Journal of Inequalities and Applications
a SpringerOpen Journal

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The existence of contingent epiderivative for a set-valued mapping and vector variational-like inequalities

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Abstract

In this paper we analyze the existence of the contingent epiderivative for a set-valued mapping in a general normed space with respect to Daniell cone. As its application, we will also establish the relationships between set-valued optimization problems and variational-like inequality problems under the conditions of pseudo invexity.

MSC: 49J52; 90C56

Keywords: set-valued optimization; ideal minimal points; contingent epiderivative; existence conditions; strong variational-like inequalities

1 Introduction

In [1], Jahn and Raüh introduced the notion of contingent epiderivative for a set-valued mapping, which modifies a notion introduced by Aubin [2] as upper contingent derivative, and also established the existence theory of this contingent epiderivative for a single-valued function. In [3], Jahn and Khan obtained the existence of this kind of contingent epiderivative for a real set-valued function. It has been shown that this notion of contingent epiderivative is a fundamental concept for the formulation of optimality conditions in set-valued optimization, but there are few works that study its existence for a set-valued mapping in general conditions. Although in [4] Rodríguez-Marín and Sama derived the existence of contingent epiderivative, this can only be assured if a set-valued mapping F has the *LBD* (lower bounded derivative) property. In the last decades, many researchers have given several other generalized notions of epiderivatives by using weak minimizers and minimizers and derived the existence theories for them; see [5] and [4, 6, 7], respectively. Using different kinds of minimal elements, one can define different kinds of epiderivatives. In our paper, we use the ideal minimal elements of a set and the concept of contingent cone to define the contingent epiderivative and analyze its domain, existence, uniqueness and other properties. Under determined conditions, we establish $\mathrm{dom}(DF(x_0, y_0)) = \mathrm{cone}(A - x_0) = T(A, x_0)$ and get the existence of $DF(x_0, y_0)$, where $\mathrm{dom}\, F$ is the domain of F, $DF(x_0, y_0)$ is the contingent epiderivative of F at $(x_0, y_0) \in \mathrm{gr}\, F$ (the graph of F is denoted by $\mathrm{gr}\, F$) and $T(A, x_0)$ is the contingent cone of A at x_0, respectively.

Our other purpose in this paper is to investigate the relationships between set-valued optimization problems and variational-like inequality problems. In fact, the relationships between vector variational-like inequality problems and optimization problems for a single-valued mapping have been studied by many authors, see [8–10] and so on; and in [11], Zeng

and Li also discussed the relationships between weak vector variational-like inequality problems and set-valued optimization problems. However, to the best of our knowledge, there are few papers discussing the solution relationships between set-valued optimization problems and strong vector variational-like inequality problems. Motivated by the works in [9] and [11], in this paper, we firstly introduce several kinds of generalized invexity for set-valued mappings and then prove that the solutions of the variational-like inequality problems are equivalent to the minima (ideal minima) of set-valued optimization problems.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries and recall the main notions of contingent cone. In Section 3, the concept of contingent epiderivatives is introduced and, under determined conditions, its existence theory is also established. In Section 4, we present that the solutions of the variational-like inequality problems are equivalent to the minima (ideal minima) of set-valued optimization problems.

2 Preliminaries and notations

Throughout the paper, if not stated otherwise, let X be a real normed space, let A be a nonempty subset of X, and let Y be a real normed space partially ordered by a closed, convex and pointed cone $D \subset Y$. The points of origin of all real normed spaces are denoted by 0_X and 0_Y.

Let $y_1, y_2 \in Y$, the orderings are defined in Y as follows:

$y_1 \nleq y_2 \Leftrightarrow y_2 - y_1 \notin D$;

$y_1 \leq y_2 \Leftrightarrow y_2 - y_1 \in D$.

Let $A \subset X$, $F : A \to 2^Y$ be a set-valued mapping. The graph, the epigraph and the domain of F are defined, respectively, by

$$\operatorname{gr} F = \big\{(x, y) \in X \times Y : y \in F(x)\big\},$$
$$\operatorname{epi} F = \big\{(x, y) \in X \times Y : y \in F(x) + D\big\},$$

and

$$\operatorname{dom} F = \big\{x \in X : F(x) \neq \emptyset\big\}.$$

We say that F is a D-convex set-valued mapping on A, if A is a convex set, and for all $x, y \in A$, all $\lambda \in [0, 1]$,

$$\lambda F(x) + (1 - \lambda)F(y) \subset F\big(\lambda x + (1 - \lambda)y\big) + D.$$

It is well known that if F is D-convex on A, then $\operatorname{epi} F$ is a convex subset in $X \times Y$.

Proposition 2.1 (see [12]) *Let $F : X \to 2^Y$. If $\operatorname{epi} F$ is a closed subset in $X \times Y$, then $F(x) + D$ is a closed set for each $x \in X$.*

Definition 2.1 Let B be a non-empty subset of Y.

(i) $y_0 \in B$ is called a minimal point of B with respect to cone D if $(B - y_0) \cap -D = \{0_Y\}$;

(ii) $y_0 \in B$ is called an ideal minimal point of B with respect to cone D if $B - y_0 \subseteq D$.

The elements of all minimal points and ideal minimal points of B are denoted by $\operatorname{Min} B$ and $\operatorname{IMin} B$, respectively.

Obviously, $\operatorname{IMin} B \subseteq \operatorname{Min} B$ if D is a pointed cone.

The following standard notions can be found in [6].

Definition 2.2 Let D be a closed, convex and pointed cone in Y.

  (i) The cone D is called Daniell cone if any decreasing sequence in Y having a lower
      bound converges to its infimum.

 (ii) A subset B of Y is said to be D-lower bounded (or be minorized) if there is an
      element $y \in Y$ such that $B \subset y + D$.

Proposition 2.2 *Let D be a closed, convex and pointed cone in Y, non-empty subset $B \subset Y$. If* $\operatorname{Min} B \neq \emptyset$ *and B is the D-lower bounded for every minimal point, i.e.,*

$$\text{for each } \bar{y} \in \operatorname{Min} B, \quad B \subset \bar{y} + D, \tag{2.1}$$

then $\operatorname{Min} B$ *is a single-point set and* $\operatorname{Min} B = \operatorname{IMin} B$.

Proof We prove this result by contradiction. Suppose that at least there are $y_1, y_2 \in \operatorname{Min} B$ and $y_1 \neq y_2$. By condition (2.1), we have

$$y_1 \in y_2 + D, \tag{2.2}$$

and

$$y_2 \in y_1 + D. \tag{2.3}$$

From (2.2), (2.3) and the assumption $y_1 \neq y_2$, one has $y_1 - y_2 \in D \setminus \{0_Y\}$ and $y_2 - y_1 \in D \setminus \{0_Y\}$, which contradicts D being a pointed cone, thus $y_1 = y_2 = \bar{y}$. Namely, $\operatorname{Min} B$ is a singleton. Again from (2.1), we have

$$B \subset \bar{y} + D,$$

i.e.,

$$B - \bar{y} \subset D,$$

this yields $\bar{y} \in \operatorname{IMin} B$. The proof is complete. □

Remark 2.1 We notice if $\operatorname{Min} B \neq \emptyset$ and (2.1) is fulfilled, then $\operatorname{IMin} B = \operatorname{Min} B \neq \emptyset$ is a single-point set. Clearly, if $\operatorname{IMin} B \neq \emptyset$, then $B \subseteq \operatorname{IMin} B + D$.

The following theorem is due to Borwein.

Theorem 2.1 (see [13]) *Assume that D is a pointed, convex and Daniell cone, and let B be a closed subset of Y. If B is D-lower bounded* (*or is minorized*), *then* $\operatorname{Min} B \neq \emptyset$.

Definition 2.3 For $A \subset X$, cone A and cl A denote its cone hull and closure of A, respectively.

Furthermore,

$$\text{cone } A = \{\lambda a : \lambda \geq 0, a \in A\}.$$

Let $A \subset X$, the contingent cone of A at x_0 is defined by

$$T(A, x_0) = \left\{ u \in X : \exists t_n \to 0^+, \exists u_n \to u, \exists n_0 \in N, \forall n \geq n_0, x_0 + t_n u_n \in A \right\}.$$

It is well known that if A is a convex set and $x_0 \in A$, then

$$T(A, x_0) = \text{cl cone}(A - x_0).$$

We say that $A \subset X$ satisfies the property F, if for any $x \in A$ and $\lambda \in [0, 1]$, one has $\lambda x \in A$. Let $0_X \in A$, we say that A satisfies the property Λ near the 0_X, if there exists neighborhood $B(0_X, \varepsilon)$ such that for any $x \in B(0_X, \varepsilon) \cap (\text{cl } A \setminus \text{int } A)$ and $\lambda \in [0, 1]$, one has $\lambda x \in (\text{cl } A \setminus \text{int } A)$, where $B(0_X, \varepsilon)$ denotes the ball centered at 0_X with radius ε.

Corollary 1 *Let $A \subset X$ be a convex set, $x_0 \in A$, then $A - x_0$ satisfies the property F.*

Proof For any $x \in A$ and $t \in [0, 1]$, $t(x - x_0) + x_0 = tx + (1 - t)x_0$, thereby, $t(x - x_0) + x_0 \in A$ follows immediately from A being a convex set, which implies $t(x - x_0) \in A - x_0$, the proof is complete. $\square$

Proposition 2.3 *Let $A \subset X$, if A satisfies the properties F and Λ near the 0_X, then $\text{cone}(\text{cl } A) = \text{cl}(\text{cone } A)$.*

Proof Obviously, $0_X \in \text{cone}(\text{cl } A)$ and $0_X \in \text{cl}(\text{cone } A)$, so we consider $u \neq 0_X$ in the sequel. First, we prove the inclusion $\text{cone}(\text{cl } A) \subseteq \text{cl}(\text{cone } A)$. Let $u \in \text{cone}(\text{cl } A)$, then there exist $a \in \text{cl } A$, $t > 0$ such that $u = ta$. Furthermore, following $a \in \text{cl } A$, there exists $\{a_n\} \subset A$, $a_n \to a$. For each n, set $u_n = ta_n$, then $u_n \in \text{cone } A$ and $u_n = ta_n \to ta = u$, i.e., $u_n \to u$, this implies $u \in \text{cl}(\text{cone } A)$.

For the contrary inclusion, we should only prove $\text{cone}(\text{cl } A)$ is a closed cone. Let $\{u_n\} \subset \text{cone}(\text{cl } A)$ and $u_n \to u$. Next, we will prove $u \in \text{cone}(\text{cl } A)$. In fact, we can conclude that there exist $x_n \in X$ and $x_n \to 0_X$ such that $u + x_n = u_n \in \text{cone}(\text{cl } A)$. So, for each n, there exist $t_n > 0$, $a_n \in \text{cl } A$ such that

$$u + x_n = u_n = t_n a_n. \tag{2.4}$$

On the one hand, if $a_n \to 0_X$, then $t_n \to +\infty$ as $n \to +\infty$. For each n, we have

$$\frac{u}{t_n} + \frac{x_n}{t_n} = a_n \in \text{cl } A.$$

For given $\varepsilon > 0$, there exists n_0 such that $a_n \in B(0_X, \varepsilon)$, $\forall n \geq n_0$. We divide it into two cases to discuss.

(i) If $\{a_n\} \subset \mathrm{cl}\,A \setminus \mathrm{int}\,A$ as $n \geq n_0$, by assumption, we can conclude that there exists a subsequence $\{a_{n_m}\} \subseteq \{a_n\}$ such that

$$a_{n_m} = \lambda_{n_m} a_{\bar{n}_0}, \tag{2.5}$$

with $\lambda_{n_m} \in (0,1]$ and $\bar{n}_0 \geq n_0$. In fact, since $t_n a_n \to u$, so for any $\varepsilon_1 > 0$, there exists $n_1 \geq n_0$ such that $\|t_n a_n - u\| \leq \varepsilon_1$ as $n \geq n_1$. If for any given $\bar{n}_0 \geq n_1$ and $\lambda \in (0,1]$,

$$\lambda a_{\bar{n}_0} \notin \{a_n\}, \tag{2.6}$$

then we can take some t_n from $\{t_n\}$ such that $\frac{t_{\bar{n}_0}}{t_n} < 1$ as $n \geq \bar{n}_0$. From (2.6) it follows that $\frac{t_{\bar{n}_0}}{t_n} a_{\bar{n}_0} \notin \{a_n\}$ as $n \geq \bar{n}_0$. Thus $\|t_n(\frac{t_{\bar{n}_0}}{t_n} a_{\bar{n}_0}) - u\| = \|t_{\bar{n}_0} a_{\bar{n}_0} - u\| > \varepsilon_1$, a contradiction. So (2.5) holds. Again from $t_n a_n \to u$, so, $t_{n_m} a_{n_m} = t_{n_m} \lambda_{n_m} a_{\bar{n}_0} \to u$. Furthermore, $a_{\bar{n}_0}$ is fixed, so $t_{n_m} \lambda_{n_m} \to t_0$. That is, $t_{n_m} \lambda_{n_m} = t_0 + \bar{t}_{n_m}$ with $\bar{t}_{n_m} \to 0$. Thus, $t_{n_m} a_{n_m} = (t_0 + \bar{t}_{n_m}) a_{\bar{n}_0} = t_0 a_{\bar{n}_0} + \bar{t}_{n_m} a_{\bar{n}_0} \to t_0 a_{\bar{n}_0}$. From the uniqueness of limits, we have $u = t_0 a_{\bar{n}_0}$ with $a_{\bar{n}_0} \in \mathrm{cl}\,A \setminus \mathrm{int}\,A$, so $u \in \mathrm{cone}(\mathrm{cl}\,A)$.

(ii) If $\{a_n\} \subset \mathrm{int}\,A$, then for any n, $\exists \varepsilon_n$ such that $B(a_n, \varepsilon_n) \subset A$. From $t_n a_n = u + x_n$, one has $a_n - \frac{u}{t_n} = \frac{x_n}{t_n}$ and $a_n \in B(0_X, \varepsilon)$ as n is large enough. If there exists n_0 such that $\|a_{n_0} - \frac{u}{t_{n_0}}\| = \frac{\|x_{n_0}\|}{t_{n_0}} \leq \varepsilon_{n_0}$, then $\frac{u}{t_{n_0}} \in A$, so $u \in \mathrm{cone}(\mathrm{cl}\,A)$. If, for any n, $\|a_n - \frac{u}{t_n}\| = \|\frac{x_n}{t_n}\| > \varepsilon_n$, then this implies there exist $\bar{a}_n \in \mathrm{cl}\,A \setminus A$ and $\|\bar{a}_n - a_n\| \leq \frac{\|x_n\|}{t_n}$. However, $\|t_n \bar{a}_n - u\| = \|t_n \bar{a}_n - t_n a_n + t_n a_n - u\| \leq t_n \frac{\|x_n\|}{t_n} + \|x_n\| = 2\|x_n\| \to 0$. Thus, $t_n \bar{a}_n \to u$. By $\bar{a}_n \in \mathrm{cl}\,A \setminus A$ and (i), we conclude that $u \in \mathrm{cone}(\mathrm{cl}\,A)$.

On the other hand, if $a_n \nrightarrow 0_X$, then $\{t_n\}$ is bounded. Set $t = \sup_n\{t_n\}$, then $t \geq t_n$ and $t \in R^+$. Dividing t by (2.4) and for each $n \in N$, set $b_n = \frac{t_n a_n}{t} = \frac{u}{t} + \frac{x_n}{t}$. According to assumption A satisfying the property F, we get $b_n \in A$ and $b_n \to \frac{u}{t}$ as $n \to \infty$. Thus, $\frac{u}{t} \in \mathrm{cl}\,A$, *i.e.*, $u \in \mathrm{cone}(\mathrm{cl}\,A)$. From the above two parts, we get $\mathrm{cone}(\mathrm{cl}\,A)$ is a closed cone, so $\mathrm{cl}(\mathrm{cone}\,A) \subseteq \mathrm{cone}(\mathrm{cl}\,A)$. The proof is complete. $\qquad\square$

Corollary 2 *Let A be a cone of X and satisfy the property Λ near the 0_X, then $\mathrm{cone}(\mathrm{cl}\,A) = \mathrm{cl}\,A$.*

Corollary 3 *Let $x_0 \in A$, if A is a closed convex subset of X such that $A - x_0$ satisfy the property Λ near the 0_X, then $\mathrm{cone}(A - x_0) = \mathrm{cl}\,\mathrm{cone}(A - x_0)$.*

Proposition 2.4 (see Chapter 4 of [3]) *Let $x_0 \in A \subset X$, if A is star-shaped at x_0, then $T(A, x_0) = \mathrm{cl}(\mathrm{cone}(A - x_0))$.*

Corollary 4 *Let $x_0 \in A \subset X$, if A is a closed convex subset such that $A - x_0$ satisfy the property Λ near the 0_X, then $T(A, x_0) = \mathrm{cone}(A - x_0)$.*

Proof Since $x_0 \in A$ and A is a convex subset, we have $T(A, x_0) = \mathrm{cl}(\mathrm{cone}(A - x_0))$. To apply Corollary 3, one has

$$\mathrm{cl}\big(\mathrm{cone}(A - x_0)\big) = \mathrm{cone}(A - x_0),$$

it is clear that $T(A, x_0) = \mathrm{cone}(A - x_0)$. $\qquad\square$

3 The existence theory for contingent epiderivative

The aim of this section is to discuss the existence of the contingent epiderivative for a set-valued map defined from a real normed space to a real normed space. Let $A \subset X$, $F : A \to 2^Y$ be a set-valued mapping.

Definition 3.1 (see [1]) Let $x_0 \in A$ and a pair $(x_0, y_0) \in \mathrm{gr}\,F$ be given. The contingent epiderivative $DF(x_0, y_0)$ of F at (x_0, y_0) is the single-valued mapping from X to Y defined by

$$\mathrm{epi}\big(DF(x_0, y_0)\big) = T\big(\mathrm{epi}\,F, (x_0, y_0)\big). \tag{3.1}$$

Next, we establish $DF(x_0, y_0)$ in the following (3.6) and also prove it is a single-valued mapping.

Let a pair $(x_0, y_0) \in \mathrm{gr}\,F$, for any $x \in X$, set

$$DF(x_0, y_0)(x) = \mathrm{IMin}\big\{ y \mid (x, y) \in T\big(\mathrm{epi}\,F, (x_0, y_0)\big) \big\}. \tag{3.2}$$

Notice that if $A \subset X$ is a closed convex set, F is D-convex and $\mathrm{epi}\,F$ is a closed subset in $X \times Y$ such that $(\mathrm{epi}\,F - (x_0, y_0))$ satisfies the property Λ near the $(0_X, 0_Y)$, then

$$\begin{aligned}
T\big(\mathrm{epi}\,F, (x_0, y_0)\big) &= \mathrm{cone}\big(\mathrm{epi}\,F - \big\{(x_0, y_0)\big\}\big) \\
&= \mathrm{cone}\big(\big(\mathrm{gr}\,F + \{0\} \times D\big) - \big\{(x_0, y_0)\big\}\big) \\
&= \mathrm{cone}\big(\big(A \times F(A) + \{0\} \times D\big) - \big\{(x_0, y_0)\big\}\big) \\
&= \bigcup_{a \in A} \mathrm{cone}\big(\big(\{a\} \times F(a) + \{0\} \times D\big) - \big\{(x_0, y_0)\big\}\big),
\end{aligned}$$

is a closed convex cone. So, for every $(x, y) \in \mathrm{epi}(DF(x_0, y_0))$ and $(x, y) \neq (0_X, 0_Y)$, if $(x, y) \in T(\mathrm{epi}\,F, (x_0, y_0))$, then there exist $l > 0$, $a \in A$, $v \in F(a)$ and $d \in D$ such that

$$(x, y) = l(a - x_0, v - y_0 + d). \tag{3.3}$$

By (3.3), we have

$$x = l(a - x_0) \in \mathrm{cone}(A - x_0), \tag{3.4}$$

and

$$y = l(v - y_0 + d) \in \mathrm{cone}\big(F(x_0 + \lambda x) - y_0 + D\big), \tag{3.5}$$

where $\lambda = \frac{1}{l}$.

Otherwise, if $(x, y) = (0_X, 0_Y)$, then we can conclude that $a = x_0$, $v = y_0$ and $d = y_0$. Therefore, formulas (3.4) and (3.5) are clearly established.

Remark 3.1 From the above discussion and Corollary 4, if $A \subset X$ is a closed convex set and $A - x_0$ satisfies the property Λ near the 0_X, $F : A \to 2^Y$ is D-convex and $\mathrm{epi}\,F$ is closed in $X \times Y$ such that $(\mathrm{epi}\,F - (x_0, y_0))$ satisfies the property Λ near the $(0_X, 0_Y)$, then $DF(x_0, y_0)$ can be defined on $T(A, x_0)$.

Furthermore, we can conclude that formula (3.2) is equivalent to the following:

$$DF(x_0, y_0)(x) = \text{IMin}\big(\text{cone}\big(F(x_0 + \lambda x) - y_0 + D\big)\big), \tag{3.6}$$

where $x \in T(A, x_0)$ and $\lambda \geq 0$.

From Theorem 2.1, we can obtain the following existence theorem.

Theorem 3.1 *Let $A \subset X$ be a closed convex set, let $F : A \to 2^Y$ be D-convex, and let* epi F *be a closed subset in $X \times Y$. Let $(x_0, y_0) \in \text{gr}\, F$. If for any $x \in T(A, x_0)$ and $\lambda \geq 0$, the following conditions hold*: (1) $(F(x_0 + \lambda x) - y_0 + D)$ *satisfies the properties $\digamma$ and Λ near the 0_Y;* (2) $\text{cone}(F(x_0 + \lambda x) - y_0 + D)$ *is D-lower bounded;* (3) *the cone D is Daniell cone;* (4) $\text{cone}(F(x_0 + \lambda x) - y_0 + D)$ *satisfies* (2.1). *Then* $\text{IMin}(\text{cone}(F(x_0 + \lambda x) - y_0 + D)) \neq \emptyset$ *is a single-point set.*

Proof Since epi F a closed convex subset in $X \times Y$, by Proposition 2.1, one has $F(x_0 + \lambda x) + D$ a closed set for each $x \in T(A, x_0)$, so is $(F(x_0 + \lambda x) - y_0 + D)$. By assumption (1) and Proposition 2.3, we can conclude that $\text{cone}(F(x_0 + \lambda x) - y_0 + D)$ is a closed set. Considering Theorem 2.1 and given conditions (2) and (3), we have $\text{Min}(\text{cone}(F(x_0 + \lambda x) - y_0 + D)) \neq \emptyset$. Combining Remark 2.1 and assumption (4), we can conclude $\text{IMin}(\text{cone}(F(x_0 + \lambda x) - y_0 + D)) \neq \emptyset$ is a single-point set. □

Remark 3.2 From (3.6), we know that $DF(x_0, y_0)$ exists if $\text{IMin}(\text{cone}(F(x_0 + \lambda x) - y_0 + D)) \neq \emptyset$.

The following theorem shows that our definition of the contingent epiderivative for a set-valued mapping is well defined.

Theorem 3.2 *If all the conditions of Theorem 3.1 are fulfilled, then* $\text{epi}(DF(x_0, y_0)) = T(\text{epi}\, F, (x_0, y_0))$.

Proof Combining (3.6) and Theorem 3.1, we get $DF(x_0, y_0)(x)$ exists for each $x \in T(A, x_0)$. Since $(x, DF(x_0, y_0)(x)) \in \text{cone}(\text{epi}\, F - (x_0, y_0))$, thus

$$
\begin{aligned}
\text{epi}\big(DF(x_0, y_0)\big) &\subset \text{cone}\big(\text{epi}\, F - \big\{(x_0, y_0)\big\}\big) + \{0\} \times D \\
&= \text{cone}\big(\text{epi}\, F - \big\{(x_0, y_0)\big\}\big) \\
&= T\big(\text{epi}\, F, (x_0, y_0)\big).
\end{aligned}
$$

Now, let us show the converse inclusion. From the definition of the $DF(x_0, y_0)$, for any $(x, y) \in T(\text{epi}\, F, (x_0, y_0))$, we have

$$
\begin{aligned}
(x, y) &\in \big((x, DF(x_0, y_0)) + \{0\} \times D\big) \\
&\subset \text{gr}\big(DF(x_0, y_0)\big) + \{0_Y\} \times D \\
&= \text{epi}\big(DF(x_0, y_0)\big).
\end{aligned}
$$

Therefore,

$$T\big(\text{epi}\, F, (x_0, y_0)\big) \subset \text{epi}\big(DF(x_0, y_0)\big).$$

The proof is complete. □

Next we show under appropriate assumptions that the contingent epiderivative is a strictly positive homogeneous and subadditive map in the case of A, a convex set, and epi F, a closed convex subset.

Definition 3.2 Let X be a real linear space and let Y be a real linear space partially ordered by a closed convex pointed cone $D \subset Y$. A map $f : X \to 2^Y$ is called

(a) strictly positive homogeneous if

$$f(\alpha x) = \alpha f(x) \quad \text{for all } \alpha > 0 \text{ and all } x \in X,$$

(b) subadditive if

$$f(x_1) + f(x_2) \subset f(x_1 + x_2) + D \quad \text{for all } x_1, x_2 \in X.$$

If the properties under (a) with $\alpha \geq 0$ and (b) hold, then f is called sublinear.

Theorem 3.3 *Let A be a closed convex set in X, $x_0 \in A$ such that $A - x_0$ satisfies the property Λ near the 0_X. Let D be a closed convex pointed cone in Y and $(x_0, y_0) \in \mathrm{gr}\, F$ such that epi F is a closed convex set and $(\mathrm{epi}\, F - (x_0, y_0))$ satisfies the property Λ near the $(0_X, 0_Y)$. If for all $x \in T(A, x_0)$, $DF(x_0, y_0)(x)$ exists, then $DF(x_0, y_0)(x)$ is sublinear.*

Proof We take any $\alpha > 0$ and any $x \in T(A, x_0)$. Then we obtain

$$
\begin{aligned}
DF(x_0, y_0)(\alpha x) &= \mathrm{IMin}\big\{ y \mid (\alpha x, y) \in T\big(\mathrm{epi}\, F, (x_0, y_0)\big) \big\} \\
&= \mathrm{IMin}\big\{ y \mid y \in \big(\mathrm{cone}\big(F(x_0 + \lambda \alpha x) - y_0 + D\big)\big) \big\} \\
&= \mathrm{IMin}\big\{ \alpha v \mid \alpha v \in \big(\mathrm{cone}\big(F(x_0 + \lambda \alpha x) - y_0 + D\big)\big) \big\} \\
&= \alpha\, \mathrm{IMin}\big\{ v \mid v \in \big(\mathrm{cone}\big(F(x_0 + \lambda x) - y_0 + D\big)\big) \big\}.
\end{aligned}
$$

Thus $DF(x_0, y_0)$ is strictly positive homogeneous.

Next, for $x_1, x_2 \in T(A, x_0)$, we have $(x_1, DF(x_0, y_0)(x_1)) \in T(\mathrm{epi}\, F, (x_0, y_0))$ and $(x_2, DF(x_0, y_0)(x_2)) \in T(\mathrm{epi}\, F, (x_0, y_0))$. Since F is D-convex and epi F is a closed subset, then $T(\mathrm{epi}\, F, (x_0, y_0))$ is a closed convex cone. Thus,

$$\big(x_1 + x_2, DF(x_0, y_0)(x_1) + DF(x_0, y_0)(x_2)\big) \in T\big(\mathrm{epi}\, F, (x_0, y_0)\big),$$

implies

$$DF(x_0, y_0)(x_1) + DF(x_0, y_0)(x_2) \in \mathrm{cone}\big(F\big(x_0 + \lambda(x_1 + x_2)\big) - y_0 + D\big).$$

By Remark 2.1, we have

$$
\begin{aligned}
DF(x_0, y_0)(x_1) + DF(x_0, y_0)(x_2) &\in \mathrm{cone}\big(F\big(x_0 + \lambda(x_1 + x_2)\big) - y_0 + D\big) \\
&\subset \mathrm{IMin}\big(\mathrm{cone}\big(F\big(x_0 + \lambda(x_1 + x_2)\big) - y_0 + D\big)\big) + D \\
&= DF(x_0, y_0)(x_1 + x_2) + D.
\end{aligned}
$$

The proof is complete. $\qquad\square$

4 Relationships between vector variational-like inequalities and optimization problems

In this section, we restrict ourselves to dealing with relationships between two kinds of vector variational-like inequality problems and set-valued optimization problems.

The set-valued vector optimization problem under our consideration is

$$\text{Min}\{F(x) : x \in A\}, \tag{4.1}$$

where $F : X \to 2^Y$, $F(A) = \bigcup_{x \in A} F(x)$. We denote set-valued optimization problem (4.1) as (SOP).

Definition 4.1 Consider the above problem (SOP), let $x_0 \in A$, $y_0 \in F(x_0)$.
  (i) A pair $(x_0, y_0) \in \text{gr}\, F$ is called a minimal solution of F on A if $(F(A) - y_0) \cap -D = \{0_Y\}$;
  (ii) A pair $(x_0, y_0) \in \text{gr}\, F$ is called an ideal minimal solution of F on A if $(F(A) - y_0) \subseteq D$.

The sets of all minimal solutions and ideal minimal solutions of (SOP) are denoted by $\text{Min}(F, A)$ and $\text{IMin}(F, A)$, respectively.

In the following, we always assume that contingent epiderivative of F exists.

Definition 4.2 (see [14]) Let the set A be convex, let the set-valued mapping $F : A \to 2^Y$ be D-convex. Let $(x_0, y_0) \in \text{gr}\, F$ and let the contingent epiderivative $DF(x_0, y_0)$ of F at (x_0, y_0) exist.
  (i) A linear map $L : X \to Y$, with $L(x) \leq DF(x_0, y_0)(x)$, for all $x \in T(A, x_0)$ is called a subgradient of F at (x_0, y_0).
  (ii) The set $\partial F(x_0, y_0) = \{L : X \to Y : L(x) \leq DF(x_0, y_0)(x), \forall x \in T(A, x_0)\}$ of all subgradients L of F at (x_0, y_0) is called the subdifferential of F at (x_0, y_0).

Definition 4.3 A set $A \subset X$ is said to be an invex set if there exists a function $\eta : X \times X \to X$ such that $\forall x, y \in A$, $\forall \lambda \in [0, 1]$, $y + \lambda \eta(x, y) \in A$.

Throughout the paper, we always assume that A is an invex subset of X, the function η defined on A, *i.e.*, $\eta : A \times A \to X$ and the subdifferential of F exists at every $(x, y) \in \text{gr}\, F$.

Definition 4.4 Let $F : A \to 2^Y$ be a set-valued mapping, $(x_0, y_0) \in \text{gr}\, F$. F is called strong pseudo-invex at (x_0, y_0) with respect to η on A if $\forall x \in A$, $\forall y \in F(x)$ and $\forall L \in \partial F(x_0, y_0)$, $y - y_0 \notin D \Rightarrow L\eta(x, x_0) \notin D$.

F is said to be strong pseudo-invex with respect to η on A if for every pair $(x, y) \in \text{gr}\, F$, F is strong pseudo-invex at (x, y) with respect to η.

Definition 4.5 Let $F : A \to 2^Y$ be a set-valued mapping, $(x_0, y_0) \in \text{gr}\, F$. F is called pseudo-invex at (x_0, y_0) with respect to η on A if $\forall x \in A$, $\forall y \in F(x)$ and $\forall L \in \partial F(x_0, y_0)$, $y - y_0 \in -D \Rightarrow L\eta(x, x_0) \in -D$.

F is said to be pseudo-invex with respect to η on A if for every pair $(x, y) \in \text{gr}\, F$, F is pseudo-invex at (x, y) with respect to η.

Definition 4.6 Let $F : A \to 2^Y$ be a set-valued mapping. F is said to be D-preinvex with respect to η on A if $\forall x, y \in A$, $\forall \lambda \in [0, 1]$, $\lambda F(x) + (1 - \lambda)F(y) \subset F(y + \lambda \eta(x, y)) + D$.

From the above definitions, we have the following useful proposition.

Proposition 4.1 *Let $F : A \to 2^Y$ be D-preinvex with respect to η on A, $(x_0, y_0) \in \operatorname{gr} F$. If $DF(x_0, y_0)$ exists on $\operatorname{cone}(A - x_0)$, then for all $L \in \partial F(x_0, y_0)$, all $x \in A$, all $y \in F(x)$, $y - y_0 \in \langle L, \eta(x, x_0) \rangle + D$.*

Proof For all $x \in A$, since F is D-preinvex with respect to η on A and $x_0 \in A$, for any $\lambda \in [0, 1]$, we obtain

$$(1 - \lambda)F(x_0) + \lambda F(x) \subset F\big(x_0 + \lambda \eta(x, x_0)\big) + D,$$

which implies that

$$y - y_0 \in \big(F\big(x_0 + \lambda \eta(x, x_0)\big) + D - y_0\big)/\lambda. \tag{4.2}$$

Thus, we get

$$y - y_0 \in DF(x_0, y_0)\big(\eta(x, x_0)\big) + D. \tag{4.3}$$

It is clear that

$$\forall x \in A, \quad \eta(x, x_0) \in \operatorname{dom} DF(x_0, y_0). \tag{4.4}$$

By the definition of subdifferential $\partial F(x_0, y_0)$, for all $L \in \partial F(x_0, y_0)$, we have

$$L(x) \leq DF(x_0, y_0)(x), \quad \forall x \in \operatorname{dom} DF(x_0, y_0),$$

that is,

$$DF(x_0, y_0)(x) \in L(x) + D. \tag{4.5}$$

Combining (4.3), (4.4) and (4.5), we can conclude, $y - y_0 \in \langle L, \eta(x, x_0) \rangle + D$. □

Remark 4.1 It is easy to see that F is pseudo-invex with respect to η on A if F is D-preinvex with respect to η on A.

A vector variational inequality has been shown to be a useful tool in vector optimization. Some authors have proved the equivalence between them, see [9, 10, 15]. The vector variational-like inequality problem is a generalized form of the vector variational inequality problem, which was introduced and studied by [8].

Now, let us offer the following definitions.

A vector variational-like inequality problem (VVLI) is to find $(x_0, y_0) \in \operatorname{gr} F$ such that $\langle L, \eta(x, x_0) \rangle \notin -D \setminus \{0_Y\}$ for all $x \in A \setminus \{x_0\}$ and all $L \in \partial F(x_0, y_0)$. A strong vector variational-like inequality problem (SVVLI) is to find $(x_0, y_0) \in \operatorname{gr} F$ such that $\langle L, \eta(x, x_0) \rangle \in D$ for all $x \in A$ and all $L \in \partial F(x_0, y_0)$.

In the following, using the tools of non-smooth analysis and the concept of non-smooth (strong) vector pseudo-invexity, we shall obtain the stronger results than those of [11].

Theorem 4.1 *Let $x_0 \in A$, $(x_0, y_0) \in \operatorname{gr} F$. If the pair (x_0, y_0) solves the (SVVIL) and F is strong pseudo-invex at (x_0, y_0) with respect to η on A, then (x_0, y_0) is an ideal minimizer of the (SOP).*

Proof Since the pair (x_0, y_0) solves the (SVVIL), we have that

$$\langle L, \eta(x, x_0) \rangle \in D, \quad \forall x \in A, \forall L \in \partial F(x_0, y_0). \tag{4.6}$$

If (x_0, y_0) is not an ideal minimizer of the (SOP), then there exist $\bar{x} \in A$, $(\bar{x}, \bar{y}) \in \operatorname{gr} F$ such that

$$\bar{y} - y_0 \notin D. \tag{4.7}$$

Since F is strong pseudo-invex at (x_0, y_0) with respect to η on A, we get $\bar{x} \in A$, $\forall L \in \partial F(x_0, y_0)$ such that

$$\bar{y} - y_0 \notin D \Rightarrow \langle L, \eta(\bar{x}, x_0) \rangle \notin D. \tag{4.8}$$

By (4.7) and (4.8), we obtain $\langle L, \eta(\bar{x}, x_0) \rangle \notin D$, which contradicts (4.6). $\qquad \square$

In order to see the converse of the above theorem, we must impose stronger conditions, as can be observed in the following theorem.

Theorem 4.2 *Suppose that $F : A \to 2^Y$ and $-F$ is D-preinvex with respect to η on A, $(x_0, y_0) \in \operatorname{gr} F$. If $-\partial F(x_0, y_0) \subseteq \partial(-F)(x_0, -y_0)$ and the pair (x_0, y_0) is an ideal minimizer of (SOP), then (x_0, y_0) solves the (SVVLI).*

Proof Let (x_0, y_0) be an ideal minimizer of the (SOP), then we have

$$y - y_0 \in D, \quad \forall x \in A, \forall y \in F(x). \tag{4.9}$$

Since $-F$ is D-preinvex with respect to η and $-\partial F(x_0, y_0) \subseteq \partial(-F)(x_0, -y_0)$, we have that $\forall x \in A$, $\forall y \in F(x)$ and $\forall L \in \partial F(x_0, y_0)$ such that

$$-y + y_0 \in \langle -L, \eta(x, x_0) \rangle + D,$$

which implies

$$\langle L, \eta(x, x_0) \rangle \in y - y_0 + D. \tag{4.10}$$

Combining (4.9) and (4.10), we can conclude

$$\langle L, \eta(x, x_0) \rangle \in D + D \subset D$$

for all $x \in A$ and all $L \in \partial F(x_0, y_0)$. That is, (x_0, y_0) solves the (SVVLI). $\qquad \square$

Theorem 4.3 *Suppose that $F : A \to 2^Y$ and $-F$ is D-preinvex with respect to η on A, $(x_0, y_0) \in \mathrm{gr}\, F$. If $-\partial F(x_0, y_0) \subseteq \partial(-F)(x_0, -y_0)$ and the pair (x_0, y_0) is a minimizer of (SOP), then (x_0, y_0) solves the (VVLI).*

Proof Let (x_0, y_0) be a minimizer of the (SOP), then we have

$$y - y_0 \notin \left(-D \setminus \{0_Y\}\right), \quad \forall x \in A \setminus \{x_0\}, \forall y \in F(x). \tag{4.11}$$

If (x_0, y_0) does not solve the (VVIL), then there exist $\bar{x} \in A \setminus \{x_0\}$ and $\bar{L} \in \partial F(x_0, y_0)$ such that

$$\left\langle \bar{L}, \eta(\bar{x}, x_0) \right\rangle \in \left(-D \setminus \{0_Y\}\right). \tag{4.12}$$

Since $-F$ is D-preinvex with respect to η and $-\partial F(x_0, y_0) \subseteq \partial(-F)(x_0, -y_0)$, we have that $\forall x \in A$, $\forall y \in F(x)$ and $\bar{L} \in \partial F(x_0, y_0)$ such that

$$-y + y_0 \in \left\langle -\bar{L}, \eta(x, x_0) \right\rangle + D. \tag{4.13}$$

Combining (4.12) and (4.13), for all $\bar{y} \in F(\bar{x})$, one has

$$-\bar{y} + y_0 \in D \setminus \{0_Y\} + D \subset D \setminus \{0_Y\},$$

which contradicts (4.11). The proof is complete. □

The converse case of the above theorem (see Theorem 4.4 of [11]) only requires F to be pseudo-invex.

Authors' contributions
YC carried out the main preliminaries, the existence theory for contingent epiderivative and the relationships between vector variational-like inequalities and optimization problems. SL carried out the introduction and some preliminaries. YL participated in the design of the study and some preliminaries.

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