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Some results for terminating ${}_2F_1(2)$ series

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Abstract

The aim of this research paper is to find explicit expressions of

$${}_2F_1[-n, a; 2a+i; 2]$$

and

$${}_2F_1\left[\begin{matrix}-n, a\\ 2n+i\end{matrix}; 2\right]$$

each for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. Known results earlier obtained by Kim *et al.* and Chu follow special cases of our main findings. The results are derived with the help of generalizations of Gauss second, Kummer and Bailey summation theorems for the series ${}_2F_1$ obtained earlier by Lavoie *et al.*

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1 Introduction

The generalized hypergeometric functions with p numeratorial and q denominatorial parameters are defined by [1]

$$\begin{aligned}{}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p\\ \beta_1, \ldots, \beta_q\end{matrix}; z\right] &= {}_pF_q[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z]\\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!},\end{aligned} \tag{1.1}$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined for any complex number α by

$$(\alpha)_n = \begin{cases}\alpha(\alpha+1)\cdots(\alpha+n-1), & \text{if } n \in \mathbb{N} = \{1, 2, \ldots\},\\ 1, & \text{if } n = 0.\end{cases}$$

Using the fundamental functional relation $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $(\alpha)_n$ can be written as

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \quad \bigl(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\bigr),$$

where Γ is the well-known gamma function.

It is well known that the series ${}_pF_q$ converges for all z if $p \leq q$ and for $|z| < 1$ if $p = q + 1$, and the series diverges for all $z \neq 0$ if $p > q + 1$. The convergence of the series for the case $|z| = 1$ when $p = q + 1$ is of much interest.

The series ${}_{q+1}F_q[\alpha_1, \ldots, \alpha_{q+1}; \beta_1, \ldots, \beta_q;\ z]$ *with* $|z| = 1$ *converges absolutely if* $(\sum \beta_j - \sum \alpha_j) > 0$.

The series converges conditionally if $z \neq 1$ *and* $0 \geq (\sum \beta_j - \sum \alpha_j) > -1$. *The series diverges if* $(\sum \beta_j - \sum \alpha_j) \leq -1$.

It should be remarked here that whenever the hypergeometric function ${}_2F_1$ and the generalized hypergeometric functions ${}_pF_q$ can be summed in terms of gamma functions, the results are very important from the application point of view. This function has been extensively studied by many authors (see, *e.g.*, Slater [2] and Exton [3]). We begin by recalling the well-known and classical Gauss second summation theorem (see [4])

$$ {}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+1) \end{matrix}; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2})}, \tag{1.2} $$

the Kummer summation theorem (see [4])

$$ {}_2F_1\left[\begin{matrix} a, b \\ 1+a-b \end{matrix}; -1\right] = \frac{\Gamma(1 + \frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(1 + \frac{1}{2}a - b)} \tag{1.3} $$

provided $\Re(b) < 1$ for convergence, and the Bailey summation theorem (see [4])

$$ {}_2F_1\left[\begin{matrix} a, 1-a \\ c \end{matrix}; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2}c)\Gamma(\frac{1}{2}c + \frac{1}{2})}{\Gamma(\frac{1}{2}c + \frac{1}{2}a)\Gamma(\frac{1}{2}c - \frac{1}{2}a + \frac{1}{2})}. \tag{1.4} $$

In 1996, Lavoie, Grondin and Rathie [5] generalized the above mentioned classical summation theorems in the following form.

Generalization of the Gauss second summation theorem:

$$ \begin{aligned} &{}_2F_1\left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+i+1) \end{matrix}; \frac{1}{2}\right] \\ &\quad = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}i + \frac{1}{2})\Gamma(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}i + \frac{1}{2})}{\Gamma(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2} + \frac{1}{2}|i|)} \\ &\qquad \times \left\{ \frac{A_i}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2}i + \frac{1}{2} - [\frac{1+i}{2}])} + \frac{B_i}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b + \frac{1}{2}i - [\frac{i}{2}])} \right\} \end{aligned} \tag{1.5} $$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$.

As usual, $[x]$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$. For $i = 0$, we get Gauss second summation theorem (1.2). The coefficients A_i and B_i are given in Table 1.

Table 1 The coefficients A_i and B_i for $-5 \le i \le 5$

i	A_i	B_i
5	$-[(a+b+6)^2 - \frac{1}{4}(a-b-6)^2$ $+ \frac{1}{2}(a+b+6)(a-b-6) - 11(a+b+6)$ $- \frac{13}{2}(a-b-6) + 20]$	$(a+b+6)^2 - \frac{1}{4}(a-b-6)^2$ $- \frac{1}{2}(a+b+6)(a-b-6) - 17(a+b+6)$ $+ \frac{1}{2}(a-b-6) + 62$
4	$\frac{1}{2}(a+b+1)(a+b-3)$ $- \frac{1}{4}(a-b+3)(a-b-3)$	$-2(a+b-1)$
3	$-\frac{1}{2}(3a+b-2)$	$\frac{1}{2}(a+3b-2)$
2	$\frac{1}{2}(a+b-1)$	-2
1	-1	1
0	1	0
-1	1	1
-2	$\frac{1}{2}(a+b-1)$	2
-3	$\frac{1}{2}(3a+b-2)$	$\frac{1}{2}(a+3b-2)$
-4	$\frac{1}{2}(a+b+1)(a+b-3)$ $- \frac{1}{4}(a-b+3)(a-b-3)$	$2(a+b-1)$
-5	$(a+b-4)^2 - \frac{1}{4}(a-b+4)^2$ $+ \frac{1}{2}(a+b-4)(a-b+4)$ $+ 4(a+b-4) + \frac{7}{2}(a-b+4)$	$(a+b-4)^2 - \frac{1}{4}(a-b+4)^2$ $- \frac{1}{2}(a+b-4)(a-b+4)$ $+ 8(a+b-4) + \frac{1}{2}(a-b+4) + 12$

Table 2 The coefficients C_i and D_i for $-5 \le i \le 5$

i	C_i	D_i
5	$-4(a-b+6)^2 + 2b(a-b+6) + b^2$ $+ 22(a-b+6) - 13b - 20$	$4(a-b+6)^2 + 2b(a-b+6) - b^2$ $- 34(a-b+6) - b + 62$
4	$2(a-b+3)(a-b+1) - (b-1)(b-4)$	$-4(a-b+2)$
3	$(3b-2a-5)$	$(2a-b+1)$
2	$(a-b+1)$	-2
1	-1	1
0	1	0
-1	1	1
-2	$(a-b-1)$	2
-3	$(2a-3b-4)$	$(2a-b-2)$
-4	$2(a-b-3)(a-b-1) - b(b+3)$	$4(a-b-2)$
-5	$4(a-b-4)^2 - 2b(a-b-4)^2 - b^2$ $+ 8(a-b-4) - 7b$	$4(a-b-4)^2 + 2b(a-b-4) - b^2$ $+ 16(a-b-4) - b + 12$

The generalization of the Kummer summation theorem:

$$
{}_2F_1\left[\begin{matrix} a, b \\ 1+a-b+i \end{matrix}; -1\right] = \frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(1-b)\Gamma(1+a-b+i)}{\Gamma(1-b+\frac{1}{2}(i+|i|))} \times \left\{ \frac{C_i}{\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+1)\Gamma(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])} + \frac{D_i}{\Gamma(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}i-[\frac{i}{2}])} \right\} \tag{1.6}
$$

provided $\Re(b) < 1 + \frac{i}{2}$ for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. For $i = 0$, we get Kummer summation theorem (1.3). The coefficients C_i and D_i are given in Table 2.

The generalization of the Bailey summation theorem:

$$
{}_2F_1\left[\begin{matrix} a, 1-a+i \\ b \end{matrix}; \frac{1}{2}\right] = \frac{2^{1+i-b}\Gamma(\frac{1}{2})\Gamma(b)\Gamma(1-a)}{\Gamma(1-a+\frac{1}{2}(i+|i|))} \left\{ \frac{E_i}{\Gamma(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2}a-[\frac{1+i}{2}])} + \frac{F_i}{\Gamma(\frac{1}{2}b-\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2}a-\frac{1}{2}-[\frac{i}{2}])} \right\} \tag{1.7}
$$

Table 3 The coefficients E_i and F_i for $-5 \le i \le 5$

i	E_i	F_i
5	$-(4b^2-2ab-a^2-22b+13a+20)$	$4b^2+2ab-a^2-34b-a+62$
4	$2(b-2)(b-4)-(a-1)(a-4)$	$-4(b-3)$
3	$(a-2b+3)$	$(a+2b-7)$
2	$b-2$	-2
1	-1	1
0	1	0
-1	1	1
-2	b	2
-3	$2b-a$	$a+2b+2$
-4	$2b(b+2)-a(a+3)$	$4(b+1)$
-5	$(4b^2-2ab-a^2+8b-7a)$	$(4b^2+2ab-a^2+16b-a+12)$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. For $i = 0$, we get Kummer summation theorem (1.3). The coefficients E_i and F_i are given in Table 3.

On the other hand, in the investigation of a model of biological junction with quantum-like characteristic based upon Toeplitz operators, Samoletov [6] obtained, by using the principle of mathematical induction, the following sum containing factorials

$$\sum_{k=0}^{n} \frac{(-1)^k(2k+1)!!}{(n-k)!k!(k+1)!} = \frac{(-1)^n}{\sqrt{n!(n+1)!}} \left(\sqrt{n+1}\frac{(n-1)!}{n!}\right)^{(-1)^n} \quad (n \in \mathbb{N}_0), \tag{1.8}$$

where, as usual

$$\begin{aligned} &(2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1) = \frac{(2n+1)!}{2^n n!}, \\ &(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!, \\ &0!! = (-1)!! = 1. \end{aligned}$$

In terms of the familiar gamma function and the Gauss hypergeometric function, Samoletov [6] also rewrote the sum (1.8) in its equivalent form

$${}_2F_1\left[\begin{matrix} -n, \frac{3}{2} \\ 2 \end{matrix}; 2\right] = \frac{1}{\sqrt{\pi}} \begin{cases} \frac{\Gamma(\frac{1}{2}n+\frac{1}{2})}{\Gamma(\frac{1}{2}n+1)}, & \text{if } n \text{ is even}, \\ -\frac{\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2}n+\frac{3}{2})}, & \text{if } n \text{ is odd}. \end{cases}$$

Later on, Srivastava [7] pointed out that result (1.8) can be obtained quite simply from the known result [8]

$$\begin{aligned} &{}_2F_1\left[\begin{matrix} -n, a \\ 2a-1 \end{matrix}; 2\right] \\ &\quad = \frac{\Gamma(a-\frac{1}{2})}{\sqrt{\pi}} \left\{ \frac{1+(-1)^n}{2} \frac{\Gamma(\frac{1}{2}n+\frac{1}{2})}{\Gamma(a+\frac{1}{2}n-\frac{1}{2})} - \frac{1-(-1)^n}{2} \frac{\Gamma(\frac{1}{2}n+1)}{\Gamma(a+\frac{1}{2}n)} \right\} \end{aligned} \tag{1.9}$$

by setting $a = \frac{3}{2}$. In the same paper, Srivastava [7] obtained for $a = \frac{3}{2}$, equivalent expressions of (1.9) by using Plaff-Kummer transformation and Euler transformation.

The aim of this paper is to find explicit expressions of

$$ {}_2F_1\left[\begin{matrix}-n, a\\ 2a+i\end{matrix}; 2\right] $$

and

$$ {}_2F_1\left[\begin{matrix}-n, a\\ 2n+i\end{matrix}; 2\right] $$

each for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. The results are derived with the help of the generalized Gauss second summation theorem, the generalized Kummer summation theorem and the generalized Bailey summation theorem for the series ${}_2F_1$ obtained earlier by Lavoie *et al.* Several known as well as new results have also been obtained from our main findings.

2 Main results

The results to be established are

$$ \begin{aligned} &{}_2F_1\left[\begin{matrix}-n, a\\ 2a+i\end{matrix}; 2\right] \\ &\quad = \frac{2^n\Gamma(\frac{1}{2})\Gamma(a)\Gamma(1-a)}{(2a+i)_n\Gamma(a+\frac{1}{2}(i+|i|))} \\ &\qquad \times \left\{\frac{A_i}{\Gamma(-\frac{1}{2}n+\frac{1}{2})\Gamma(1-a-\frac{1}{2}n-[\frac{1+i}{2}])} + \frac{B_i}{\Gamma(-\frac{1}{2}n)\Gamma(\frac{1}{2}-a-\frac{1}{2}n-[\frac{i}{2}])}\right\} \end{aligned} \tag{2.1} $$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. The coefficients A_i and B_i can be obtained from Table 1 by changing a by $-n$ and b by $1 - 2a - i - n$, respectively, and

$$ \begin{aligned} &{}_2F_1\left[\begin{matrix}-n, a\\ -2n-i\end{matrix}; 2\right] \\ &\quad = \frac{2^{2n+i}\Gamma(n+1)\Gamma(n+i+1)}{\Gamma(2n+i+1)\Gamma(n+1+\frac{1}{2}(i+|i|))} \\ &\qquad \times \left\{E_i\frac{\Gamma(\frac{1}{2}-\frac{1}{2}a)(\frac{1}{2}+\frac{1}{2}a+[\frac{1+i}{2}])_n}{\Gamma(\frac{1}{2}-\frac{1}{2}a-[\frac{1+i}{2}])} + F_i\frac{\Gamma(1-\frac{1}{2}a)(1+\frac{1}{2}a+[\frac{i}{2}])_n}{\Gamma(-\frac{1}{2}a-[\frac{i}{2}])}\right\} \end{aligned} \tag{2.2} $$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. The coefficients E_i and F_i can be obtained from Table 3 by changing a by $-n$ and b by $1 - a - n$, respectively.

3 Proofs of (2.1) and (2.2)

In order to derive (2.1), we proceed as follows. Expressing ${}_2F_1$ as a series, we have

$$ {}_2F_1\left[\begin{matrix}-n, a\\ 2a+i\end{matrix}; 2\right] = \sum_{r=0}^{n}\frac{(-n)_r(a)_r}{(2a+i)_r}\frac{2^r}{r!} $$

On reversing the series, we have

$$ {}_2F_1\left[\begin{matrix}-n, a\\ 2a+i\end{matrix}; 2\right] = \sum_{r=0}^{n} \frac{(-n)_{n-r}(a)_{n-r}}{(2a+i)_{n-r}} \frac{2^{n-r}}{(n-r)!}. $$

Using the well-known identities

$$ (a)_{n-r} = \frac{(-1)^r (a)_n}{(1-a-n)_r} $$

and

$$ (n-r)! = \frac{(-1)^r n!}{(-n)_r} \quad (0 \leqq r \leqq n), $$

we have, after a little algebra,

$$ {}_2F_1\left[\begin{matrix}-n, a\\ 2a+i\end{matrix}; 2\right] = \frac{(-2)^n (a)_n}{(2a+i)_n} \sum_{r=0}^{n} \frac{(-n)_r (1-2a-i-n)_r}{(1-a-n)_r 2^r r!}. $$

Summing up the series, we finally have

$$ {}_2F_1\left[\begin{matrix}-n, a\\ 2a+i\end{matrix}; 2\right] = \frac{(-2)^n (a)_n}{(2a+i)_n} {}_2F_1\left[\begin{matrix}-n, 1-2a-i-n\\ 1-a-n\end{matrix}; \frac{1}{2}\right]. \tag{3.1} $$

Now it can be easily seen that the ${}_2F_1$ on the right-hand side can be evaluated with the help of generalized Gauss second summation theorem (1.5) and after a little simplification, we easily arrive at the right-hand side of (2.1). Further, in (3.1), if we use the Plaff-transformation [1]

$$ {}_2F_1\left[\begin{matrix}a, b\\ c\end{matrix}; z\right] = (1-z)^{-a} {}_2F_1\left[\begin{matrix}a, c-b\\ c\end{matrix}; -\frac{z}{1-z}\right], \tag{3.2} $$

it takes the following form

$$ {}_2F_1\left[\begin{matrix}-n, a\\ 2a+i\end{matrix}; 2\right] = \frac{(-2)^n (a)_n}{(2a+i)_n} {}_2F_1\left[\begin{matrix}-n, a+i\\ 1-a-n\end{matrix}; -1\right]. \tag{3.3} $$

The ${}_2F_1$ on the right-hand side can now be evaluated with the help of generalized Kummer summation theorem (1.6) and after a little algebra, we obtain the following equivalent form of our first result (2.1)

$$ \begin{aligned} {}_2F_1\left[\begin{matrix}-n, a\\ 2a+i\end{matrix}; 2\right] &= \frac{2^n \Gamma(\frac{1}{2})\Gamma(1-a-i)\Gamma(1-a)}{(2a+i)_n \Gamma(1-a-\frac{1}{2}(i-|i|))} \\ &\quad \times \left\{ \frac{C_i}{\Gamma(1-a-\frac{1}{2}i-\frac{1}{2}n)\Gamma(\frac{1}{2}+\frac{1}{2}i-\frac{1}{2}n-[\frac{1+i}{2}])} \right. \\ &\quad \left. + \frac{D_i}{\Gamma(\frac{1}{2}-\frac{1}{2}i-a-\frac{1}{2}n)\Gamma(\frac{1}{2}i-\frac{1}{2}n-[\frac{i}{2}])} \right\} \end{aligned} \tag{3.4} $$

for $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$. The coefficients C_i and D_i can be obtained from Table 2 of C_i and D_i by changing a by $-n$ and b by $a+i$, respectively. Similarly, in (3.1), if we use the Euler transformation [1]

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = (1-z)^{c-a-b} {}_2F_1\left[\begin{matrix} c-a, c-b \\ c \end{matrix}; z\right], \tag{3.5} $$

it takes the following form

$$ {}_2F_1\left[\begin{matrix} -n, a \\ 2a+i \end{matrix}; 2\right] = \frac{(-1)^n (2)^{-a-i} (a)_n}{(2a+i)_n} {}_2F_1\left[\begin{matrix} 1-a, a+i \\ 1-a-n \end{matrix}; \frac{1}{2}\right]. \tag{3.6} $$

The ${}_2F_1$ on the right-hand side can now be evaluated with the help of generalized Bailey summation theorem (1.7) and after a little simplification, we get the following equivalent form of our first result (2.1)

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} -n, a \\ 2a+i \end{matrix}; 2\right] = {} & \frac{2^n \Gamma(\frac{1}{2}) \Gamma(a) \Gamma(1-a)}{(2a+i)_n \Gamma(a + \frac{1}{2}(i+|i|))} \\ & \times \left\{ \frac{E_i}{\Gamma(-\frac{1}{2}n + \frac{1}{2}) \Gamma(1-a-\frac{1}{2}n - [\frac{1+i}{2}])} \right. \\ & \left. + \frac{F_i}{\Gamma(-\frac{1}{2}n) \Gamma(\frac{1}{2} - a - \frac{1}{2}n - [\frac{i}{2}])} \right\}. \end{aligned} \tag{3.7} $$

The coefficients E_i and F_i can be obtained from the table of E_i and F_i by changing a by $1-a$ and c by $1-a-n$, respectively. It is not out of place to mention here that in (2.1) or its equivalent forms (3.4) or (3.7), if we take $i = 0, \pm 1, \pm 2, \pm 3, \pm 4$, we get the following results

$$ {}_2F_1\left[\begin{matrix} -n, a \\ 2a \end{matrix}; 2\right] = \frac{2^n \Gamma(\frac{1}{2}) \Gamma(1-a)}{(2a)_n \Gamma(-\frac{1}{2}n + \frac{1}{2}) \Gamma(1-a-\frac{1}{2}n)}; \tag{3.8} $$

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} -n, a \\ 2a+1 \end{matrix}; 2\right] = {} & \frac{2^n \Gamma(\frac{1}{2}) \Gamma(1-a)}{a(2a+1)_n} \\ & \times \left\{ \frac{1}{\Gamma(-\frac{1}{2}n) \Gamma(\frac{1}{2} - a - \frac{1}{2}n)} - \frac{1}{\Gamma(-\frac{1}{2}n + \frac{1}{2}) \Gamma(-a - \frac{1}{2}n)} \right\}; \end{aligned} \tag{3.9} $$

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} -n, a \\ 2a-1 \end{matrix}; 2\right] = {} & \frac{2^n \Gamma(\frac{1}{2}) \Gamma(1-a)}{(2a-1)_n} \\ & \times \left\{ \frac{1}{\Gamma(-\frac{1}{2}n) \Gamma(\frac{3}{2} - a - \frac{1}{2}n)} + \frac{1}{\Gamma(-\frac{1}{2}n + \frac{1}{2}) \Gamma(1 - a - \frac{1}{2}n)} \right\}; \end{aligned} \tag{3.10} $$

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} -n, a \\ 2a+2 \end{matrix}; 2\right] = {} & -\frac{2^n \Gamma(\frac{1}{2}) \Gamma(1-a)}{a(a+1)(2a-1)_n} \\ & \times \left\{ \frac{a+n+1}{\Gamma(-\frac{1}{2}n + \frac{1}{2}) \Gamma(-a - \frac{1}{2}n)} + \frac{2}{\Gamma(-\frac{1}{2}n) \Gamma(-\frac{1}{2} - a - \frac{1}{2}n)} \right\}; \end{aligned} \tag{3.11} $$

$$
{}_2F_1\left[\begin{matrix}-n,a\\2a-2\end{matrix};2\right]=\frac{2^n\Gamma(\frac{1}{2})\Gamma(1-a)}{(2a-2)_n}\times\left\{\frac{2}{\Gamma(-\frac{1}{2}n)\Gamma(\frac{3}{2}-a-\frac{1}{2}n)}-\frac{(a+n-1)}{\Gamma(-\frac{1}{2}n+\frac{1}{2})\Gamma(2-a-\frac{1}{2}n)}\right\}; \tag{3.12}
$$

$$
{}_2F_1\left[\begin{matrix}-n,a\\2a+3\end{matrix};2\right]=\frac{2^n\Gamma(\frac{1}{2})\Gamma(1-a)}{a(a+1)(a+2)(2a+3)_n}\times\left\{\frac{(a+2n+2)}{\Gamma(-\frac{1}{2}n+\frac{1}{2})\Gamma(-1-a-\frac{1}{2}n)}-\frac{(3a+2n+4)}{\Gamma(-\frac{1}{2}n)\Gamma(\frac{3}{2}-a-\frac{1}{2}n)}\right\}; \tag{3.13}
$$

$$
{}_2F_1\left[\begin{matrix}-n,a\\2a-3\end{matrix};2\right]=-\frac{2^n\Gamma(\frac{1}{2})\Gamma(1-a)}{(2a-3)_n}\times\left\{\frac{(a+2n-1)}{\Gamma(-\frac{1}{2}n+\frac{1}{2})\Gamma(2-a-\frac{1}{2}n)}-\frac{(3a+2n+5)}{\Gamma(-\frac{1}{2}n)\Gamma(\frac{5}{2}-a-\frac{1}{2}n)}\right\}; \tag{3.14}
$$

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix}-n,a\\2a+4\end{matrix};2\right]&=\frac{2^n\Gamma(\frac{1}{2})\Gamma(1-a)}{a(a+1)(a+2)(a+3)(2a+4)_n}\\
&\quad\times\left\{\frac{(a^2+4an+5a+2n^2+8n+6)}{\Gamma(-\frac{1}{2}n+\frac{1}{2})\Gamma(-1-a-\frac{1}{2}n)}\right.\\
&\quad\left.+\frac{4(a+n+2)}{\Gamma(-\frac{1}{2}n)\Gamma(-\frac{3}{2}-a-\frac{1}{2}n)}\right\};
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix}-n,a\\2a-4\end{matrix};2\right]&=\frac{2^n\Gamma(\frac{1}{2})\Gamma(1-a)}{(2a-4)_n}\\
&\quad\times\left\{\frac{(a^2+4an-3a+2n^2-8n+2)}{\Gamma(-\frac{1}{2}n+\frac{1}{2})\Gamma(3-a-\frac{1}{2}n)}\right.\\
&\quad\left.+\frac{4(2-a-n)}{\Gamma(-\frac{1}{2}n)\Gamma(\frac{5}{2}-a-\frac{1}{2}n)}\right\}.
\end{aligned} \tag{3.16}
$$

We remark in passing that the results (3.8), (3.9) and (3.10) are recorded in [8] in another form which can be easily seen to be equivalent by using the reflection property of the gamma function. Other results (3.11) to (3.16) are believed to be new.

Special cases

In (2.1) or its equivalent forms (3.3) or (3.7), it is not difficult to obtain the following very interesting results:

$$
{}_2F_1\left[\begin{matrix}-2n,a\\2a\end{matrix};2\right]=\frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}; \tag{3.17}
$$

$$
{}_2F_1\left[\begin{matrix}-2n-1,a\\2a\end{matrix};2\right]=0; \tag{3.18}
$$

$$
{}_2F_1\left[\begin{matrix}-2n,a\\2a+1\end{matrix};2\right]=\frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n}; \tag{3.19}
$$

$$
{}_2F_1\left[\begin{matrix}-2n-1,a\\2a+1\end{matrix};2\right]=\frac{(\frac{3}{2})_n}{(2a+1)(a+\frac{3}{2})_n}; \tag{3.20}
$$

$$ {}_2F_1\left[\begin{matrix}-2n, a\\ 2a-1\end{matrix}; 2\right] = \frac{(\frac{1}{2})_n}{(a-\frac{1}{2})_n}; \tag{3.21}$$

$$ {}_2F_1\left[\begin{matrix}-2n-1, a\\ 2a-1\end{matrix}; 2\right] = -\frac{(\frac{3}{2})_n}{(2a-1)(a-\frac{1}{2})_n}; \tag{3.22}$$

$$ {}_2F_1\left[\begin{matrix}-2n, a\\ 2a+2\end{matrix}; 2\right] = \frac{(\frac{1}{2})_n(\frac{1}{2}a+\frac{3}{2})_n}{(a+\frac{3}{2})_n(\frac{1}{2}a+\frac{1}{2})_n}; \tag{3.23}$$

$$ {}_2F_1\left[\begin{matrix}-2n-1, a\\ 2a+2\end{matrix}; 2\right] = \frac{(\frac{3}{2})_n}{(a+1)(a+\frac{3}{2})_n}; \tag{3.24}$$

$$ {}_2F_1\left[\begin{matrix}-2n, a\\ 2a-2\end{matrix}; 2\right] = \frac{(\frac{1}{2})_n(\frac{1}{2}a+\frac{1}{2})_n}{(a-\frac{1}{2})_n(\frac{1}{2}a-\frac{1}{2})_n}; \tag{3.25}$$

$$ {}_2F_1\left[\begin{matrix}-2n-1, a\\ 2a-2\end{matrix}; 2\right] = -\frac{(\frac{3}{2})_n}{(a-1)(a-\frac{1}{2})_n}; \tag{3.26}$$

$$ {}_2F_1\left[\begin{matrix}-2n, a\\ 2a+3\end{matrix}; 2\right] = \frac{(\frac{1}{2})_n(\frac{1}{4}a+\frac{3}{2})_n}{(a+\frac{3}{2})_n(\frac{1}{4}a+\frac{1}{2})_n}; \tag{3.27}$$

$$ {}_2F_1\left[\begin{matrix}-2n-1, a\\ 2a+3\end{matrix}; 2\right] = \frac{3(\frac{3}{2})_n(\frac{3}{4}a+\frac{5}{2})_n}{(2a+3)(a+\frac{5}{2})_n(\frac{3}{4}a+\frac{3}{2})_n}; \tag{3.28}$$

$$ {}_2F_1\left[\begin{matrix}-2n, a\\ 2a-3\end{matrix}; 2\right] = \frac{(\frac{1}{2})_n(\frac{1}{4}a+\frac{3}{4})_n}{(a-\frac{3}{2})_n(\frac{1}{4}a-\frac{1}{4})_n}; \tag{3.29}$$

$$ {}_2F_1\left[\begin{matrix}-2n-1, a\\ 2a-3\end{matrix}; 2\right] = -\frac{3(\frac{3}{2})_n(\frac{3}{4}a+\frac{1}{4})_n}{(2a-3)(a-\frac{1}{2})_n(\frac{3}{4}a-\frac{3}{4})_n}; \tag{3.30}$$

$$ {}_2F_1\left[\begin{matrix}-2n, a\\ 2a+4\end{matrix}; 2\right] = \frac{(\frac{1}{2})_n}{(a+\frac{5}{2})_n}\left\{\frac{2(a+1)(\frac{1}{2}a+\frac{5}{2})_n}{(a+2)(\frac{1}{2}a+\frac{1}{2})_n} - \frac{a}{a+2}\right\}; \tag{3.31}$$

$$ {}_2F_1\left[\begin{matrix}-2n-1, a\\ 2a+4\end{matrix}; 2\right] = \frac{2(\frac{3}{2})_n(\frac{1}{2}a+\frac{5}{2})_n}{(a+2)(a+\frac{5}{2})_n(\frac{1}{2}a+\frac{3}{2})_n}; \tag{3.32}$$

$$ {}_2F_1\left[\begin{matrix}-2n, a\\ 2a-4\end{matrix}; 2\right] = \frac{(\frac{1}{2})_n}{(a-\frac{3}{2})_n}\left\{-\frac{2(a-3)(\frac{1}{2}a+\frac{1}{2})_n}{(a-2)(\frac{1}{2}a-\frac{3}{2})_n} + \frac{(a-4)}{(a-2)}\right\}; \tag{3.33}$$

$$ {}_2F_1\left[\begin{matrix}-2n-1, a\\ 2a-4\end{matrix}; 2\right] = -\frac{2(\frac{3}{2})_n(\frac{1}{2}a+\frac{1}{2})_n}{(a-2)(a-\frac{3}{2})_n(\frac{1}{2}a-\frac{1}{2})_n}. \tag{3.34}$$

Remarks (1) The results (3.17) to (3.34) were also obtained by Kim *et al.* [9, 10] by other means.

(2) In 2010, the results (3.17) to (3.34) were again obtained by Chu [11].

(3) The results (3.17) and (3.18) are also recorded in [1].

(4) It is interesting to compare the results (3.17) and (3.19).

In order to derive (2.2), we proceed as follows. Expressing ${}_2F_1$ as a series, we have

$$ {}_2F_1\left[\begin{matrix}-n, a\\ -2n-i\end{matrix}; 2\right] = \sum_{r=0}^{n} \frac{(-n)_r(a)_r}{(-2n-i)_r}\frac{2^r}{r!}.$$

On reversing the series, we have

$$ {}_2F_1\left[\begin{matrix}-n, a\\ -2n-i\end{matrix}; 2\right] = \sum_{r=0}^{n} \frac{(-n)_{n-r}(a)_{n-r}}{(-2n-i)_{n-r}} \frac{2^{n-r}}{(n-r)!}. $$

Using appropriate identities, after a little algebra, we have

$$ {}_2F_1\left[\begin{matrix}-n, a\\ -2n-i\end{matrix}; 2\right] = \frac{2^n (a)_n \Gamma(n+i+1)}{\Gamma(2n+i+1)} {}_2F_1\left[\begin{matrix}-n, n+i+1\\ 1-a-n\end{matrix}; \frac{1}{2}\right]. \tag{3.35} $$

The ${}_2F_1$ on the right-hand side can now be evaluated with the help of generalized Bailey summation theorem (1.7), and after some simplification, we easily arrive at the right-hand side of (2.2). This completes the proof of (2.2).

Remark As mentioned in (2.1), we can also get two more equivalent forms of our second main result (2.2) by employing Plaff's transformation (3.2) and Euler's transformation (3.5), but the details are left as an exercise to the interested reader.

Special cases

In (2.2), if we take $i = 0, \pm 1, \pm 2, \pm 3, \pm 4$, we get the following interesting results:

$$ {}_2F_1\left[\begin{matrix}-n, a\\ -2n\end{matrix}; 2\right] = \frac{(\frac{1}{2}a + \frac{1}{2})_n}{(\frac{1}{2})_n}; \tag{3.36} $$

$$ {}_2F_1\left[\begin{matrix}-n, a\\ -2n+1\end{matrix}; 2\right] = \frac{1}{(\frac{1}{2})_n}\left\{\left(\frac{1}{2}a\right)_n + \left(\frac{1}{2}a + \frac{1}{2}\right)_n\right\}; \tag{3.37} $$

$$ {}_2F_1\left[\begin{matrix}-n, a\\ -2n-1\end{matrix}; 2\right] = \frac{1}{(\frac{3}{2})_n}\left\{(a+1)\left(\frac{1}{2}a + \frac{3}{2}\right)_n - a\left(\frac{1}{2}a + 1\right)_n\right\}; \tag{3.38} $$

$$ {}_2F_1\left[\begin{matrix}-n, a\\ -2n+2\end{matrix}; 2\right] = \frac{(2n-1)}{(n-1)(\frac{1}{2})_n}\left\{\left(\frac{1}{2}a\right)_n + \frac{(a+n-1)(\frac{1}{2}a - \frac{1}{2})_n}{(a-1)}\right\}; \tag{3.39} $$

$$ \begin{aligned} {}_2F_1\left[\begin{matrix}-n, a\\ -2n-2\end{matrix}; 2\right] &= \frac{1}{(n+1)(\frac{3}{2})_n} \\ &\quad \times \left\{(a+1)(a+n+1)\left(\frac{1}{2}a + \frac{3}{2}\right)_n - a(a+2)\left(\frac{1}{2}a + 2\right)_n\right\}; \end{aligned} \tag{3.40} $$

$$ \begin{aligned} {}_2F_1\left[\begin{matrix}-n, a\\ -2n+3\end{matrix}; 2\right] &= \frac{(2n-1)}{(n-2)(\frac{1}{2})_n} \\ &\quad \times \left\{\frac{(2a+3n-4)(\frac{1}{2}a - 1)_n}{(a-2)} + \frac{(n+2a-2)(\frac{1}{2}a - \frac{1}{2})_n}{(a-1)}\right\}; \end{aligned} \tag{3.41} $$

$$ \begin{aligned} {}_2F_1\left[\begin{matrix}-n, a\\ -2n-3\end{matrix}; 2\right] &= \frac{1}{3(n+1)(\frac{5}{2})_n} \\ &\quad \times \left\{(a+1)(a+3)(2a+n+1)\left(\frac{1}{2}a + \frac{5}{2}\right)_n \right. \\ &\quad \left. - a(a+2)(2a+3n+5)\left(\frac{1}{2}a + 2\right)_n\right\}; \end{aligned} \tag{3.42} $$

$$
{}_2F_1\left[\begin{matrix}-n, a\\ -2n+4\end{matrix}; 2\right] = \frac{(2n-1)(2n-3)}{(n-2)(n-3)(\frac{1}{2})_n} \times \left\{\frac{(n^2+2a^2-5n+4an-8a+6)(\frac{1}{2}a-\frac{3}{2})_n}{(a-1)(a-3)} + \frac{2(a+n-2)(\frac{1}{2}a-1)_n}{(a-2)}\right\}; \tag{3.43}
$$

$$
{}_2F_1\left[\begin{matrix}-n, a\\ -2n-4\end{matrix}; 2\right] = \frac{1}{3(n+1)(n+2)(\frac{5}{2})_n} \times \left\{(a+1)(a+3)\bigl(n^2+2a^2+4an+3n+8a+2\bigr) \cdot \left(\frac{1}{2}a+\frac{5}{2}\right)_n - 2a(a+2)(a+4)(a+n+2)\left(\frac{1}{2}a+3\right)_n\right\}. \tag{3.44}
$$

Remark The results (3.36) to (3.44) have also been recently obtained by Chu [11] by following an entirely different method.

4 Concluding remark

By employing one of the results, that is, (1.9) or its equivalent form (3.10), very recently Sofo and Srivastava [12] obtained a general sum containing factorials. Generalization of the general sum obtained by Sofo and Srivastava [12] is under investigation and will be published soon.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have equal contributions to each part of this paper. All the authors have read and approved the final manuscript.

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