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Fixed point theorems for multivalued mappings in *G*-cone metric spaces

Akbar Azam* and Nayyar Mehmood

*Correspondence:
akbarazam@yahoo.com
Department of Mathematics,
COMSATS Institute of Information
Technology, Chak Shahzad,
Islamabad 44000, Pakistan

Abstract
We extend the idea of Hausdorff distance function in *G*-cone metric spaces and obtain fixed points of multivalued mappings in *G*-cone metric spaces.
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1 Introduction

The main revolution in the existence theory of many linear and non-linear operators happened after the Banach contraction principle. After this principle many researchers put their efforts into studying the existence and solutions for nonlinear equations (algebraic, differential and integral), a system of linear (nonlinear) equations and convergence of many computational methods [1]. Banach contraction gave us many important theories like variational inequalities, optimization theory and many computational theories [1, 2]. Due to wide spreading importance of Banach contraction, many authors generalized it in several directions [3–9]. Nadler [10] was first to present it in a multivalued case, and then many authors extended Nadler's multivalued contraction. One of the real generalizations of Nadler's theorem was given by Mizoguchi and Takahashi in the following way.

Theorem 1.1 [11] *Let* (X, d) *be a complete metric space, and let* $T : X \to 2^X$ *be a multivalued map such that* Tx *is a closed bounded subset of* X *for all* $x \in X$. *If there exists a function* $\varphi : (0, \infty) \to [0, 1)$ *such that* $\limsup_{r\to t^+} \varphi(r) < 1$ *for all* $t \in [0, \infty)$ *and if*

$$H(Tx, Ty) \le \varphi\big(d(x, y)\big)\big(d(x, y)\big) \quad \text{for all } x, y (x \ne y) \in X,$$

then T *has a fixed point in* X.

Suzuki [12] proved that Mizoguchi and Takahashi's theorem is a real generalization of Nadler's theorem. Recently Huang and Zhang [13] introduced a cone metric space with a normal cone with a constant K, which is generalization of a metric space. After that Rezapour and Hamlbarani [14] generalized a cone metric space with a non-normal cone. Afterwards many researchers [15–24] have studied fixed point results in cone metric spaces. In [25] Mustafa *et al.* generalized the metric space and introduced the notion of *G*-metric space which recovered the flaws of Dhage's generalization [26, 27] of a metric space. Many researchers proved many fixed point results using a G-metric space [28, 29]. Anchalee Kaewcharoen and Attapol Kaewkhao [28] and Nedal *et al.* [30] proved fixed point results for multivalued maps in G-metric spaces. In 2009, Beg *et al.* [31] introduced the notion of G-cone metric space and generalized some results. Chi-Ming Cheng [32] proved Nadler-type results in tvs G-cone metric spaces.

In 2011 Cho and Bae [33] generalized a Mizoguchi Takahashi-type theorem in a cone metric space. In the present paper, we introduce the notion of Hausdorff distance function on G-cone metric spaces and exploit it to study some fixed point results in G-cone metric spaces. Our result generalizes many results in literature.

2 Preliminaries

Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, nonempty and $P \neq \{\theta\}$,

(b) $a, b \in R$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$, more generally, if $a, b, c \in R$, $a, b, c \geq 0$, $x, y, z \in P \Longrightarrow ax + by + cz \in P$,

(c) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P$.

A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$

$$\theta \preccurlyeq x \preccurlyeq y \quad \text{implies} \quad \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P, while $x \ll y$ stands for $y - x \in \operatorname{int} P$ (interior of P), while $x \prec y$ means $x \preccurlyeq y$ and $x \neq y$.

Rezapour [14] proved that there are no normal cones with normal constants $K < 1$ and for each $k > 1$, there are cones with normal constants $K > 1$.

Remark 2.1 [34] The results concerning fixed points and other results, in the case of cone spaces with non-normal solid cones, cannot be provided by reducing to metric spaces, because in this case neither of the conditions of Lemmas 1-4 in [13] hold. Further, the vector cone metric is not continuous in a general case, *i.e.*, from $x_n \to x$, $y_n \to y$ it need not follow that $d(x_n, y_n) \to d(x, y)$.

For the case of non-normal cones, we have the following properties.

(PT1) If $u \preccurlyeq v$ and $v \ll w$, then $u \ll w$.

(PT2) If $u \ll v$ and $v \preccurlyeq w$, then $u \ll w$.

(PT3) If $u \ll v$ and $v \ll w$, then $u \ll w$.

(PT4) If $\theta \preccurlyeq u \ll c$ for each $c \in \operatorname{int} P$, then $u = \theta$.

(PT5) If $a \preccurlyeq b + c$ for each $c \in \operatorname{int} P$, then $a \preccurlyeq b$.

(PT6) If E is a real Banach space with a cone P, and if $a \preccurlyeq \lambda a$, where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.

(PT7) If $c \in \operatorname{int} P$, $a_n \in \mathbb{E}$ and $a_n \to \theta$, then there exists an n_0 such that, for all $n > n_0$, we have $a_n \ll c$.

In the following we shall always assume that the cone P is solid and non-normal.

Definition 2.1 [31] Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \to E$ satisfies:

(G1) $G(x,y,z) = \theta$ if $x = y = z$,
(G2) $\theta \prec G(x,x,y)$, whenever $x \neq y$, for all $x,y \in X$,
(G3) $G(x,x,y) \preccurlyeq G(x,y,z)$, whenever $y \neq z$,
(G4) $G(x,y,z) = G(x,z,y) = G(y,x,z) = \cdots$ (symmetric in all three variables),
(G5) $G(x,y,z) \preccurlyeq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$.

Then G is called a generalized cone metric on X, and X is called a generalized cone metric space or, more specifically, a G-cone metric space.

The concept of a G-cone metric space is more general than that of G-metric spaces and cone metric spaces (see [31]).

Definition 2.2 [31] A G-cone metric space X is symmetric if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.

Example 2.1 [31] Let (X,d) be a cone metric space. Define $G : X \times X \times X \to E$ by $G(x,y,z) = d(x,y) + d(y,z) + d(z,x)$. Then (X,G) is a G-cone metric space.

Proposition 2.1 [31] *Let X be a G-cone metric space, define $d_G : X \times X \to E$ by*

$$d_G(x,y) = G(x,y,y) + G(y,x,x).$$

Then (X,d_G) is a cone metric space.

It can be noted that $G(x,y,y) \preccurlyeq \frac{2}{3} d_G(x,y)$. If X is a symmetric G-cone metric space, then $d_G(x,y) = 2G(x,y,y)$ for all $x,y \in X$.

Definition 2.3 [31] Let X be a G-cone metric space and let $\{x_n\}$ be a sequence in X.

We say that $\{x_n\}$ is:

(a) a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there is N such that for all $n,m,l > N$, $G(x_n,x_m,x_l) \ll c$.
(b) a convergent sequence if for every c in E with $\theta \ll c$, there is N such that for all $m,n > N$, $G(x_m,x_n,x) \ll c$ for some fixed x in X. Here x is called the limit of a sequence $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

A G-cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Proposition 2.2 [31] *Let X be a G-cone metric space, then the following are equivalent.*

(i) *$\{x_n\}$ converges to x.*
(ii) *$G(x_n,x_n,x) \to \theta$ as $n \to \infty$.*
(iii) *$G(x_n,x,x) \to \theta$ as $n \to \infty$.*
(iv) *$G(x_m,x_n,x) \to \theta$ as $m,n \to \infty$.*

Lemma 2.1 [31] *Let $\{x_n\}$ be a sequence in a G-cone metric space X. If $\{x_n\}$ converges to $x \in X$, then $G(x_m,x_n,x) \to \theta$ as $m,n \to \infty$.*

Lemma 2.2 [31] *Let $\{x_n\}$ be a sequence in a G-cone metric space X and $x \in X$. If $\{x_n\}$ converges to $x \in X$, then $\{x_n\}$ is a Cauchy sequence.*

Lemma 2.3 [31] *Let $\{x_n\}$ be a sequence in a G-cone metric space X. If $\{x_n\}$ is a Cauchy sequence in X, then $G(x_m, x_n, x_l) \to \theta$, as $m, n, l \to \infty$.*

3 Main result

Denote by $N(X)$, $B(X)$ and $CB(X)$ the set of nonempty, bounded, sequentially closed bounded subsets of G-cone metric spaces, respectively.

Let (X, G) be a G-cone metric space. We define (see [33])

$$s(p) = \{q \in E : p \preccurlyeq q\} \quad \text{for } q \in E,$$

and

$$s(a, B) = \bigcup_{b \in B} s\big(d_G(a, b)\big) = \bigcup_{b \in B} \big\{x \in E : \ d_G(a, b) \preccurlyeq x\big\} \quad \text{for } a \in X \text{ and } B \in N(X).$$

For $A, B \in B(X)$, we define

$$\hat{s}(A, B) = \bigcup_{a \in A, b \in B} s\big(d_G(a, b)\big),$$

$$s(a, B, C) = s(a, B) + \hat{s}(B, C) + s(a, C) = \big\{u + v + w : u \in s(a, B), v \in \hat{s}(B, C), w \in s(a, C)\big\},$$

and

$$s(A, B, C) = \left(\bigcap_{a \in A} s(a, B, C)\right) \cap \left(\bigcap_{b \in B} s(b, A, C)\right) \cap \left(\bigcap_{c \in C} s(c, A, B)\right).$$

Lemma 3.1 *Let (X, G) be a G-cone metric space, let P be a cone in a Banach space E.*

(i) *Let $p, q \in E$. If $p \preccurlyeq q$, then $s(q) \subset s(p)$.*
(ii) *Let $x \in X$ and $A \in N(X)$. If $0 \in s(x, A)$, then $x \in A$.*
(iii) *Let $q \in P$ and let $A, B, C \in B(X)$ and $a \in A$. If $q \in s(A, B, C)$, then $q \in s(a, B, C)$.*

Remark 3.1 Recently, Kaewcharoen and Kaewkhao [28] (see also [30]) introduced the following concepts. Let X be a G-metric space and let $CB(X)$ be the family of all nonempty closed bounded subsets of X. Let $H_G(\cdot, \cdot, \cdot)$ be the Hausdorff G-distance on $CB(X)$, *i.e.*,

$$H_G(A, B, C) = \max\Big\{\sup_{a \in A} G(a, B, C), \sup_{b \in B} G(b, A, C), \sup_{c \in C} G(c, A, B)\Big\},$$

$$H_{d_G}(A, B) = \max\Big\{\sup_{a \in A} d_G(a, B), \sup_{b \in B} d_G(b, A)\Big\},$$

where

$$G(x, B, C) = d_G(x, B) + d_G(B, C) + d_G(x, C),$$

$$d_G(x, B) = \inf\big\{d_G(x, y), y \in B\big\},$$

$$d_G(A,B) = \inf\{d_G(a,b), a \in A, b \in B\},$$
$$G(a,b,C) = \inf\{G(a,b,c), c \in C\}.$$

The above expressions show a relation between H_G and H_{d_G}. Moreover, note that if (X,G) is a G-cone metric space, $E = R$, and $P = [0,\infty)$, then (X,G) is a G-metric space. Also, for $A,B,C \in CB(X)$, $H_G(A,B,C) = \inf s(A,B,C)$.

Remark 3.2 Let (X,G) be a G-cone metric space. Then
(a) $\hat{s}(\{a\},\{b\}) = s(d_G(a,b))$ for $a,b \in X$.
(b) If $x \in s(a,B,B)$ then $x \in 2s(d_G(a,b))$.

Proof (a) By definition

$$\hat{s}(\{a\},\{b\}) = \bigcup_{a\in\{a\},b\in\{b\}} s(d_G(a,b)) = s(d_G(a,b)).$$

(b) Now let

$$x \in s(a,B,B), \quad \text{then}$$
$$x \in s(a,B,B) = s(a,B) + \hat{s}(B,B) + s(a,B)$$
$$\Rightarrow \quad x \in 2s(a,B) + \hat{s}(B,B)$$
$$\Rightarrow \quad x \in 2s(d_G(a,b)) + s(\theta).$$

Let $x = y + z$ for $y \in 2s(d_G(a,b))$ and $z \in s(\theta)$. Then by definition $\theta \preccurlyeq z$ and $2d_G(a,b) \preccurlyeq y$, which implies $\theta + 2d_G(a,b) \preccurlyeq y + z = x$. Hence $2d_G(a,b) \preccurlyeq x$, so $x \in 2s(d_G(a,b))$. □

In the following theorem, we use the generalized Hausdorff distance on G-cone metric spaces to find fixed points of a multivalued mapping.

Remark 3.3 If (X,G) is a G-metric space, then (X,d_G) is a metric space, where

$$d_G(x,y) = G(x,y,y) + G(y,x,x).$$

It is noticed in [35] that in the symmetric case ((X,G) is symmetric), many fixed point theorems on G-metric spaces are particular cases of existing fixed point theorems in metric spaces. In these deductions, the fact $G(Tx,Ty,Ty) + G(Ty,Tx,Tx) = 2G(Tx,Ty,Ty) = d_G(Tx,Ty)$ is exploited for a single-valued mapping T on X. Whereas in the case of multivalued mapping $T : X \to 2^X$ on a G-cone metric space,

$$\begin{aligned} s(Tx,Ty,Ty) &= \Big(\bigcap_{a\in Tx} s(a,Ty,Ty)\Big) \cap \Big(\bigcap_{b\in Ty} s(b,Tx,Ty)\Big) \cap \Big(\bigcap_{b\in Ty} s(b,Tx,Ty)\Big) \\ &= \Big(\bigcap_{a\in Tx} s(a,Ty,Ty)\Big) \cap \Big(\bigcap_{b\in Ty} s(b,Tx,Ty)\Big) \end{aligned}$$

$$= \left(\bigcap_{a\in Tx} 2s(a, Ty)\right) \cap \left(\bigcap_{b\in Ty} s(b, Tx) + \hat{s}(Tx, Ty) + s(b, Ty)\right)$$
$$\neq s(Ty, Tx, Tx).$$

Therefore,

$$\left(\bigcap_{a\in Tx} s(a, Ty)\right) \cap \left(\bigcap_{b\in Ty} s(b, Tx)\right) \neq s(Tx, Ty, Ty) + s(Ty, Tx, Tx)$$

and even in a symmetric case, we cannot follow a similar technique to deduce G-cone metric multivalued fixed point results from similar results of metric spaces.

In a non-symmetric case, the authors [35] deduce some G-metric fixed point theorems from similar results of metric spaces by using the fact that if (X,G) is a G-metric on X, then

$$\delta(x,y) = \max\{G(x,y,y), G(y,x,x)\}$$

is a metric on X. Whereas, in the case of a G-cone metric space, the expression $\max\{G(x,y,y), G(y,x,x)\}$ is meaningless as $G(x,y,y)$, $G(y,x,x)$ are vectors, not essentially comparable, and we cannot find maximum of these elements. That is, (X,δ) may not be a cone metric space if (X,G) is a G-cone metric space. In the explanation of this fact, we refer to Example 3.1 below, from [31]. Hence multivalued fixed point results on G-cone metric spaces cannot be deduced from similar fixed point theorems on metric spaces.

Example 3.1 [31] Let $X = \{a,b\}$, $E = R^3$,

$$P = \{(x,y,z) \in E : x,y,z \geq 0\}.$$

Define $G : X \times X \times X \to E$ by

$$G(a,a,a) = (0,0,0) = G(b,b,b),$$
$$G(a,b,b) = (0,1,1) = G(b,a,b) = G(b,b,a),$$
$$G(b,a,a) = (0,1,0) = G(a,b,a) = G(a,a,b).$$

Note that $\delta(a,b) = \max\{G(a,a,b), G(a,b,b)\} = \max\{(1,0,0),(0,1,1)\}$ has no meaning as discussed above.

Theorem 3.1 *Let (X,G) be a complete cone metric space, and let $T : X \longrightarrow CB(X)$ be a multivalued mapping. If there exists a function $\varphi : P \to [0,1)$ such that*

$$\limsup_{n\to\infty} \varphi(r_n) < 1 \tag{a}$$

for any decreasing sequence $\{r_n\}$ in P, and if

$$\varphi\big(G(x,y,z)\big)G(x,y,z) \in s(Tx, Ty, Tz) \tag{1}$$

for all $x,y,z \in X$, then T has a fixed point in X.

Proof Let x_0 be an arbitrary point in X and $x_1 \in Tx_0$. From (1), we have

$$\varphi\big(G(x_0,x_1,x_1)\big)G(x_0,x_1,x_1) \in s(Tx_0, Tx_1, Tx_1).$$

Thus, by Lemma 3.1(iii), we get

$$\varphi\big(G(x_0,x_1,x_1)\big)G(x_0,x_1,x_1) \in s(x_1, Tx_1, Tx_1).$$

By Remark 3.2, we can take $x_2 \in Tx_1$ such that

$$\varphi\big(G(x_0,x_1,x_1)\big)G(x_0,x_1,x_1) \in 2s\big(d_G(x_1,x_2)\big).$$

Thus,

$$2d_G(x_1,x_2) \preccurlyeq \varphi\big(G(x_0,x_1,x_1)\big)G(x_0,x_1,x_1).$$

Again, by (1), we have

$$\varphi\big(G(x_1,x_2,x_2)\big)G(x_1,x_2,x_2) \in s(Tx_1, Tx_2, Tx_2),$$

and by Lemma 3.1(iii)

$$\varphi\big(G(x_1,x_2,x_2)\big)G(x_1,x_2,x_2) \in s(x_2, Tx_2, Tx_2).$$

By Remark 3.2, we can take $x_3 \in Tx_2$ such that

$$\varphi\big(G(x_1,x_2,x_2)\big)G(x_1,x_2,x_2) \in 2s\big(d_G(x_2,x_3)\big).$$

Thus,

$$2d_G(x_2,x_3) \preccurlyeq \varphi\big(G(x_1,x_2,x_2)\big)G(x_1,x_2,x_2).$$

It implies that

$$\begin{aligned}
2d_G(x_2,x_3) &\preccurlyeq \varphi\big(G(x_1,x_2,x_2)\big)G(x_1,x_2,x_2) \\
&\preccurlyeq \varphi\big(G(x_1,x_2,x_2)\big)G(x_1,x_2,x_2) + \varphi\big(G(x_1,x_2,x_2)\big)G(x_2,x_1,x_1) \\
&\preccurlyeq \varphi\big(G(x_1,x_2,x_2)\big)\big[G(x_1,x_2,x_2) + G(x_2,x_1,x_1)\big] \\
&= \varphi\big(G(x_1,x_2,x_2)\big)d_G(x_1,x_2) \\
&\Rightarrow \quad d_G(x_2,x_3) \preccurlyeq \frac{1}{2}\varphi\big(G(x_1,x_2,x_2)\big)d_G(x_1,x_2).
\end{aligned}$$

By induction we can construct a sequence $\{x_n\}$ in X such that

$$d_G(x_n,x_{n+1}) \preccurlyeq \frac{1}{2}\varphi\big(G(x_{n-1},x_n,x_n)\big)d_G(x_{n-1},x_n), \quad x_{n+1} \in Tx_n, \text{ for } n = 1,2,3\ldots. \tag{2}$$

Assume that $x_{n+1} \neq x_n$ for all $n \in N$. From (2) the sequence $\{d_G(x_n, x_{n+1})\}_{n\in N}$ is a decreasing sequence in P. So, there exists $l \in (0,1)$ such that

$$\limsup_{n\to\infty} \varphi\bigl(d_G(x_n, x_{n+1})\bigr) = l.$$

Thus, there exists $n_0 \in N$ such that for all $n \geq n_0$, $\varphi(d_G(x_n, x_{n+1})) \prec l_0$ for some $l_0 \in (l, 1)$. Choose $n_0 = 1$, then we have

$$\begin{aligned} d_G(x_n, x_{n+1}) &\preccurlyeq \frac{1}{2}\varphi\bigl(d_G(x_{n-1}, x_n)\bigr)d_G(x_{n-1}, x_n) \\ &\prec l_0 d_G(x_{n-1}, x_n) \\ &\prec (l_0)^n d_G(x_0, x_1) \quad \text{for all } n \geq 1. \end{aligned}$$

Moreover, for $m > n \geq 1$, we have that

$$d_G(x_n, x_m) \preccurlyeq \frac{(l_0)^n}{1 - l_0} d_G(x_0, x_1).$$

According to (PT1) and (PT7), it follows that $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $v \in X$ such that $x_n \to v$. Assume $k_1 \in N$ such that $d_G(x_n, v) \ll \frac{c}{2}$ for all $n \geq k_1$.

We now show that $v \in Tv$. So, for $x_n, v \in X$ and by using (2), we have

$$\varphi\bigl(G(x_n, v, v)\bigr)G(x_n, v, v) \in s(Tx_n, Tv, Tv).$$

By Lemma 3.1(iii) we have

$$\varphi\bigl(G(x_n, v, v)\bigr)G(x_n, v, v) \in s(x_{n+1}, Tv, Tv).$$

Thus there exists $u_n \in Tv$ such that

$$\varphi\bigl(G(x_n, v, v)\bigr)G(x_n, v, v) \in 2s\bigl(d_G(x_{n+1}, u_n)\bigr).$$

It implies that

$$\begin{aligned} 2d_G(x_{n+1}, u_n) &\preccurlyeq \varphi\bigl(G(x_n, v, v)\bigr)G(x_n, v, v), \\ d_G(x_{n+1}, u_n) &\preccurlyeq \frac{1}{2}\varphi\bigl(G(x_n, v, v)\bigr)G(x_n, v, v) \\ &\preccurlyeq \varphi\bigl(G(x_n, v, v)\bigr)\bigl[G(x_n, v, v) + G(x_n, x_n, v)\bigr] \\ &= \varphi\bigl(G(x_n, v, v)\bigr)d_G(x_n, v). \end{aligned}$$

So

$$d_G(x_{n+1}, u_n) \preccurlyeq \varphi\bigl(G(x_n, v, v)\bigr)d_G(x_n, v). \tag{3}$$

Now consider

$$\begin{aligned} d_G(v,u_n) &\preccurlyeq d_G(x_{n+1},v) + d_G(x_{n+1},u_n) \\ &\preccurlyeq d_G(x_{n+1},v) + \varphi\big(G(x_n,v,v)\big)d_G(x_n,v) \quad \text{by using (3)} \\ &\prec d_G(x_{n+1},v) + d_G(x_n,v), \end{aligned}$$

$$d_G(v,u_n) \ll \frac{c}{2} + \frac{c}{2} = c, \quad \text{for all } n \geq k_1.$$

Therefore $\lim_{n\to\infty} u_n = v$. Since Tv is closed, so $v \in Tv$. □

The next corollary is Nadler's multivalued contraction theorem in a G-cone metric space.

Corollary 3.1 *Let (X,G) be a complete G-cone metric space, and let $T : X \longrightarrow CB(X)$ be a multivalued mapping. If there exists a constant $k \in [0,1)$ such that*

$$kG(x,y,z) \in s(Tx,Ty,Tz)$$

for all $x,y,z \in X$, then T has a fixed point in X.

By Remark 3.1, we have the following results of [30].

Corollary 3.2 [30] *Let (X,G) be a complete G-metric space, and let $T : X \longrightarrow CB(X)$ be a multivalued mapping. If there exists a function $\varphi : [0,+\infty) \to [0,1)$ such that*

$$\limsup_{r\to t^+} \varphi(r) < 1$$

for any $t \geq 0$, and if

$$H_G(Tx,Ty,Tz) \leq \varphi\big(G(x,y,z)\big)G(x,y,z)$$

for all $x,y,z \in X$, then T has a fixed point in X.

Corollary 3.3 [30] *Let (X,G) be a complete G-metric space, and let $T : X \longrightarrow CB(X)$ be a multivalued mapping. If there exists a constant $k \in [0,1)$ such that*

$$H_G(Tx,Ty,Tz) \leq kG(x,y,z)$$

for all $x,y,z \in X$, then T has a fixed point in X.

In the following we formulate an illustrative example regarding our main theorem.

Example 3.2 Let $X = [0,1]$, $E = C[0,1]$ be endowed with the strongly locally convex topology $\tau(E,E^*)$, and let $P = \{x \in E : 0 \leq x(t),\ t \in [0,1]\}$. Then the cone is $\tau(E,E^*)$-solid, and non-normal with respect to the topology $\tau(E,E^*)$. Define $G : X \times X \times X \to E$ by

$$G(x,y,z)(t) = \text{Max}\big\{|x-y|, |y-z|, |x-z|\big\}e^t.$$

Then G is a G-cone metric on X.

Consider a mapping $T : X \to CB(X)$ defined by

$$Tx = \left[0, \frac{1}{10}x\right].$$

Let $\varphi(t) = \frac{1}{5}$ for all $t \in P$. The contractive condition of the main theorem is trivial for the case when $x = y = z = 0$. Suppose, without any loss of generality, that all x, y and z are nonzero and $x < y < z$. Then

$$G(x,y,z) = |x - z|e^t,$$

and

$$d_G(x,y) = 2|x - y|e^t.$$

Now

$$s(x, Ty) = \begin{cases} 0 & \text{if } x \le \frac{y}{10}, \\ |x - \frac{y}{10}|e^t & \text{if } x > \frac{y}{10}, \end{cases}$$

$$s(y, Tz) = \begin{cases} 0 & \text{if } y \le \frac{z}{10}, \\ |y - \frac{z}{10}|e^t & \text{if } y > \frac{z}{10}. \end{cases}$$

For $s(x, Ty) = 0 = s(y, Tz)$, we have

$$s(x, Ty, Tz) = s(0),$$

$$\bigcap_{y \in Ty} s(y, Tx, Tz) = s\left(2\left|\frac{y}{10} - \frac{x}{10}\right|e^t\right),$$

and

$$\bigcap_{z \in Tz} s(z, Tx, Ty) = s\left(2\left|\frac{z}{10} - \frac{x}{10} - \frac{y}{10}\right|e^t\right).$$

Thus

$$s(Tx, Ty, Tz) = \big(s(0)\big) \cap \left(s\left(2\left|\frac{y}{10} - \frac{x}{10}\right|e^t\right)\right) \cap \left(s\left(2\left|\frac{z}{10} - \frac{x}{10} - \frac{y}{10}\right|e^t\right)\right).$$

Now

$$\text{If } s(Tx, Ty, Tz) = s\left(2\left|\frac{z}{10} - \frac{x}{10} - \frac{y}{10}\right|e^t\right), \quad \text{then}$$

$$\begin{aligned} 2\left|\frac{z}{10} - \frac{x}{10} - \frac{y}{10}\right|e^t &\le 2\left|\frac{z}{10} - \frac{x}{10}\right|e^t, \quad \text{for } t \in [0,1] \\ &= \frac{1}{5}|z - x|e^t = \frac{1}{5}\operatorname{Max}\big\{|x - y|, |y - z|, |x - z|\big\}e^t \\ &= \frac{1}{5}G(x,y,z); \end{aligned}$$

$$\text{If } s(Tx, Ty, Tz) = s\left(2\left|\frac{y}{10} - \frac{x}{10}\right|e^t\right), \quad \text{then}$$

$$\begin{aligned} 2\left|\frac{y}{10} - \frac{x}{10}\right|e^t &\leq 2\left|\frac{z}{10} - \frac{x}{10}\right|e^t, \quad \text{for } t \in [0,1] \\ &= \frac{1}{5}|z - x|e^t = \frac{1}{5}\operatorname{Max}\{|x-y|, |y-z|, |x-z|\}e^t \\ &= \frac{1}{5}G(x,y,z). \end{aligned}$$

Hence,

$$\frac{1}{5}G(x,y,z) \in s(Tx, Ty, Tz).$$

All the assumptions of Theorem 3.1 also hold for other possible values of $s(x, Ty)$ and $s(y, Tz)$ to obtain $0 \in T0$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors read and approved the final manuscript.

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References

1. Moore, RE, Cloud, MJ: Computational Functional Analysis, 2nd edn. Ellis Horwood Series in Mathematics and Its Applications. Woodhead Publishing, Cambridge (2007)
2. Noor, A: Principles of Variational Inequalities. Lambert Academic Publishing, Saarbrücken (2009)
3. Abbas, M, Hussain, N, Rhoades, BE: Coincidence point theorems for multivalued f-weak contraction mappings and applications. RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. a Mat. **105**(2), 261-272 (2011)
4. Agarwal, RP, O'Regan, DO, Shahzad, N: Fixed point theorems for generalized contractive maps of Mei-Keeler type. Math. Nachr. **276**, 3-12 (2004)
5. Agarwal, RP, Karapinar, E: Remarks on some coupled fixed point theorems in G-metric spaces. Fixed Point Theory Appl. **2013**, Article ID 10 (2013). doi:10.1186/1687-1812-2013-2
6. Azam, A, Arshad, M, Beg, I: Existence of fixed points in complete cone metric spaces. Int. J. Mod. Math. **5**(1), 91-99 (2010)
7. Hussain, N, Abbas, M: Common fixed point results for two new classes of hybrid pairs in symmetric spaces. Appl. Math. Comput. **218**, 542-547 (2011)
8. Hussain, N, Karapinar, E, Salimi, P, Vetro, P: Fixed point results for G^m-Meir-Keeler contractive and $G-(\alpha,\psi)$-Meir-Keeler contractive mappings. Fixed Point Theory Appl. **2013**, Article ID 34 (2013)
9. Wang, T: Fixed point theorems and fixed point stability for multivalued mappings on metric spaces. Nanjing Daxue Xuebao Shuxue Bannian Kan **6**, 16-23 (1989)
10. Nadler, SB Jr: Multi-valued contraction mappings. Pac. J. Math. **30**, 475-478 (1969)
11. Mizoguchi, N, Takahashi, W: Fixed point theorems for multi-valued mappings on complete metric spaces. J. Math. Anal. Appl. **141**, 177-188 (1989)
12. Suzuki, T: Mizoguchi-Takahashi's fixed point theorem is a real generalization of Nadler's. J. Math. Anal. Appl. **340**(1), 752-755 (2008)
13. Huang, LG, Zhang, X: Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. Appl. **332**(2), 1468-1476 (2007). doi:10.1016/j.jmaa.2005.03.087
14. Rezapour, S, Hamlbarani, R: Some notes on the paper 'Cone metric spaces and fixed point theorems of contractive mappings'. J. Math. Anal. Appl. **345**, 719-724 (2008)
15. Abbas, M, Rhoades, BE: Fixed and periodic point results in cone metric spaces. Appl. Math. Lett. **22**(4), 511-515 (2009)
16. Azam, A, Arshad, M, Beg, I: Common fixed points of two maps in cone metric spaces. Rend. Circ. Mat. Palermo **57**, 433-441 (2008)
17. Bari, CD, Vetro, P: ϕ-pairs and common fixed points in cone metric spaces. Rend. Circ. Mat. Palermo **57**, 279-285 (2008)
18. Bari, CD, Vetro, P: Weakly ϕ-pairs and common fixed points in cone metric spaces. Rend. Circ. Mat. Palermo **58**, 125-132 (2009)
19. Branciari, A: A fixed point theorem for mappings satisfying a general contractive condition of integral type. Int. J. Math. Math. Sci. **29**, 531-536 (2002)

20. Cho, SH, Bae, JS: Fixed point theorems for multivalued contractive mappings and multivalued Caristi type mappings in cone metric spaces. Fixed Point Theory Appl. **2012**, Article ID 133 (2012). doi:10.1186/1687-1812-2011-87
21. Du, WS: A note on cone metric fixed point theory and its equivalence. Nonlinear Anal., Theory Methods Appl. **72**(5), 2259-2261 (2010)
22. Kadelburg, Z, Radenovič, S: Some results on set-valued contractions in abstract metric spaces. Comput. Math. Appl. **62**, 342-350 (2011)
23. Klim, D, Wardowski, D: Dynamic processes and fixed points of set-valued nonlinear contractions in cone metric spaces. Nonlinear Anal. **71**, 5170-5175 (2009)
24. Shatanawi, W: Some common coupled fixed point results in cone metric spaces. Int. J. Math. Anal. **4**, 2381-2388 (2010)
25. Mustafa, Z, Sims, B: A new approach to generalized metric spaces. J. Nonlinear Convex Anal. **7**(2), 289-297 (2006)
26. Dhage, BC: Generalized metric space and mapping with fixed point. Bull. Calcutta Math. Soc. **84**, 329-336 (1992)
27. Dhage, BC: Generalized metric space and topological structure. I. An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **46**(1), 3-24 (2000)
28. Kaewcharoen, A, Kaewkhao, A: Common fixed points for single-valued and multi-valued mappings in *G*-metric spaces. Int. J. Math. Anal. **5**(36), 1775-1790 (2011)
29. Mustafa, Z, Sims, B: Some remarks concerning *D*-metric spaces. In: Proc. Int. Conf. on Fixed Point Theory and Appl. pp. 189-198. Valencia, Spain, July 2003 (2003)
30. Nedal, T, Hassen, A, Karapinar, E, Shatanawi, W: Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in *G*-metric spaces. Fixed Point Theory Appl. **2012**, Article ID 48 (2012)
31. Beg, I, Abbas, M, Nazir, T: Generalized cone metric spaces. J. Nonlinear Sci. Appl. **3**(1), 21-31 (2010)
32. Chen, CM: On set-valued contractions of Nadler type in tvs-*G*-cone metric spaces. Fixed Point Theory Appl. **2012**, Article ID 52 (2012). doi:10.1186/1687-1812-2012-52
33. Cho, SH, Bae, JS: Fixed point theorems for multi-valued maps in cone metric spaces. Fixed Point Theory Appl. **2011**, Article ID 87 (2011). doi:10.1186/1687-1812-2011-87
34. Janković, S, Kadelburg, Z, Radenović, S: On cone metric spaces: a survey. Nonlinear Anal. **74**, 2591-2601 (2011)
35. Jlelli, M, Samet, B: Remarks on *G*-metric spaces and fixed point theorems. Fixed Point Theory Appl. **2012**, Article ID 210 (2012)

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