Journal of Inequalities and Applications
a SpringerOpen Journal

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Strong convergence properties for ψ-mixing random variables

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Abstract

In this paper, by using the Rosenthal-type maximal inequality for ψ-mixing random variables, we obtain the Khintchine-Kolmogorov-type convergence theorem, which can be applied to establish the three series theorem and the Chung-type strong law of large numbers for ψ-mixing random variables. In addition, the strong stability for weighted sums of ψ-mixing random variables is studied, which generalizes the corresponding one of independent random variables.

MSC: 60F15

Keywords: strong stability; Khintchine-Kolmogorov-type convergence theorem; ψ-mixing random variables

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space. The random variables we deal with are all defined on $(\Omega, \mathcal{F}, P)$. Throughout the paper, let $I(A)$ be the indicator function of the set A. For random variable X, denote $X^{(c)} = XI(|X| \le c)$ for some $c > 0$. Denote $\log^+ x \doteq \ln \max(e, x)$. C and c denote positive constants, which may be different in various places.

Let $\{X_n, n \ge 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$, and let $S_n = \sum_{i=1}^n X_i$ for each $n \ge 1$. Let n and m be positive integers. Write $\mathcal{F}_n^m = \sigma(X_i, n \le i \le m)$. Given σ-algebras $\mathcal{B}$, $\mathcal{R}$ in $\mathcal{F}$, let

$$\psi(\mathcal{B}, \mathcal{R}) = \sup_{A \in \mathcal{B}, B \in \mathcal{R}, P(A)P(B) > 0} \frac{|P(AB) - P(A)P(B)|}{P(A)P(B)}. \tag{1.1}$$

Define the mixing coefficients by

$$\psi(n) = \sup_{k \ge 1} \psi\left(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty\right), \quad n \ge 0.$$

Definition 1.1 A sequence $\{X_n, n \ge 1\}$ of random variables is said to be a sequence of ψ-mixing random variables if $\psi(n) \downarrow 0$ as $n \to \infty$.

The concept of ψ-mixing random variables was introduced by Blum *et al.* [1] and some applications have been found. See, for example, Blum *et al.* [1] for strong law of large numbers, Yang [2] for almost sure convergence of weighted sums, Wu [3] for strong consistency of M estimator in linear model, Wang *et al.* [4] for maximal inequality and Hájek-Rényi-type inequality, strong growth rate and the integrability of the supremum, Zhu *et al.* [5] for strong convergence properties, Pan *et al.* [6] for strong convergence of weighted sums, and so on. When these are compared with the corresponding results of independent random variable sequences, there still remains much to be desired. The main purpose of this paper is to establish the Khintchine-Kolmogorov-type convergence theorem, which can be applied to obtain the three series theorem and the Chung-type strong law of large numbers for ψ-mixing random variables. In addition, we will study the strong stability for weighted sums of ψ-mixing random variables, which generalizes the corresponding one of independent random variables.

For independent and identically distributed random variable sequences, Jamison *et al.* [7] proved the following theorem.

Theorem A *Let* $\{X, X_n, n \geq 1\}$ *be an independent and identically distributed sequence with the same distribution function* $F(x)$, *and let* $\{w_n, n \geq 1\}$ *be a sequence of positive numbers. Write* $W_n = \sum_{i=1}^n \omega_n$ *and* $N(x) = \mathrm{Card}\{n : W_n/\omega_n \leq x\}$, $x > 0$. *If*

(i) $W_n \to \infty$ *and* $\omega_n W_n^{-1} \to 0$ *as* $n \to \infty$,

(ii) $E|X| < \infty$ *and* $EN(|X|) < \infty$,

(iii) $\int_{-\infty}^{\infty} x^2 (\int_{y \geq |x|} N(y)/y^3 \, dy) \, dF(x) < \infty$,

then

$$W_n^{-1} \sum_{i=1}^n \omega_i X_i \to c \quad a.s., \tag{1.2}$$

where c is a constant.

The result of Theorem A for independent and identically distributed sequences has been generalized to some dependent sequences, such as negatively associated sequences, negatively superadditive dependent sequences, $\tilde{\rho}$-mixing sequences, $\tilde{\varphi}$-mixing sequences, and so forth. We will further study the strong stability for weighted sums of ψ-mixing random variables, which generalizes corresponding one of independent sequences. The main results of the paper depend on the following important lemma - Rosenthal-type maximal inequality for ψ-mixing random variables.

Lemma 1.1 (*cf.* Wang *et al.* [4]) *Let* $\{X_n, n \geq 1\}$ *be a sequence of* ψ*-mixing random variables satisfying* $\sum_{n=1}^{\infty} \psi(n) < \infty$, $q \geq 2$. *Assume that* $EX_n = 0$ *and* $E|X_n|^q < \infty$ *for each* $n \geq 1$. *Then there exists a constant C depending only on q and* $\psi(\cdot)$ *such that*

$$E\left(\max_{1 \leq j \leq n} \left|\sum_{i=a+1}^{a+j} X_i\right|^q\right) \leq C\left[\sum_{i=a+1}^{a+n} E|X_i|^q + \left(\sum_{i=a+1}^{a+n} EX_i^2\right)^{q/2}\right] \tag{1.3}$$

for every $a \geq 0$ *and* $n \geq 1$. *In particular, we have*

$$E\left(\max_{1 \leq j \leq n} \left|\sum_{i=1}^{j} X_i\right|^q\right) \leq C\left[\sum_{i=1}^{n} E|X_i|^q + \left(\sum_{i=1}^{n} EX_i^2\right)^{q/2}\right] \tag{1.4}$$

for every $n \geq 1$.

The following concept of stochastic domination will be used frequently throughout the paper.

Definition 1.2 A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a constant C such that

$$P(|X_n| > x) \leq CP(|X| > x) \tag{1.5}$$

for all $x \geq 0$ and $n \geq 1$.

By the definition of stochastic domination and integration by parts, we can get the following basic property for stochastic domination. For the proof, one can refer to Wang *et al.* [8], Tang [9] or Shen and Wu [10].

Lemma 1.2 *Let $\{X_n, n \geq 1\}$ be a sequence of random variables, which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following statement holds*

$$E|X_n|^\alpha I(|X_n| \leq b) \leq C\{E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)\},$$

where C is a positive constant.

2 Khintchine-Kolmogorov-type convergence theorem

In this section, we will prove the Khintchine-Kolmogorov-type convergence theorem for ψ-mixing random variables. By using the Khintchine-Kolmogorov-type convergence theorem, we can get the three series theorem and the Chung-type strong law of large numbers for ψ-mixing random variables.

Theorem 2.1 (Khintchine-Kolmogorov-type convergence theorem) *Let $\{X_n, n \geq 1\}$ be a sequence of ψ-mixing random variables satisfying $\sum_{n=1}^{\infty} \psi(n) < \infty$. Assume that*

$$\sum_{n=1}^{\infty} \operatorname{Var}(X_n) < \infty, \tag{2.1}$$

then $\sum_{n=1}^{\infty}(X_n - EX_n)$ converges a.s.

Proof Without loss of generality, we assume that $EX_n = 0$ for all $n \geq 1$. For any $\varepsilon > 0$, it can be checked that

$$\begin{aligned} P\Big(\sup_{k,m\geq n} |S_k - S_m| > \varepsilon\Big) &\leq P\left(\sup_{k\geq n} |S_k - S_n| > \frac{\varepsilon}{2}\right) + P\left(\sup_{m\geq n} |S_m - S_n| > \frac{\varepsilon}{2}\right) \\ &\leq 2 \lim_{N\to\infty} P\left(\max_{n\leq k\leq N} |S_k - S_n| > \frac{\varepsilon}{2}\right) \\ &\leq 2 \lim_{N\to\infty} \frac{2}{(\frac{\varepsilon}{2})^2} \sum_{i=n+1}^{N} \operatorname{Var}(X_i) \\ &= \frac{16}{\varepsilon^2} \sum_{i=n+1}^{\infty} \operatorname{Var}(X_i) \to 0, \quad n \to \infty, \end{aligned}$$

where the last inequality follows from Lemma 1.1. Thus, the sequence $\{S_n, n \geq 1\}$ is a.s. Cauchy, and, therefore, we can obtain the desired result immediately. This completes the proof of the theorem. □

With the Khintchine-Kolmogorov-type convergence theorem in hand, we can get the three series theorem and the Chun-type strong law of large numbers for ψ-mixing random variables.

Theorem 2.2 (Three series theorem) *Let $\{X_n, n \geq 1\}$ be a sequence of ψ-mixing random variables satisfying $\sum_{n=1}^{\infty} \psi(n) < \infty$. For some $c > 0$, if*

$$\sum_{n=1}^{\infty} P(|X_n| > c) < \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} EX_n^{(c)} \quad \textit{converges}, \tag{2.3}$$

$$\sum_{n=1}^{\infty} \operatorname{Var}(X_n^{(c)}) < \infty, \tag{2.4}$$

then $\sum_{n=1}^{\infty} X_n$ converges almost surely.

Proof According to (2.4) and Theorem 2.1, we have

$$\sum_{n=1}^{\infty} (X_n^{(c)} - EX_n^{(c)}) \quad \text{converges a.s.} \tag{2.5}$$

It follows by (2.3) and (2.5) that

$$\sum_{n=1}^{\infty} X_n^{(c)} \quad \text{converges a.s.} \tag{2.6}$$

Obviously, (2.2) implies that

$$\sum_{n=1}^{\infty} P(X_n \neq X_n^{(c)}) = \sum_{n=1}^{\infty} P(|X_n| > c) < \infty. \tag{2.7}$$

It follows by (2.7) and Borel-Cantelli lemma that

$$P(X_n \neq X_n^{(c)}, \text{i.o.}) = 0. \tag{2.8}$$

Finally, combining (2.6) with (2.8), we can get that $\sum_{n=1}^{\infty} X_n$ converges a.s. The proof is completed. □

Theorem 2.3 (Chung-type strong law of large numbers) *Let $\{X_n, n \geq 1\}$ be a sequence of mean zero ψ-mixing random variables satisfying $\sum_{n=1}^{\infty} \psi(n) < \infty$, and let $\{a_n, n \geq 1\}$ be a sequence of positive numbers satisfying $0 < a_n \uparrow \infty$. If there exists some $p \in [1, 2]$ such that*

$$\sum_{n=1}^{\infty} \frac{E|X_n|^p}{a_n^p} < \infty, \tag{2.9}$$

then

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{i=1}^{n}X_i=0\quad a.s. \tag{2.10}$$

Proof It follows by (2.9) that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\operatorname{Var}(X_n^{(a_n)})}{a_n^2}&\le\sum_{n=1}^{\infty}\frac{E(X_n^{(a_n)})^2}{a_n^2}\\&=\sum_{n=1}^{\infty}\frac{EX_n^2I(|X_n|\le a_n)}{a_n^2}\\&\le\sum_{n=1}^{\infty}\frac{E|X_n|^p}{a_n^p}<\infty.\end{aligned}$$

Therefore, we have by Theorem 2.1 that

$$\sum_{n=1}^{\infty}\frac{X_n^{(a_n)}-EX_n^{(a_n)}}{a_n}\quad\text{converges a.s.} \tag{2.11}$$

Since $p\in[1,2]$, it follows by $EX_n=0$ that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{|EX_n^{(a_n)}|}{a_n}&=\sum_{n=1}^{\infty}\frac{|EX_nI(|X_n|\le a_n)|}{a_n}\\&=\sum_{n=1}^{\infty}\frac{|EX_nI(|X_n|>a_n)|}{a_n}\\&\le\sum_{n=1}^{\infty}\frac{E|X_n|^p}{a_n^p}<\infty,\end{aligned}$$

which implies that

$$\sum_{n=1}^{\infty}\frac{EX_n^{(a_n)}}{a_n}\quad\text{converges.} \tag{2.12}$$

Together with (2.11) and (2.12), we can see that

$$\sum_{n=1}^{\infty}\frac{X_n^{(a_n)}}{a_n}\quad\text{converges a.s.} \tag{2.13}$$

By Markov's inequality and (2.9), we have

$$\sum_{n=1}^{\infty}P\bigl(X_n\ne X_n^{(a_n)}\bigr)=\sum_{n=1}^{\infty}P\bigl(|X_n|>a_n\bigr)\le\sum_{n=1}^{\infty}\frac{E|X_n|^p}{a_n^p}<\infty. \tag{2.14}$$

Hence, the desired result (2.10) follows from (2.13), (2.14), Borel-Cantelli lemma and Kronecker's lemma immediately. □

3 Strong stability for weighted sums of ψ-mixing random variables

In the previous section, we were able to get the Khintchine-Kolmogorov-type convergence theorem for ψ-mixing random variables. In this section, we will study the strong stability for weighted sums of ψ-mixing random variables by using the Khintchine-Kolmogorov-type convergence theorem.

The concept of strong stability is as follows.

Definition 3.1 A sequence $\{Y_n, n \geq 1\}$ is said to be strongly stable if there exist two constant sequences $\{b_n, n \geq 1\}$ and $\{d_n, n \geq 1\}$ with $0 < b_n \uparrow \infty$ such that

$$b_n^{-1} Y_n - d_n \to 0 \quad \text{a.s.}$$

For the definition of strong stability, one can refer to Chow and Teicher [11]. Many authors have extended the strong law of large numbers for sequences of random variables to the case of triangular array of rowwise random variables and arrays of rowwise random variables. See, for example, Hu and Taylor [12], Bai and Cheng [13], Gan and Chen [14], Kuczmaszewska [15], Wu [16–18], Sung [19], Wang *et al.* [20–24], Zhou [25], Shen [26], Shen *et al.* [27], and so on.

Our main results are as follows.

Theorem 3.1 *Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be two sequences of positive numbers with $c_n = b_n/a_n$ and $b_n \uparrow \infty$. Let $\{X_n, n \geq 1\}$ be a sequence of ψ-mixing random variables, which is stochastically dominated by a random variable X. Assume that $\sum_{n=1}^{\infty} \psi(n) < \infty$. Denote $N(x) = \operatorname{Card}\{n : c_n \leq x\}$, $x > 0$, $1 \leq p \leq 2$. If the following conditions are satisfied*

(i) $EN(|X|) < \infty$,

(ii) $\int_0^{\infty} t^{p-1} P(|X| > t) (\int_t^{\infty} N(y)/y^{p+1}\, dy)\, dt < \infty$,

then there exist $d_n \in \mathbf{R}$, $n = 1, 2, \ldots$, such that

$$b_n^{-1} \sum_{i=1}^{n} a_i X_i - d_n \to 0 \quad \textit{a.s.} \tag{3.1}$$

Proof Let $S_n = \sum_{i=1}^{n} a_i X_i$, $T_n = \sum_{i=1}^{n} a_i X_i^{(c_i)}$. By Definition 1.2 and (i), we can see that

$$\sum_{i=1}^{\infty} P\bigl(X_i \neq X_i^{(c_i)}\bigr) = \sum_{i=1}^{\infty} P\bigl(|X_i| > c_i\bigr) \leq C \sum_{i=1}^{\infty} P\bigl(|X| > c_i\bigr) \leq CEN\bigl(|X|\bigr) < \infty. \tag{3.2}$$

By Borel-Cantelli lemma, for any sequence $\{d_n, n \geq 1\} \subset \mathbf{R}$, the sequences $\{b_n^{-1} T_n - d_n\}$ and $\{b_n^{-1} S_n - d_n\}$ converge on the same set and to the same limit. We will show that $b_n^{-1} \sum_{i=1}^{n} a_i (X_i^{(c_i)} - EX_i^{(c_i)}) \to 0$ a.s., which gives the theorem with $d_n = b_n^{-1} \sum_{i=1}^{n} a_i EX_i^{(c_i)}$. Note that $\{a_i (X_i^{(c_i)} - EX_i^{(c_i)}), i \geq 1\}$ is a sequence of mean zero ψ-mixing random variables. It follows from C_r inequality, Jensen's inequality and Lemma 1.2 that

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{E|a_n (X_n^{(c_n)} - EX_n^{(c_n)})|^p}{b_n^p} \\ &\quad \leq C \sum_{n=1}^{\infty} c_n^{-p} E\bigl(|X_n|^p I\bigl(|X_n| \leq c_n\bigr)\bigr) \end{aligned}$$

$$\le C\sum_{n=1}^{\infty} c_n^{-p}\left[c_n^p P\left(|X|>c_n\right)+E|X|^p I\left(|X|\le c_n\right)\right]$$

$$\le C\sum_{n=1}^{\infty} P\left(|X|>c_n\right)+C\sum_{n=1}^{\infty} c_n^{-p}\int_0^{c_n} t^{p-1}P\left(|X|>t\right)dt \tag{3.3}$$

and

$$\sum_{n=1}^{\infty} c_n^{-p}\int_0^{c_n} t^{p-1}P\left(|X|>t\right)dt \le \int_0^{\infty} t^{p-1}P\left(|X|>t\right)\sum_{n:c_n\ge t} c_n^{-p}\,dt$$
$$\le C\int_0^{\infty} t^{p-1}P\left(|X|>t\right)\left(\int_t^{\infty} N(y)/y^{p+1}\,dy\right)dt. \tag{3.4}$$

The last inequality above follows from the fact that

$$\begin{aligned}\sum_{n:c_n\ge t} c_n^{-p} &= \lim_{u\to\infty}\sum_{n:t\le c_n\le u} c_n^{-p}\\ &= \lim_{u\to\infty}\int_t^u y^{-p}\,dN(y)\\ &= \lim_{u\to\infty}\left(u^{-p}N(u)-t^{-p}N(t)+p\int_t^u y^{-(p+1)}N(y)\,dy\right)\end{aligned}$$

and

$$u^{-p}N(u)\le p\int_u^{\infty} y^{-(p+1)}N(y)\,dy\to 0 \quad \text{as } u\to\infty.$$

Obviously,

$$\sum_{n=1}^{\infty} P\left(|X|>c_n\right)\le EN\left(|X|\right)<\infty. \tag{3.5}$$

Thus, by (3.3)-(3.5) and condition (ii), we can see that

$$\sum_{n=1}^{\infty}\frac{E|a_n(X_n^{(c_n)}-EX_n^{(c_n)})|^p}{b_n^p}<\infty. \tag{3.6}$$

Therefore,

$$b_n^{-1}\sum_{i=1}^{n} a_i\left(X_i^{(c_i)}-EX_i^{(c_i)}\right)\to 0 \quad \text{a.s.},$$

following from (3.6), Theorem 2.3 and Kronecker's lemma immediately. The desired result is obtained. □

Corollary 3.1 *Let the conditions of Theorem* 3.1 *be satisfied, and let* $EX_n = 0$ *for* $n \ge 1$. *Assume that* $\int_1^{\infty} EN(|X|/s)\,ds<\infty$. *Then* $b_n^{-1}\sum_{i=1}^n a_iX_i \to 0$ *a.s.*

Proof By Theorem 3.1, we only need to prove that

$$b_n^{-1}\sum_{i=1}^{n} a_i EX_i^{(c_i)}\to 0 \quad \text{a.s.} \tag{3.7}$$

In fact,

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{a_i|EX_i^{(c_i)}|}{b_i} &= \sum_{i=1}^{\infty}c_i^{-1}\bigl|EX_iI\bigl(|X_i|\le c_i\bigr)\bigr| \le \sum_{i=1}^{\infty}c_i^{-1}E|X_i|I\bigl(|X_i|>c_i\bigr)\\ &\le \sum_{i=1}^{\infty}c_i^{-1}\biggl(c_iP\bigl(|X_i|>c_i\bigr)+\int_{c_i}^{\infty}P\bigl(|X_i|>t\bigr)\,dt\biggr)\\ &\le CEN\bigl(|X|\bigr)+C\int_1^{\infty}EN\bigl(|X|/s\bigr)\,ds<\infty,\end{aligned}$$

which implies (3.7) by Kronecker's lemma. We complete the proof of the corollary. □

Theorem 3.2 *Let $\{a_n, n\ge 1\}$ and $\{b_n, n\ge 1\}$ be two sequences of positive numbers with $c_n = b_n/a_n$ and $b_n\uparrow\infty$. Let $\{X_n, n\ge 1\}$ be a sequence of mean zero ψ-mixing random variables, which is stochastically dominated by a random variable X. Assume that $\sum_{n=1}^{\infty}\psi(n)<\infty$. Denote $N(x) = \mathrm{Card}\{n : c_n\le x\}$, $x>0$, $1\le p\le 2$. If the following conditions are satisfied*

(i) $EN(|X|)<\infty$,
(ii) $\int_1^{\infty}EN(|X|/s)\,ds<\infty$,
(iii) $\max_{1\le j\le n}c_j^p\sum_{i=n}^{\infty}c_i^{-p}=O(n)$,

then

$$b_n^{-1}\sum_{i=1}^{n}a_iX_i\to 0\quad a.s. \tag{3.8}$$

Proof By condition (i) and (3.2), we only need to prove that $b_n^{-1}\sum_{i=1}^{n}a_iX_i^{(c_i)}\to 0$ a.s. For this purpose, it suffices to show that

$$b_n^{-1}\sum_{i=1}^{n}a_i\bigl(X_i^{(c_i)}-EX_i^{(c_i)}\bigr)\to 0\quad\text{a.s.} \tag{3.9}$$

and

$$b_n^{-1}\sum_{i=1}^{n}a_iEX_i^{(c_i)}\to 0\quad\text{as } n\to\infty. \tag{3.10}$$

Equation (3.10) follows from the proof of Corollary 3.1 immediately.

To prove (3.9), we set $\varepsilon_0=0$ and $\varepsilon_n=\max_{1\le j\le n}c_j$ for $n\ge 1$. It follows from C_r inequality, Jensen's inequality and Lemma 1.2 that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{E|a_n(X_n^{(c_n)}-EX_n^{(c_n)})|^p}{b_n^p} &\le C\sum_{n=1}^{\infty}c_n^{-p}E\bigl(|X_n|^pI\bigl(|X_n|\le c_n\bigr)\bigr)\\ &\le C\sum_{n=1}^{\infty}P\bigl(|X|>c_n\bigr)+C\sum_{n=1}^{\infty}c_n^{-p}E|X|^pI\bigl(|X|\le c_n\bigr).\end{aligned}$$

Obviously,

$$\sum_{n=1}^{\infty}P\bigl(|X|>c_n\bigr)\le EN\bigl(|X|\bigr)<\infty \tag{3.11}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} c_n^{-p} E|X|^p I(|X| \le c_n) &\le \sum_{n=1}^{\infty} c_n^{-p} E|X|^p I(|X| \le \varepsilon_n) \\ &\le \sum_{j=1}^{\infty} \varepsilon_j^p P(\varepsilon_{j-1} < |X| \le \varepsilon_j) \sum_{n=j}^{\infty} c_n^{-p} \le C \sum_{j=1}^{\infty} P(|X| > \varepsilon_{j-1}) \\ &\le C\left(1 + \sum_{n=1}^{\infty} P(|X| > c_n)\right) \le C(1 + EN(|X|)) < \infty.\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{E|a_n(X_n^{(c_n)} - EX_n^{(c_n)})|^p}{b_n^p} < \infty, \tag{3.12}$$

following from the statements above. By Theorem 2.3 and Kronecker's lemma, we can obtain (3.9) immediately. The proof is completed. □

Theorem 3.3 *Let $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ be two sequences of positive numbers with $c_n = b_n/a_n$ and $b_n \uparrow \infty$. Let $\{X_n, n \ge 1\}$ be a sequence of ψ-mixing random variables, which is stochastically dominated by a random variable X. Assume that $\sum_{n=1}^{\infty} \psi(n) < \infty$. Define $N(x) = \mathrm{Card}\{n : c_n \le x\}$, $R(x) = \int_x^{\infty} N(y)y^{-3}\,dy$, $x > 0$. If the following conditions are satisfied*

(i) *$N(x) < \infty$ for any $x > 0$,*
(ii) *$R(1) = \int_1^{\infty} N(y)y^{-3}\,dy < \infty$,*
(iii) *$EX^2R(|X|) < \infty$,*

then there exist $d_n \in \mathbf{R}$, $n = 1, 2, \ldots$, such that

$$b_n^{-1} \sum_{i=1}^{n} a_i X_i - d_n \to 0 \quad a.s. \tag{3.13}$$

Proof Since $N(x)$ is nondecreasing, then for any $x > 0$

$$R(x) \ge N(x) \int_x^{\infty} y^{-3}\,dy = \frac{1}{2} x^{-2} N(x), \tag{3.14}$$

which implies that $EN(|X|) \le 2EX^2R(|X|) < \infty$. Therefore,

$$\begin{aligned}\sum_{i=1}^{\infty} P(X_i \ne X_i^{(c_i)}) &= \sum_{i=1}^{\infty} P(|X_i| > c_i) \\ &\le C \sum_{i=1}^{\infty} P(|X| > c_i) \le CEN(|X|) < \infty.\end{aligned} \tag{3.15}$$

By Borel-Cantelli lemma for any sequence $\{d_n, n \ge 1\} \subset \mathbf{R}$, $\{b_n^{-1}S_n - d_n\}$ and $\{b_n^{-1}T_n - d_n\}$ converge on the same set and to the same limit. We will show that $b_n^{-1}\sum_{i=1}^{n} a_i(X_i^{(c_i)} - EX_i^{(c_i)}) \to 0$ a.s., which gives the theorem with $d_n = b_n^{-1}\sum_{i=1}^{n} a_i EX_i^{(c_i)}$. It follows from

Lemma 1.2 that

$$\sum_{n=1}^{\infty} \frac{\operatorname{Var}(a_n X_n^{(c_n)})}{b_n^2} \le \sum_{n=1}^{\infty} c_n^{-2} E\big(X_n^{(c_n)}\big)^2 = \sum_{n=1}^{\infty} c_n^{-2} E X_n^2 I\big(|X_n| \le c_n\big)$$
$$\le CEN\big(|X|\big) + C \sum_{n=1}^{\infty} c_n^{-2} E X^2 I\big(|X| \le c_n\big) \tag{3.16}$$

and

$$\sum_{n=1}^{\infty} c_n^{-2} E X^2 I\big(|X| \le c_n\big) = \sum_{n: c_n \le 1} c_n^{-2} E X^2 I\big(|X| \le c_n\big) + \sum_{n: c_n > 1} c_n^{-2} E X^2 I\big(|X| \le c_n\big)$$
$$\doteq I_1 + I_2. \tag{3.17}$$

Since $N(1) = \operatorname{Card}\{n : c_n \le 1\} \le 2R(1) < \infty$ from (3.14) and (ii), it follows that $I_1 < \infty$. For I_2, we have

$$I_2 = \sum_{n: c_n > 1} c_n^{-2} E X^2 I\big(|X| \le c_n\big) = \sum_{k=2}^{\infty} \sum_{k-1 < c_n \le k} c_n^{-2} E X^2 I\big(|X| \le c_n\big)$$
$$\le \sum_{k=2}^{\infty} \big(N(k) - N(k-1)\big)(k-1)^{-2} E X^2 I\big(|X| \le 1\big)$$
$$+ \sum_{k=2}^{\infty} \big(N(k) - N(k-1)\big)(k-1)^{-2} E X^2 I\big(1 < |X| \le k\big)$$
$$\doteq I_{21} + I_{22},$$
$$I_{21} \le C \sum_{k=2}^{\infty} \big(N(k) - N(k-1)\big) \sum_{j=k-1}^{\infty} j^{-3} = C \sum_{j=1}^{\infty} j^{-3} \sum_{k=2}^{j+1} \big(N(k) - N(k-1)\big)$$
$$\le C \sum_{j=1}^{\infty} (j+1)^{-3} N(j+1) \le C \int_1^{\infty} y^{-3} N(y)\, dy < \infty.$$

Since $N(x)$ is nondecreasing and $R(x)$ is nonincreasing, we have

$$I_{22} \le \sum_{m=2}^{\infty} E X^2 I\big(m-1 < |X| \le m\big) \sum_{k=m}^{\infty} N(k)\big((k-1)^{-2} - k^{-2}\big)$$
$$\le C \sum_{m=2}^{\infty} E X^2 I\big(m-1 < |X| \le m\big) \sum_{k=m}^{\infty} \int_k^{k+1} N(x) x^{-3}\, dx$$
$$\le C \sum_{m=2}^{\infty} E X^2 R\big(|X|\big) I\big(m-1 < |X| \le m\big) \le CEX^2 R\big(|X|\big) < \infty.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{\operatorname{Var}(a_n X_n^{(c_n)})}{b_n^2} < \infty \tag{3.18}$$

following from the above statements. By Theorem 2.1 and Kronecker's lemma, we have

$$b_n^{-1}\sum_{i=1}^{n} a_i\big(X_i^{(c_i)} - EX_i^{(c_i)}\big) \to 0 \quad \text{a.s.} \tag{3.19}$$

Taking $d_n = b_n^{-1}\sum_{i=1}^{n} a_i EX_i^{(c_i)}$, we have $b_n^{-1}\sum_{i=1}^{n} a_i X_i^{(c_i)} - d_n \to 0$ a.s. The proof is completed. □

Corollary 3.2 *Let the conditions of Theorem* 3.3 *be satisfied. If* $EX_n = 0$, $n \geq 1$ *and* $\int_1^{\infty} EN(|X|/s)\,ds < \infty$, *then* $b_n^{-1}\sum_{i=1}^{n} a_i X_i \to 0$ *a.s.*

In the following, we denote $\alpha(x) : \mathbf{R}_+ \to \mathbf{R}_+$ as a positive and nonincreasing function with $a_n = \alpha(n)$, $b_n = \sum_{i=1}^{n} a_i$, $c_n = b_n/a_n$, $n \geq 1$, where

$$0 < b_n \uparrow \infty, \tag{3.20}$$

$$0 < \liminf_{n\to\infty} n^{-1} c_n \alpha(\log c_n) \leq \limsup_{n\to\infty} n^{-1} c_n \alpha(\log c_n) < \infty, \tag{3.21}$$

$$x\alpha\big(\log^+ x\big) \quad \text{is nonincreasing for } x > 0. \tag{3.22}$$

Theorem 3.4 *Let* $\{X_n, n \geq 1\}$ *be a sequence of identically distributed* ψ*-mixing random variables with* $\sum_{n=1}^{\infty} \psi(n) < \infty$. *If* $E|X_1|\alpha(\log^+ |X_1|) < \infty$, *then there exist* $d_n \in \mathbf{R}$, $n = 1, 2, \ldots,$ *such that* $b_n^{-1}\sum_{i=1}^{n} a_i X_i - d_n \to 0$ *a.s.*

Proof Since $\alpha(x)$ is positive and nonincreasing for $x > 0$ and $0 < b_n \uparrow \infty$, it follows that $c_n \uparrow \infty$. By (3.21), we can choose constants $m \in \mathbf{N}$, $C_1 > 0$, $C_2 > 0$ such that for $n \geq m$,

$$C_1 n \leq c_n \alpha(\log c_n) \leq C_2 n. \tag{3.23}$$

Therefore, for $n \geq m$, we have $\frac{1}{c_n} \leq \frac{\alpha(\log c_m)}{C_1 n}$, which implies that

$$\sum_{j=m}^{\infty} c_j^{-2} \leq \sum_{j=m}^{\infty} \frac{\alpha^2(\log c_m)}{C_1^2 j^2} \leq \frac{\alpha^2(\log c_m)}{C_1^2 m}. \tag{3.24}$$

By (3.22)-(3.24), it follows that

$$\begin{aligned}
\sum_{j=m}^{\infty} \frac{E(a_j X_j^{(c_j)})^2}{b_j^2} &\leq \sum_{j=m}^{\infty} c_j^{-2}\left(\int_{\{|X_1|\leq c_{m-1}\}} X_1^2\,dP + \sum_{i=m}^{j}\int_{\{c_{i-1}<|X_1|\leq c_i\}} X_1^2\,dP\right) \\
&\leq C + \sum_{j=m}^{\infty} c_j^{-2} \sum_{i=m}^{j} \int_{\{c_{i-1}<|X_1|\leq c_i\}} X_1^2\,dP \\
&\leq C + C\sum_{i=m}^{\infty} i^{-1}\alpha^2(\log c_i)\int_{\{c_{i-1}<|X_1|\leq c_i\}} X_1^2\,dP \\
&\leq C + C\sum_{i=m}^{\infty} \alpha(\log c_i)\int_{\{c_{i-1}<|X_1|\leq c_i\}} |X_1|\,dP \\
&\leq C + C\sum_{i=m}^{\infty} \int_{\{c_{i-1}<|X_1|\leq c_i\}} |X_1|\alpha\big(\log^+ |X_1|\big)\,dP < \infty.
\end{aligned}$$

Therefore,

$$\sum_{j=1}^{\infty} \frac{\mathrm{Var}(a_j X_j^{(c_j)})}{b_j^2} \le \sum_{j=1}^{\infty} \frac{E(a_j X_j^{(c_j)})^2}{b_j^2} < \infty, \tag{3.25}$$

which implies that

$$b_n^{-1} \sum_{i=1}^{n} a_i \bigl(X_i^{(c_i)} - EX_i^{(c_i)}\bigr) \to 0 \quad \text{a.s.} \tag{3.26}$$

from Theorem 2.1 and Kronecker's lemma. By (3.22) and (3.23) again, we have

$$\begin{aligned}\sum_{j=m}^{\infty} P\bigl(|X_j| > c_j\bigr) &\le \sum_{j=m}^{\infty} P\bigl(|X_j|\alpha\bigl(\log^+ |X_j|\bigr) \ge c_j \alpha(\log c_j)\bigr) \\ &\le \sum_{j=m}^{\infty} P\bigl(|X_1|\alpha\bigl(\log^+ |X_1|\bigr) \ge C_1 j\bigr) < \infty,\end{aligned}$$

$$\sum_{j=1}^{\infty} P\bigl(X_j \ne X_j^{(c_j)}\bigr) = \sum_{j=1}^{\infty} P\bigl(|X_j| > c_j\bigr) = \sum_{j=1}^{m-1} P\bigl(|X_j| > c_j\bigr) + \sum_{j=m}^{\infty} P\bigl(|X_j| > c_j\bigr) < \infty.$$

By Borel-Cantelli lemma, we have $P(X_j \ne X_j^{(c_j)}, \text{i.o.}) = 0$. Together with (3.26), we can see that

$$b_n^{-1} \sum_{i=1}^{n} a_i \bigl(X_i - EX_i^{(c_i)}\bigr) \to 0 \quad \text{a.s.} \tag{3.27}$$

Taking $d_n = b_n^{-1} \sum_{i=1}^{n} a_i EX_i^{(c_i)}$ for $n \ge 1$, we get the desired result. □

Theorem 3.5 *Let* $\{X_n, n \ge 1\}$ *be a sequence of* ψ*-mixing random variables with* $\sum_{n=1}^{\infty} \psi(n) < \infty$. *If for some* $1 \le p \le 2$,

$$\sum_{n=1}^{\infty} n^{-p} E\bigl|X_n \alpha\bigl(\log^+ |X_n|\bigr)\bigr|^p < \infty,$$

then there exist $d_n \in \mathbf{R}$, $n = 1, 2, \ldots$, *such that* $b_n^{-1} \sum_{i=1}^{n} a_i X_i - d_n \to 0$ *a.s.*

Proof Similar to the proof of Theorem 3.4, it is easily seen that

$$\begin{aligned}\sum_{j=1}^{\infty} P\bigl(X_j \ne X_j^{(c_j)}\bigr) &\le m - 1 + \sum_{j=m}^{\infty} P\bigl(|X_j|\alpha\bigl(\log^+ |X_j|\bigr) \ge c_j \alpha(\log c_j)\bigr) \\ &\le m - 1 + \sum_{j=m}^{\infty} P\bigl(|X_j|\alpha\bigl(\log^+ |X_j|\bigr) \ge C_1 j\bigr) < \infty.\end{aligned}$$

By Borel-Cantelli lemma for any sequence $\{d_n, n \ge 1\} \subset \mathbf{R}$, the sequences $\{b_n^{-1} \sum_{i=1}^{n} a_i X_i - d_n\}$ and $\{b_n^{-1} \sum_{i=1}^{n} a_i X_i^{(c_i)} - d_n\}$ converge on the same set and to the same limit. We will show that $b_n^{-1}\sum_{i=1}^n a_i(X_i^{(c_i)}-EX_i^{(c_i)})\to 0$ a.s., which gives the theorem with $d_n = b_n^{-1}\sum_{i=1}^n a_iEX_i^{(c_i)}$. Note that $\{a_i(X_i^{(c_i)}-EX_i^{(c_i)})/b_i, i\ge 1\}$ is a sequence of mean zero ψ-mixing random variables. By C_r inequality and Jensen's inequality, we can see that

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{E|a_j(X_j^{(c_j)}-EX_j^{(c_j)})|^p}{b_j^p} &\le C(m-1)+C\sum_{j=m}^{\infty}c_j^{-p}E|X_j|^pI\bigl(|X_j|\le c_j\bigr)\\ &\le C(m-1)+C\sum_{j=m}^{\infty}j^{-p}\bigl(\alpha(\log c_j)\bigr)^pE|X_j|^pI\bigl(|X_j|\le c_j\bigr)\\ &\le C(m-1)+C\sum_{j=1}^{\infty}j^{-p}E\bigl|X_j\alpha\bigl(\log^+|X_j|\bigr)\bigr|^p<\infty.\end{aligned}$$

It follows by Theorem 2.3 that $b_n^{-1}\sum_{i=1}^n a_i(X_i^{(c_i)}-EX_i^{(c_i)})\to 0$ a.s. The proof is completed. □

Corollary 3.3 *Let the conditions of Theorem* 3.5 *be satisfied. Furthermore, suppose that* $EX_n=0$ *and* $\sum_{n=1}^{\infty}\int_1^{\infty}P(|X_n|>sc_n)\,ds<\infty$, *then* $b_n^{-1}\sum_{i=1}^n a_iX_i\to 0$ *a.s.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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Acknowledgements
The authors are most grateful to the editor Andrei Volodin and an anonymous referee for careful reading of the manuscript and valuable suggestions, which helped in improving an earlier version of this paper. This work was supported by the Natural Science Project of Department of Education of Anhui Province (KJ2011z056) and the National Natural Science Foundation of China (11201001).

Received: 23 April 2013 Accepted: 24 July 2013 Published: 2 August 2013

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doi:10.1186/1029-242X-2013-360
Cite this article as: Xu and Tang: Strong convergence properties for ψ-mixing random variables. *Journal of Inequalities and Applications* 2013 **2013**:360.