V_1$, we have

$$f^{-1}(V_2)(x) \wedge \mu = V_2\big(f(x)\big) \wedge \mu \leq F(1) \vee \lambda.$$

So, $f^{-1}(V_2)$ is a (λ, μ)-fuzzy linear subspace of V_1 over F. □

4 Further research

The present work can be extended in several directions. Let us indicate some possibilities.

1. One can define (λ, μ)-fuzzy hypervector spaces and study their properties (definitions of fuzzy hypervector spaces can be found in [10]).
2. One may give the definition of (λ_1, μ_1)-fuzzy linear subspaces over (λ_2, μ_2)-fuzzy subfields, where $0 \leq \lambda_1 < \mu_1 \leq 1$ and $0 \leq \lambda_2 < \mu_2 \leq 1$. Then explore the properties of them.
3. One can investigate the interval-valued (type 2, lattice-valued, *etc.*) (λ_1, μ_1)-fuzzy linear subspaces.
4. One can also research (λ, μ)-anti-fuzzy linear subspaces [11].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CL first gave the idea of generalizing the fuzzy linear space to (λ, μ)-fuzzy linear subspace, YF gave some good ideas on improving the paper, and he finished the latex version of the paper. All authors read and approved the final manuscript.

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