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A posteriori error estimates of mixed finite element methods for general optimal control problems governed by integro-differential equations

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Abstract
In this paper, we study the mixed finite element methods for general convex optimal control problems governed by integro-differential equations. The state and the co-state are discretized by the lowest order Raviart-Thomas mixed finite element spaces and the control is discretized by piecewise constant elements. We derive *a posteriori* error estimates for the coupled state and control approximation. Such estimates are obtained for some model problems which frequently appear in many applications.
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1 Introduction

The finite element discretization of optimal control problems has been extensively investigated in early literature. There are two early papers on the numerical approximation of linear quadratic elliptic optimal control problems by Falk [1] and Geveci [2]. In [3], the authors derived *a posteriori* error estimators for a class of distributed elliptic optimal control problems. These error estimators are shown to be useful in adaptive finite element approximation for the optimal control problems and are implemented in the adaptive approach. Brunner and Yan [4] discussed finite element Galerkin discretization of a class of constrained optimal control problems governed by integral equations and integro-differential equations. The analysis focuses on the derivation of *a priori* error estimates and *a posteriori* error estimators for the approximation schemes. Systematic introduction of the finite element method for optimal control problems can be found in [5–7]. Some of the techniques directly relevant to our work can be found in [8, 9].

In many control problems, the objective functional contains the gradient of the state variables. Thus, the accuracy of the gradient is important in numerical discretization of the coupled state equations. Mixed finite element methods are appropriate for the state equations in such cases since both the scalar variable and its flux variable can be approximated to the same accuracy by using such methods. Some specialists have made many important works on some topic of mixed finite element methods for linear optimal control problems. Some realistic regularity assumptions are presented and applied to error estimation by using an operator interpolation. The authors derive L^2-superconvergence properties for the flux functions along the Gauss lines and for the scalar functions at the Gauss points via mixed projections in [10–12]. Also, L^∞-error estimates for general optimal control problems using mixed finite element methods are considered in [13, 14]. In [15, 16], *a posteriori* error estimates of mixed finite element methods for general convex optimal control problems are addressed. However, there does not seem to exist much work on theoretical analysis for mixed finite element approximation of optimal control problems governed by integro-differential equations in the literature.

In this paper we derive *a posteriori* error estimates of mixed finite element methods for general optimal control problems governed by integro-differential equations. We are concerned with the following optimal control problems:

$$\min_{u\in K\subset U}\{g_1(\mathbf{p})+g_2(y)+j(u)\} \tag{1.1}$$

subject to the state equation

$$-\operatorname{div}(A\nabla y)+\int_\Omega G(s,t)y(s)\,ds=f+Bu, \quad x\in\Omega, \tag{1.2}$$

with the boundary condition

$$y=0, \qquad x\in\partial\Omega, \tag{1.3}$$

which can be written in the form of the first-order system

$$\operatorname{div}\mathbf{p}+\int_\Omega G(s,t)y(s)\,ds=f+Bu, \quad x\in\Omega, \tag{1.4}$$

$$\mathbf{p}=-A\nabla y, \quad x\in\Omega, \tag{1.5}$$

$$y=0, \quad x\in\partial\Omega, \tag{1.6}$$

where $\Omega\subset\mathbb{R}^2$ is a regular bounded and convex open set with the boundary $\partial\Omega$, Ω_U is a bounded open set in $\mathbb{R}^2$ with the Lipschitz boundary $\partial\Omega_U$, g_1, g_2, and j are convex functionals and K is a closed convex set in $U=L^2(\Omega_U)$. Here, $f\in L^2(\Omega)$ and B is a continuous linear operator from $L^2(\Omega_U)$ to $L^2(\Omega)$, $G(\cdot,\cdot)\in H^1(\Omega\times\Omega)$, and there are constants $c_0, C_0>0$ satisfying

$$c_0\le G(s,t)\le C_0, \quad \forall s,t\in\Omega. \tag{1.7}$$

The coefficient matrix $A(x)=(a_{i,j}(x))_{2\times 2}\in L^\infty(\Omega;\mathbb{R}^{2\times 2})$ is a symmetric 2×2-matrix and there are constants $c_1,c_2>0$ satisfying, for any vector $\mathbf{X}\in\mathbb{R}^2$, $c_1\|\mathbf{X}\|^2_{\mathbb{R}^2}\le\mathbf{X}^tA\mathbf{X}\le c_2\|\mathbf{X}\|^2_{\mathbb{R}^2}$.

We adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|^p_{m,p}=\sum_{|\alpha|\le m}\|D^\alpha v\|^p_{L^p(\Omega)}$, a semi-norm $|\cdot|_{m,p}$ given by $|v|^p_{m,p}=\sum_{|\alpha|=m}\|D^\alpha v\|^p_{L^p(\Omega)}$. We set $W^{m,p}_0(\Omega)=\{v\in W^{m,p}(\Omega): v|_{\partial\Omega}=0\}$. For $p=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega)$, $H^m_0(\Omega)=W^{m,2}_0(\Omega)$, and $\|\cdot\|_m=\|\cdot\|_{m,2}$, $\|\cdot\|=\|\cdot\|_{0,2}$.

Now, we recall a result from Kress [17].

Lemma 1.1 *We assume that $G(\cdot,\cdot)$ is such that the equations*

$$-\operatorname{div}(A\nabla\xi)+\int_\Omega G(s,t)\xi(t)\,dt=F_1\quad in\ \Omega,\xi|_{\partial\Omega}=0, \tag{1.8}$$

$$-\operatorname{div}(A\nabla\zeta)+\int_\Omega G(s,t)\zeta(s)\,ds=F_2\quad in\ \Omega,\zeta|_{\partial\Omega}=0 \tag{1.9}$$

have unique solutions $\xi,\zeta\in H^1(\Omega)$ for any $F_1,F_2\in L^2(\Omega)$, respectively. Moreover, there exists a positive constant C such that

$$\|\xi\|_{H^2(\Omega)}\le C\|F_1\|_{L^2(\Omega)}, \tag{1.10}$$

$$\|\zeta\|_{H^2(\Omega)}\le C\|F_2\|_{L^2(\Omega)}. \tag{1.11}$$

In particular, it can be proved that [18] there exist unique solutions for the above integral-differential equations if $|G(s,t)|\le a_0$, where a_0 is small enough such that

$$\int_\Omega A\nabla v\nabla v\ge(a_0+\delta)|\Omega|\|v\|_{1,\Omega}^2,\quad\forall v\in H^1(\Omega),$$

where $|\Omega|=\int_\Omega 1$.

The outline of this paper is as follows. In the next section, we construct the mixed finite element discretization for the optimal control problems governed by integro-differential equations and briefly state the definitions and properties of some interpolation operators. Then we discuss *a posteriori* error estimates for the intermediate error in Section 3. In Section 4, we derive *a posteriori* error estimates for the control and state approximations. Finally, some applications are presented in Section 5.

2 Mixed methods for optimal control problems

In this section we briefly discuss the mixed finite element discretization of convex optimal control problems (1.1)-(1.3). Let

$$\mathbf{V}=H(\operatorname{div};\Omega)=\{\mathbf{v}\in(L^2(\Omega))^2,\operatorname{div}\mathbf{v}\in L^2(\Omega)\},\qquad W=L^2(\Omega).$$

The Hilbert space $\mathbf{V}$ is equipped with the following norm:

$$\|\mathbf{v}\|_{\operatorname{div}}=\|\mathbf{v}\|_{H(\operatorname{div};\Omega)}=\left(\|\mathbf{v}\|_{0,\Omega}^2+\|\operatorname{div}\mathbf{v}\|_{0,\Omega}^2\right)^{1/2}.$$

Then, the weak formulation of the optimal control problems (1.1)-(1.3) is to find $(\mathbf{p},y,u)\in\mathbf{V}\times W\times U$ such that

$$\min_{u\in K\subset U}\{g_1(\mathbf{p})+g_2(y)+j(u)\}, \tag{2.1}$$

$$\left(A^{-1}\mathbf{p},\mathbf{v}\right)-(y,\operatorname{div}\mathbf{v})=0,\quad\forall\mathbf{v}\in\mathbf{V}, \tag{2.2}$$

$$(\operatorname{div}\mathbf{p},w)+\int_\Omega\int_\Omega G(s,t)y(s)w(t)\,ds\,dt=(f+Bu,w),\quad\forall w\in W, \tag{2.3}$$

where the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$ is denoted by $(\cdot,\cdot)$. It is well known (see, *e.g.*, [19]) that the optimal control problem (2.1)-(2.3) has a unique solution $(\mathbf{p},y,u)$, and that a triplet $(\mathbf{p}, y, u)$ is the solution of (2.1)-(2.3) if and only if there is a co-state $(\mathbf{q}, z) \in \mathbf{V} \times W$ such that $(\mathbf{p}, y, \mathbf{q}, z, u)$ satisfies the following optimality conditions:

$$\left(A^{-1}\mathbf{p}, \mathbf{v}\right) - (y, \operatorname{div} \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.4}$$

$$(\operatorname{div} \mathbf{p}, w) + \int_\Omega \int_\Omega G(s,t) y(s) w(t)\, ds\, dt = (f + Bu, w), \quad \forall w \in W, \tag{2.5}$$

$$\left(A^{-1}\mathbf{q}, \mathbf{v}\right) - (z, \operatorname{div} \mathbf{v}) = -\left(g_1'(\mathbf{p}), \mathbf{v}\right), \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.6}$$

$$(\operatorname{div} \mathbf{q}, w) + \int_\Omega \int_\Omega G(s,t) w(s) z(t)\, ds\, dt = \left(g_2'(y), w\right), \quad \forall w \in W, \tag{2.7}$$

$$\left(j'(u) + B^* z, \tilde{u} - u\right)_U \geq 0, \quad \forall \tilde{u} \in K, \tag{2.8}$$

where g_1', g_2', and j' are the derivatives of g_1, g_2, and j, B^* is the adjoint operator of B, and $(\cdot,\cdot)_U$ is the inner product of U. In the rest of the paper, we shall simply write the product as $(\cdot,\cdot)$ whenever no confusion should be caused.

We are now able to introduce the discretized problem. To this aim, we consider a family of triangulations or rectangulations $\mathcal{T}_h$ of $\bar{\Omega}$. With each element $T \in \mathcal{T}_h$, we associate two parameters $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of the set T and $\sigma(T)$ is the diameter of the largest ball contained in T. The mesh size of the grid is defined by $h = \max_{T\in\mathcal{T}_h} \rho(T)$. We suppose that the regularity assumptions are satisfied. There exist two positive constants ϱ_1 and ϱ_2 such that

$$\frac{\rho(T)}{\sigma(T)} \leq \varrho_1, \qquad \frac{h}{\rho(T)} \leq \varrho_2$$

hold for all $T \in \mathcal{T}_h$ and all $h > 0$. In addition, C or c denotes a general positive constant independent of h.

Let us define $\bar{\Omega}_h = \bigcup_{T\in\mathcal{T}_h} T$, and let Ω_h and Γ_h denote its interior and its boundary, respectively. We assume that $\bar{\Omega}_h$ is convex and the vertices of $\mathcal{T}_h$ placed on the boundary of Γ_h are points of $\partial\Omega$. We also assume that $|\Omega \setminus \Omega_h| \leq Ch^2$.

Similarly, we assume that $\mathcal{T}_h(\Omega_U)$ are triangulations or rectangulations of Ω_U. With each element $s \in \mathcal{T}_h(\Omega_U)$, the two parameters $\rho(s)$ and $\sigma(s)$ are assumed to satisfy the regularity assumptions. Next, to every boundary triangle or rectangle T (s) of $\mathcal{T}_h$ $(\mathcal{T}_h(\Omega_U))$, we associate another triangle or rectangle $\hat{T}$ $(\hat{s})$ with curved boundary. We denote by $\hat{\mathcal{T}}_h$ $(\hat{\mathcal{T}}_h(\Omega_U))$ the union of these curved boundary triangles with interior triangles of $\mathcal{T}_h$ $(\mathcal{T}_h(\Omega_U))$ such that

$$\bar{\Omega} = \bigcup_{\hat{T}\in\hat{\mathcal{T}}_h} \hat{T}, \qquad \bar{\Omega}_U = \bigcup_{\hat{s}\in\hat{\mathcal{T}}_h(\Omega_U)} \hat{s}.$$

Let $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ denote the Raviart-Thomas space [20] of the lowest order associated with the triangulations or rectangulations $\mathcal{T}_h$ of $\bar{\Omega}$. P_k denotes the space of polynomials of total degree at most k, $Q_{m,n}$ indicates the space of polynomials of degree no more than m and n in x and y, respectively. If T is a triangle, $\mathbf{V}(T) = \{\mathbf{v} \in P_0^2(T) + x \cdot P_0(T)\}$, and if T is a rectangle, $\mathbf{V}(T) = \{\mathbf{v} \in Q_{1,0}(T) \times Q_{0,1}(T)\}$. We define

$$\mathbf{V}_h := \left\{\mathbf{v}_h \in \mathbf{V} : \forall T \in \mathcal{T}_h, \mathbf{v}_h|_T \in \mathbf{V}(T); \mathbf{v}_h = 0, \text{ on } \bar{\Omega} \setminus \Omega_h\right\},$$

$$W_h := \{w_h \in W : \forall T \in \mathcal{T}_h, w_h|_T = \text{constant}; w_h = 0, \text{ on } \bar{\Omega} \setminus \Omega_h\}.$$

Associated with $\hat{\mathcal{T}}_h(\Omega_U)$ is another finite dimensional subspace U_h of U:

$$U_h := \{\tilde{u}_h \in U : \forall \hat{s} \in \hat{\mathcal{T}}_h(\Omega_U),\ \tilde{u}_h|_{\hat{s}} = \text{constant}\}.$$

The mixed finite element discretization of (2.1)-(2.3) is as follows: compute $(\mathbf{p}_h, y_h, u_h) \in \mathbf{V}_h \times W_h \times U_h$ such that

$$\min_{u_h \in K_h \subset U_h} \{g_1(\mathbf{p}_h) + g_2(y_h) + j(u_h)\}, \tag{2.9}$$

$$(A^{-1}\mathbf{p}_h, \mathbf{v}_h) - (y_h, \operatorname{div} \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.10}$$

$$(\operatorname{div} \mathbf{p}_h, w_h) + \int_\Omega \int_\Omega G(s,t) y_h(s) w_h(t)\, ds\, dt = (f + Bu_h, w_h), \quad \forall w_h \in W_h, \tag{2.11}$$

where $K_h = U_h \cap K$. Under our assumptions on the kernel $G(\cdot,\cdot)$, it can be shown that there exists an $\bar{h} > 0$ such that for $h \in (0, \bar{h})$, the mixed finite element approximation

$$(A^{-1}\mathbf{p}_h, \mathbf{v}_h) - (y_h, \operatorname{div} \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.12}$$

$$(\operatorname{div} \mathbf{p}_h, w_h) + \int_\Omega \int_\Omega G(s,t) y_h(s) w_h(t)\, ds\, dt = (F, w_h), \quad \forall w_h \in W_h \tag{2.13}$$

has a unique solution $(\mathbf{p}_h, y_h)$ for any $F \in L^2(\Omega)$.

The optimal control problem (2.9)-(2.11) again has a unique solution $(\mathbf{p}_h, y_h, u_h)$, and a triplet $(\mathbf{p}_h, y_h, u_h)$ is the solution of (2.9)-(2.11) if and only if there is a co-state $(\mathbf{q}_h, z_h) \in \mathbf{V}_h \times W_h$ such that $(\mathbf{p}_h, y_h, \mathbf{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(A^{-1}\mathbf{p}_h, \mathbf{v}_h) - (y_h, \operatorname{div} \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.14}$$

$$(\operatorname{div} \mathbf{p}_h, w_h) + \int_\Omega \int_\Omega G(s,t) y_h(s) w_h(t)\, ds\, dt = (f + Bu_h, w_h), \quad \forall w_h \in W_h, \tag{2.15}$$

$$(A^{-1}\mathbf{q}_h, \mathbf{v}_h) - (z_h, \operatorname{div} \mathbf{v}_h) = -(g_1'(\mathbf{p}_h), \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.16}$$

$$(\operatorname{div} \mathbf{q}_h, w_h) + \int_\Omega \int_\Omega G(s,t) w_h(s) z_h(t)\, ds\, dt = (g_2'(y_h), w_h), \quad \forall w_h \in W_h, \tag{2.17}$$

$$(j'(u_h) + B^* z_h, \tilde{u}_h - u_h)_U \geq 0, \quad \forall \tilde{u}_h \in K_h. \tag{2.18}$$

In the rest of the paper, we shall use some intermediate variables. For any control function $\tilde{u} \in K$, we first define the state solution $(\mathbf{p}(\tilde{u}), y(\tilde{u}), \mathbf{q}(\tilde{u}), z(\tilde{u}))$ associated with $\tilde{u}$ that satisfies

$$(A^{-1}\mathbf{p}(\tilde{u}), \mathbf{v}) - (y(\tilde{u}), \operatorname{div} \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.19}$$

$$(\operatorname{div} \mathbf{p}(\tilde{u}), w) + \int_\Omega \int_\Omega G(s,t) y(\tilde{u})(s) w(t)\, ds\, dt = (f + B\tilde{u}, w), \quad \forall w \in W, \tag{2.20}$$

$$(A^{-1}\mathbf{q}(\tilde{u}), \mathbf{v}) - (z(\tilde{u}), \operatorname{div} \mathbf{v}) = -(g_1'(\mathbf{p}(\tilde{u})), \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.21}$$

$$(\operatorname{div} \mathbf{q}(\tilde{u}), w) + \int_\Omega \int_\Omega G(s,t) w_h(s) z_h(\tilde{u})(t)\, ds\, dt = (g_2'(y(\tilde{u})), w), \quad \forall w \in W. \tag{2.22}$$

Correspondingly, we define the discrete state solution $(\mathbf{p}_h(\tilde{u}), y_h(\tilde{u}), \mathbf{q}_h(\tilde{u}), z_h(\tilde{u}))$ associated with $\tilde{u} \in K$ that satisfies

$$\big(A^{-1}\mathbf{p}_h(\tilde{u}), \mathbf{v}_h\big) - \big(y_h(\tilde{u}), \operatorname{div} \mathbf{v}_h\big) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.23}$$

$$\big(\operatorname{div} \mathbf{p}_h(\tilde{u}), w_h\big) + \int_\Omega \int_\Omega G(s,t) y_h(\tilde{u})(s) w_h(t)\, ds\, dt = (f + B\tilde{u}, w_h), \quad \forall w_h \in W_h, \tag{2.24}$$

$$\big(A^{-1}\mathbf{q}_h(\tilde{u}), \mathbf{v}_h\big) - \big(z_h(\tilde{u}), \operatorname{div} \mathbf{v}_h\big) = -\big(g_1'\big(\mathbf{p}_h(\tilde{u})\big), \mathbf{v}_h\big), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.25}$$

$$\big(\operatorname{div} \mathbf{q}_h(\tilde{u}), w_h\big) + \int_\Omega \int_\Omega G(s,t) w_h(s) z_h(\tilde{u})(t)\, ds\, dt = \big(g_2'\big(y_h(\tilde{u})\big), w_h\big), \quad \forall w_h \in W_h. \tag{2.26}$$

Thus, as we defined, the exact solution and its approximation can be written in the following way:

$$(\mathbf{p}, y, \mathbf{q}, z) = \big(\mathbf{p}(u), y(u), \mathbf{q}(u), z(u)\big),$$
$$(\mathbf{p}_h, y_h, \mathbf{q}_h, z_h) = \big(\mathbf{p}_h(u_h), y_h(u_h), \mathbf{q}_h(u_h), z_h(u_h)\big).$$

Let $\mathcal{E}_h$ denote the set of element sides in $\mathcal{T}_h$. If there is no risk of confusion, the local mesh size h is defined on both $\mathcal{T}_h$ and $\mathcal{E}_h$ by $h|_T := h_T$ for $T \in \mathcal{T}_h$ and $h|_E := h_E$ for $E \in \mathcal{E}_h$, respectively. For all $E \in \mathcal{E}_h$, we fix one direction of a unit normal on E pointing in the outside of Ω in case $E \subset \partial\Omega$. We define that an operator $[v] : H^1(\mathcal{T}_h) \to L^2(\mathcal{E}_h)$ is the jump of the function v across the edge E, and $\mathbf{t}$ is the tangential unit vector along E.

We define $S^0(\mathcal{T}_h) \subset L^2(\Omega)$ as the piecewise constant space and $S^1(\mathcal{T}_h) \subset H^1(\Omega)$ or $S_0^1(\mathcal{T}_h) \subset H_0^1(\Omega)$ as continuous and piecewise linear functions, piecewise is understood with respect to $\mathcal{T}_h$. We consider Clement's interpolation operator $I_h : H^1(\Omega) \to S^1(\mathcal{T}_h)$ which satisfies [21]

$$\|v - I_h v\|_{0,T} \le C h_T \|v\|_{1,w_T}, \quad \forall v \in H_0^1(\Omega), \tag{2.27}$$

$$\|v - I_h v\|_{0,E} \le C h_E^{1/2} \|v\|_{1,w_E}, \quad \forall v \in H_0^1(\Omega) \tag{2.28}$$

for each $T \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$, $w_T = \{T' \in \mathcal{T}_h, \bar{T} \cap \bar{T}' \neq \emptyset\}$, $w_E = \{T \in \mathcal{T}_h, E \in \bar{T}\}$.

Now, we define the standard $L^2(\Omega)$-orthogonal projection $P_h : W \to W_h$, which satisfies the approximation property [22]:

$$\big\|h^{-1} \cdot (v - P_h v)\big\|_0 \le C \|\nabla_h v\|_0, \quad \forall \mathbf{v} \in H^1(\mathcal{T}_h). \tag{2.29}$$

Let us define the interpolation operator $\Pi_h : \mathbf{V} \to \mathbf{V}_h$, which satisfies: for any $\mathbf{q} \in \mathbf{V}$,

$$\int_T (\mathbf{q} - \Pi_h \mathbf{q}) \cdot \mathbf{v}_h \, dx\, dy = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h,\ T \in \mathcal{T}_h.$$

We have the commuting diagram property

$$\operatorname{div} \circ \Pi_h = P_h \circ \operatorname{div} : \mathbf{V} \to W_h \quad \text{and} \quad \operatorname{div}(I - \Pi_h)\mathbf{V} \perp W_h, \tag{2.30}$$

where and after, I denotes an identity operator.

Next, the interpolation operator Π_h satisfies the local error estimate

$$\left\|h^{-1}\cdot(\mathbf{q}-\Pi_h\mathbf{q})\right\|_0 \le C|\mathbf{q}|_{1,\mathcal{T}_h}, \quad \mathbf{q}\in H^1(\mathcal{T}_h)\cap\mathbf{V}. \tag{2.31}$$

Furthermore, we assume that [23]

$$\begin{aligned}&W_h\subset H^1(\mathcal{T}_h), \qquad A^{-1}p_h|_T\in P_l, \qquad \nabla_h y_h|_T\in P_l, \quad \forall T\in\mathcal{T}_h,\\ &S^0(\mathcal{T}_h)^2\cap H^1(\mathrm{div};\Omega)\subset\mathbf{V}_h\subset H^1(\mathcal{T}_h)\cap H^1(\mathrm{div};\Omega).\end{aligned}$$

3 *A posteriori* error estimates for the intermediate errors

Given $u\in K$, let S_1, S_2 be the inverse operators of state equation (2.3) such that $\mathbf{p}(u)=S_1Bu$ and $y(u)=S_2Bu$ are the solutions of state equation (2.3). Similarly, for given $u_h\in K_h$, $\mathbf{p}_h(u_h)=S_{1h}Bu_h$, $y_h(u_h)=S_{2h}Bu_h$ are the solutions of discrete state equation (2.11). Let

$$\begin{aligned}&J(u)=g_1(S_1Bu)+g_2(S_2Bu)+j(u),\\ &J_h(u_h)=g_1(S_{1h}Bu_h)+g_2(S_{2h}Bu_h)+j(u_h).\end{aligned}$$

It is clear that J and J_h are well defined and continuous on K and K_h. Also, the functional J_h can be naturally extended on K. Then (2.1) and (2.9) can be represented as

$$\min_{u\in K}\bigl\{J(u)\bigr\}, \tag{3.1}$$

$$\min_{u_h\in K_h}\bigl\{J_h(u_h)\bigr\}. \tag{3.2}$$

An additional assumption is needed. We assume that the cost function J is strictly convex near the solution u, *i.e.*, for the solution u, there exists a neighborhood of u in L^2 such that J is convex in the sense that there is a constant $c>0$ satisfying

$$\bigl(J'(u)-J'(v),u-v\bigr)\ge c\|u-v\|_U^2, \tag{3.3}$$

for all v in this neighborhood of u. The convexity of $J(\cdot)$ is closely related to the second-order sufficient optimality conditions of optimal control problems, which are assumed in many studies on numerical methods of the problem. For instance, in many references, the authors assume the following second-order sufficiently optimality condition (see [21, 24]): there is $c>0$ such that $J''(u)v^2\ge c\|v\|_0^2$.

Now, we are able to derive the main result.

Lemma 3.1 *Let u and u_h be the solutions of* (3.1) *and* (3.2), *respectively. Assume that $K_h\subset K$. In addition, assume that $(J_h'(u_h))|_s\in H^1(s)$, $\forall s\in\mathcal{T}_h(\Omega_U)$, and that there is a $v_h\in K_h$ such that*

$$\bigl|J_h'(u_h),v_h-u\bigr)\bigr|\le C\sum_{s\in\mathcal{T}_h(\Omega_U)}h_s\bigl\|J_h'(u_h)\bigr\|_{H^1(s)}\|u-u_h\|_{L^2(s)}. \tag{3.4}$$

Then we have

$$\|u-u_h\|_U^2\le C\eta_1^2+C\bigl\|z(u_h)-z_h\bigr\|_0^2, \tag{3.5}$$

where

$$\eta_1^2 = \sum_{s\in\mathcal{T}_h(\Omega_U)} h_s^2 \left\| j'(u_h) + B^* z_h \right\|_{H^1(s)}^2. \tag{3.6}$$

Proof It follows from (3.1) and (3.2) that

$$\big(J'(u), u - v\big) \le 0, \quad \forall v \in K, \tag{3.7}$$

$$\big(J_h'(u_h), u_h - v_h\big) \le 0, \quad \forall v_h \in K_h \subset K. \tag{3.8}$$

Then it follows from (3.3) and (3.7)-(3.8) that

$$\begin{aligned} c\|u - u_h\|_U^2 &\le \big(J'(u) - J'(u_h), u - u_h\big)_U \\ &\le -\big(J'(u_h), u - u_h\big)_U \\ &= \big(J_h'(u_h), u_h - u\big)_U + \big(J_h'(u_h) - J'(u_h), u - u_h\big)_U \\ &\le \big(J_h'(u_h), v_h - u\big)_U + \big(J_h'(u_h) - J'(u_h), u - u_h\big)_U. \end{aligned} \tag{3.9}$$

From (3.4), (3.9), and the Schwarz inequality, we get that

$$\begin{aligned} c\|u - u_h\|_U^2 &\le C \sum_{s\in\mathcal{T}_h(\Omega_U)} h_s \left\| J_h'(u_h) \right\|_{H^1(s)} \|u - u_h\|_{L^2(s)} \\ &\quad + C\left\| J_h'(u_h) - J'(u_h) \right\|_U \|u - u_h\|_U \\ &\le C \sum_{s\in\mathcal{T}_h(\Omega_U)} h_s^2 \left\| J_h'(u_h) \right\|_{H^1(s)}^2 \\ &\quad + C\left\| J_h'(u_h) - J'(u_h) \right\|_U^2 + \delta \|u - u_h\|_U^2. \end{aligned} \tag{3.10}$$

It is not difficult to show

$$J_h'(u_h) = j'(u_h) + B^* z_h, \qquad J'(u_h) = j'(u_h) + B^* z(u_h), \tag{3.11}$$

where $z(u_h)$ is the solution of equations (2.19)-(2.22). From (3.11), it is easy to derive

$$\left\| J_h'(u_h) - J'(u_h) \right\|_U = \left\| B^*\big(z_h - z(u_h)\big) \right\|_U \le C \left\| z_h - z(u_h) \right\|_0. \tag{3.12}$$

It is clear that (3.5) can be derived from (3.10)-(3.12). □

Fix a function $u_h \in U_h$, let $(\mathbf{p}(u_h), y(u_h)) \in \mathbf{V} \times W$ be the solution of equations (2.19)-(2.20). Set some intermediate errors: $\varepsilon_1 := \mathbf{p}(u_h) - \mathbf{p}_h$, $e_1 := y(u_h) - y_h$.

To analyze the fixing u_h approach, let us first note the following error equations from (2.10)-(2.11) and (2.19)-(2.20):

$$\big(A^{-1}\varepsilon_1, \mathbf{v}_h\big) - (e_1, \operatorname{div} \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{3.13}$$

$$(\operatorname{div} \varepsilon_1, w_h) + \int_\Omega \int_\Omega G(s,t) e_1(s) w_h(t)\, ds\, dt = 0, \quad \forall w_h \in W_h. \tag{3.14}$$

Lemma 3.2 *For the Raviart-Thomas elements, there is a positive constant C, which only depends on A, Ω, and the shape of the elements and their maximal polynomial degree k, such that*

$$\|\mathbf{p}(u_h) - \mathbf{p}_h\|_{\mathrm{div}} + \|y(u_h) - y_h\|_0 \le C\eta_2, \tag{3.15}$$

where

$$\eta_2 := \Bigg[\sum_{T\in\mathcal{T}_h}\bigg(\Big\|f + Bu_h - \operatorname{div}\mathbf{p}_h - \int_\Omega G(s,t)y_h(s)\,ds\Big\|_{0,T}^2 + h_T^2\cdot\|\mathrm{curl}_h(A^{-1}\mathbf{p}_h)\|_{0,T}^2$$

$$+ h_T^2\cdot\min_{w_h\in W_h}\|\nabla_h w_h - A^{-1}\mathbf{p}_h\|_{0,T}^2 + \|h_E^{1/2}[(A^{-1}\mathbf{p}_h)\cdot\mathbf{t}]\|_{0,\partial T}^2\bigg)\Bigg]^{1/2}. \tag{3.16}$$

Proof We analyze a Helmholtz decomposition [23] of $A^{-1}\mathbf{p}_h$ with a fixing $\varphi \in H_0^1(\Omega)$ such that $-\operatorname{div}(A\nabla\varphi) = \operatorname{div}\mathbf{p}_h$. Then there is some $\psi \in H^1(\Omega)$ satisfying $\int_\Omega \psi\,dx = 0$, $\operatorname{Curl}\psi \perp \nabla H_0^1(\Omega)$ and

$$\mathbf{p}_h = -A\nabla\varphi + \operatorname{Curl}\psi. \tag{3.17}$$

From (3.17) and (1.8)-(1.11), we derive

$$\varepsilon_1 = A\nabla\chi - \operatorname{Curl}\psi \quad \text{with } \chi = \varphi - y(u_h) \in H_0^1(\Omega), \tag{3.18}$$

and hence the error decomposition

$$(A^{-1}\varepsilon_1, \varepsilon_1) = (A\nabla\chi, \nabla\chi) + (A^{-1}\operatorname{Curl}\psi, \operatorname{Curl}\psi). \tag{3.19}$$

It follows from Poincare's inequality and (2.29) that

$$\begin{aligned}(A\nabla\chi, \nabla\chi) &= (\nabla\chi, \varepsilon_1) = -(\operatorname{div}\varepsilon_1, \chi)\\ &= (\operatorname{div}\varepsilon_1, P_h\chi - \chi) - (\operatorname{div}\varepsilon_1, P_h\chi)\\ &\le C\|h_T\cdot\operatorname{div}\varepsilon_1\|_0\cdot\|A^{1/2}\nabla\chi\|_0 + C\|\operatorname{div}\varepsilon_1\|_0\cdot\|P_h\chi\|_0\\ &\le C\|h_T\cdot\operatorname{div}\varepsilon_1\|_0\cdot\|A^{1/2}\nabla\chi\|_0 + C\|\operatorname{div}\varepsilon_1\|_0\cdot\|A^{1/2}\nabla\chi\|_0.\end{aligned} \tag{3.20}$$

To estimate the second contribution to the right-hand side of (3.19), we utilize Clement's operator I_h. Note that $I_h\psi \in S^1(\mathcal{T}_h) \subset H^1(\Omega)$, $\operatorname{Curl} I_h\psi \in S^0(\mathcal{T}_h)^2 \cap H^1(\mathrm{div};\Omega) \subset \mathbf{V}_h$ and $\operatorname{Curl} I_h\psi \perp \nabla H_0^1(\Omega)$, whence $\operatorname{div}(\operatorname{Curl} I_h\psi) = 0$. Therefore, we obtain

$$(A^{-1}\operatorname{Curl}\psi_1, \operatorname{Curl} I_h\psi) = -(A^{-1}\varepsilon_1, \operatorname{Curl} I_h\psi) = -(e_1, \operatorname{div}\operatorname{Curl} I_h\psi) = 0.$$

Utilizing (3.17) and (2.27)-(2.28), we infer

$$\begin{aligned}&(A^{-1}\operatorname{Curl}\psi, \operatorname{Curl}\psi)\\ &\quad= (A^{-1}\operatorname{Curl}\psi, \operatorname{Curl}(\psi - I_h\psi)) = (A^{-1}\mathbf{p}_h, \operatorname{Curl}(\psi - I_h\psi))\end{aligned}$$

$$\begin{aligned}
&= -\big(\psi - I_h\psi, \operatorname{curl}_h\big(A^{-1}\mathbf{p}_h\big)\big) + \big(\big[\big(A^{-1}\mathbf{p}_h\big)\cdot\mathbf{t}\big], \psi - I_h\psi\big)_{\mathcal{E}_h} \\
&\le C\big(\big\|h_T\cdot\operatorname{curl}_h\big(A^{-1}\mathbf{p}_h\big)\big\|_0 + \big\|h_E^{1/2}\cdot\big[\big(A^{-1}\mathbf{p}_h\big)\cdot\mathbf{t}\big]\big\|_{0,\mathcal{E}_h}\big)\|\psi\|_1. \qquad (3.21)
\end{aligned}$$

With Poincare's inequality we deduce

$$\|\psi\|_1 \le C\|\nabla\psi\|_0 = C\|\operatorname{Curl}\psi\|_0 \le C\big\|A^{-1/2}\operatorname{Curl}\psi\big\|_0. \qquad (3.22)$$

From (2.20), we have

$$\begin{aligned}
\operatorname{div}\varepsilon_1 &= f + Bu_h - \operatorname{div}\mathbf{p}_h - \int_\Omega G(s,t)y(u_h)(s)\,ds \\
&= f + Bu_h - \operatorname{div}\mathbf{p}_h - \int_\Omega G(s,t)y_h(s)\,ds - \int_\Omega G(s,t)e_1(s)\,ds, \qquad (3.23)
\end{aligned}$$

and together with (3.19)-(3.23) we have

$$\begin{aligned}
\|\varepsilon_1\|_{\operatorname{div}} \le C\bigg(&\bigg\|f + Bu_h - \operatorname{div}\mathbf{p}_h - \int_\Omega G(s,t)y_h(s)\,ds\bigg\|_0 + \|e_1\|_0 \\
&+ h_T\cdot\big\|\operatorname{curl}_h\big(A^{-1}\mathbf{p}_h\big)\big\|_0 + \big\|h_E^{1/2}\big[\big(A^{-1}\mathbf{p}_h\big)\cdot\mathbf{t}\big]\big\|_{0,\mathcal{E}_h}\bigg). \qquad (3.24)
\end{aligned}$$

Now, let us estimate $\|e_1\|_0$. Let ξ be the solution of (1.9) with $F_1 = y(u_h) - y_h$. According to (1.8)-(1.11), we have $\xi \in H_0^1(\Omega)\cap H^2(\mathcal{T}_h)$. Then it follows from (1.9), (2.14)-(2.15) and (2.30) that

$$\begin{aligned}
\|e_1\|_0^2 &= \bigg(y(u_h) - y_h, -\operatorname{div}(A\nabla\xi) + \int_\Omega G(s,t)\xi(t)\,dt\bigg) \\
&= -\big(\mathbf{p}(u_h),\nabla\xi\big) + \big(y_h, \operatorname{div}\Pi_h(A\nabla\xi)\big) + \bigg(y(u_h) - y_h, \int_\Omega G(s,t)\xi(t)\,dt\bigg) \\
&= \big(\operatorname{div}\mathbf{p}(u_h),\xi\big) + \int_\Omega\int_\Omega G(s,t)y(u_h)(s)\xi(t)\,ds\,dt \\
&\quad + \big(A^{-1}\mathbf{p}_h, \Pi_h(A\nabla\xi)\big) - \bigg(y_h, \int_\Omega G(s,t)\xi(t)\,dt\bigg) \\
&= \bigg(f + Bu_h - \operatorname{div}\mathbf{p}_h - \int_\Omega G(s,t)y_h(s)\,ds, \xi\bigg) + \big(\nabla_h w_h - A^{-1}\mathbf{p}_h, (I - \Pi_h)(A\nabla\xi)\big) \\
&\le C\bigg(\bigg\|f + Bu_h - \operatorname{div}\mathbf{p}_h - \int_\Omega G(s,t)y_h(s)\,ds\bigg\|_0 + \big\|h\cdot\big(\nabla_h w_h - A^{-1}\mathbf{p}_h\big)\big\|_0\bigg)\cdot\|\xi\|_2 \\
&\le C\bigg(\bigg\|f + Bu_h - \operatorname{div}\mathbf{p}_h - \int_\Omega G(s,t)y_h(s)\,ds\bigg\|_0^2 + \big\|h\cdot\big(\nabla_h w_h - A^{-1}\mathbf{p}_h\big)\big\|_0^2\bigg) + \delta\|e_1\|_0^2
\end{aligned}$$

for any $w_h \in W_h$. Using the triangle inequality, we obtain

$$\|e_1\|_0 \le C\bigg(\bigg\|f + Bu_h - \operatorname{div}\mathbf{p}_h - \int_\Omega G(s,t)y_h(s)\,ds\bigg\|_0 + \big\|h\cdot\big(\nabla_h w_h - A^{-1}\mathbf{p}_h\big)\big\|_0\bigg). \qquad (3.25)$$

So, Lemma 3.2 has been proved by combining with (3.24) and (3.25). □

Moreover, we can prove the reverse inequality of (3.15).

Lemma 3.3 *For the Raviart-Thomas elements, there is a positive constant C, which only depends on A, Ω, and the shape of the elements and their maximal polynomial degree k, such that*

$$C\eta_2 \le \|\mathbf{p}(u_h) - \mathbf{p}_h\|_{\operatorname{div}} + \|y(u_h) - y_h\|_0. \tag{3.26}$$

Proof First, from (3.23) we derive that

$$f + Bu_h - \operatorname{div}\mathbf{p}_h - \int_\Omega G(s,t)y_h(s)\,ds = \operatorname{div}\varepsilon_1 + \int_\Omega G(s,t)e_1(s)\,ds, \tag{3.27}$$

then we have

$$\left\| f + Bu_h - \operatorname{div}\mathbf{p}_h - \int_\Omega G(s,t)y_h(s)\,ds \right\|_{0,T} \le C\bigl(\|\varepsilon_1\|_{H(\operatorname{div};T)} + \|e_1\|_{0,T}\bigr). \tag{3.28}$$

Next, using the standard Bubble function technique, we fix $\varrho_T \in P_3$ with $0 \le \varrho_T \le 1 = \max \varrho_T$ and zero boundary values on T to derive

$$C\bigl\|\operatorname{curl}\bigl(A^{-1}\mathbf{p}_h\bigr)\bigr\|_{0,T}^2 \le \bigl\|\varrho_T^{1/2}\cdot\operatorname{curl}\bigl(A^{-1}\mathbf{p}_h\bigr)\bigr\|_{0,T}^2. \tag{3.29}$$

Using (3.17) and (3.18), we obtain

$$\begin{aligned}\bigl\|\varrho_T^{1/2}\cdot\operatorname{curl}\bigl(A^{-1}\mathbf{p}_h\bigr)\bigr\|_{0,T}^2 &= \int_T \bigl(A^{-1}\varepsilon_1\bigr)\cdot\operatorname{Curl}\bigl(\varrho_T^{1/2}\cdot\operatorname{curl}\bigl(A^{-1}\mathbf{p}_h\bigr)\bigr)\,dx \\ &\le C\bigl\|A^{-1}\varepsilon_1\bigr\|_{0,T}\cdot\bigl|\varrho_T^{1/2}\cdot\operatorname{curl}\bigl(A^{-1}\mathbf{p}_h\bigr)\bigr|_{1,T} \\ &\le C\|\varepsilon_1\|_{H(\operatorname{div};T)}\cdot h_T^{-1}\cdot\bigl\|\varrho_T^{1/2}\cdot\operatorname{curl}\bigl(A^{-1}\mathbf{p}_h\bigr)\bigr\|_{0,T},\end{aligned} \tag{3.30}$$

since $\varrho_T^{1/2}\cdot\operatorname{curl}(A^{-1}\mathbf{p}_h) \in P_{l+2}$ with zero boundary values on T. Combining (3.29) and (3.30), we have

$$h_T\cdot\bigl\|\operatorname{curl}\bigl(A^{-1}\mathbf{p}_h\bigr)\bigr\|_{0,T} \le C\|\varepsilon_1\|_{H(\operatorname{div};T)}. \tag{3.31}$$

Now, let ϱ_E denote the continuous function satisfying $\varrho_E \in P_2$ with $0 \le \varrho_E \le 1 = \max\varrho_E$ on w_E. Let $\sigma = [(A^{-1}\mathbf{p}_h)\cdot\mathbf{t}]$. Using continuous extension on the reference element in [25], there exists an extension operator $P : C(E) \to C(w_E)$ satisfying $P\sigma|_E = \sigma$ and

$$c_1 h_E^{1/2}\|\sigma\|_{0,E} \le \bigl\|\varrho_E^{1/2}P\sigma\bigr\|_{0,w_E} \le c_2 h_E^{1/2}\|\sigma\|_{0,E}, \tag{3.32}$$

where c_1 and c_2 are positive constants. By the integration by parts formula and (3.31)-(3.32), we obtain

$$\begin{aligned}C\|\sigma\|_{0,E}^2 \le \bigl\|\varrho_E^{1/2}\sigma\bigr\|_{0,E}^2 &= -\int_E (\varrho_E P\sigma)\cdot\bigl[A^{-1}\varepsilon_1\cdot\mathbf{t}\bigr]\,ds \\ &= -\int_{w_E}\bigl(A^{-1}\varepsilon_1\bigr)\cdot\operatorname{Curl}(\varrho_E P\sigma)\,dx - \int_{w_E}(\varrho_E P\sigma)\operatorname{curl}\bigl(A^{-1}\varepsilon_1\bigr)\,dx \\ &= -\int_{w_E}\bigl(A^{-1}\varepsilon_1\bigr)\cdot\operatorname{Curl}(\varrho_E P\sigma)\,dx - \int_{w_E}(\varrho_E P\sigma)\operatorname{curl}\bigl(A^{-1}\mathbf{p}_h\bigr)\,dx\end{aligned}$$

$$\begin{aligned}
&\le \|\varepsilon_1\|_{0,w_E}|\varrho_E P\sigma|_{1,w_E} + \|\varrho_E P\sigma\|_{0,w_E}\left\|\operatorname{curl}\left(A^{-1}\mathbf{p}_h\right)\right\|_{0,w_E} \\
&\le Ch_E^{-1/2}\|\sigma\|_{0,E}\cdot\|\varepsilon_1\|_{H(\operatorname{div};w_E)},
\end{aligned} \tag{3.33}$$

where the inverse estimates have been used. Then we obtain that

$$\left\|h^{1/2}\left[\left(A^{-1}\mathbf{p}_h\right)\cdot\mathbf{t}\right]\right\|_{0,E} \le C\|\varepsilon_1\|_{H(\operatorname{div};w_E)}. \tag{3.34}$$

Finally, as in (3.29) and with integration by parts, we derive that

$$\begin{aligned}
&C\left\|A^{-1}\mathbf{p}_h - \nabla_h y_h\right\|_{0,T}^2 \\
&\quad\le \left\|\varrho_T^{1/2}\left(A^{-1}\mathbf{p}_h - \nabla_h y_h\right)\right\|_{0,T}^2 \\
&\quad= -\int_T \varrho_T A^{-1}\varepsilon_1\left(A^{-1}\mathbf{p}_h - \nabla_h y_h\right)dx - \int_T e_1 \operatorname{div}\left(\varrho_T\left(A^{-1}\mathbf{p}_h - \nabla_h y_h\right)\right)dx \\
&\quad\le \left\|A^{-1}\varepsilon_1\right\|_{0,T}\left\|\varrho_T\left(A^{-1}\mathbf{p}_h - \nabla_h y_h\right)\right\|_{0,T} + \|e_1\|_{0,T}\left|\varrho_T\left(A^{-1}\mathbf{p}_h - \nabla_h y_h\right)\right|_{1,T} \\
&\quad\le \left(\left\|A^{-1}\varepsilon_1\right\|_{0,T} + \|e_1\|_{0,T}\cdot h_T^{-1}\right)\left\|\varrho_T\left(A^{-1}\mathbf{p}_h - \nabla_h y_h\right)\right\|_{0,T},
\end{aligned} \tag{3.35}$$

where the inverse inequality has been used. From (3.35) it is clear that

$$h_T \min_{w_h\in W_h}\left\|A^{-1}\mathbf{p}_h - \nabla_h w_h\right\|_{0,T} \le C\left(\|e_1\|_{0,T} + h_T\left\|A^{-1}\varepsilon_1\right\|_{0,T}\right). \tag{3.36}$$

Then Lemma 3.3 is proved by combining (3.28), (3.31), (3.34) and (3.36). □

Arguing as in the proof of Lemma 3.2, we obtain the following results.

Lemma 3.4 *For the Raviart-Thomas elements, there is a positive constant C, which only depends on A, Ω, and the shape of the elements and their maximal polynomial degree k, such that*

$$\left\|\mathbf{q}(u_h) - \mathbf{q}_h\right\|_{\operatorname{div}} + \left\|z(u_h) - z_h\right\|_0 \le C(\eta_2 + \eta_3), \tag{3.37}$$

where

$$\begin{aligned}
\eta_3 := \Bigg[\sum_{T\in\mathcal{T}_h}\Bigg(&\left\|g_2'(y_h) - \operatorname{div}\mathbf{q}_h - \int_\Omega G(s,t)z_h(t)\,dt\right\|_{0,T}^2 + h_T^2\cdot\left\|\operatorname{curl}_h\left(A^{-1}\mathbf{q}_h + g_1'(\mathbf{p}_h)\right)\right\|_{0,T}^2 \\
&+ h_T^2\cdot\min_{w_h\in W_h}\left\|\nabla_h w_h - A^{-1}\mathbf{q}_h - g_1'(\mathbf{p}_h)\right\|_{0,T}^2 \\
&+ \left\|h_E^{1/2}\left[\left(A^{-1}\mathbf{q}_h + g_1'(\mathbf{p}_h)\right)\cdot\mathbf{t}\right]\right\|_{0,\partial T}^2\Bigg)\Bigg]^{1/2}.
\end{aligned} \tag{3.38}$$

Using Lemma 3.1, Lemma 3.2, and Lemma 3.4, we derive the following results.

Theorem 3.1 *Let u and u_h be the solutions of* (3.1) *and* (3.2), *respectively. Assume that $K_h \subset K$. In addition, assume that $(J_h'(u_h))|_s \in H^s(s)$, $\forall s\in\mathcal{T}_h(\Omega_U)$ ($s = 0$ or 1), and that there* *is a* $v_h \in K_h$ *such that*

$$\left|J_h'(u_h), v_h - u\right| \le C \sum_{s \in \mathcal{T}_h(\Omega_U)} h_s \left\|J_h'(u_h)\right\|_{H^1(s)} \|u - u_h\|_{L^2(s)}^s.$$

Then, for the Raviart-Thomas elements, there is a positive constant C*, which only depends on* A*,* Ω*, and the shape of the elements and their maximal polynomial degree* k*, such that*

$$\begin{aligned} &\left\|\mathbf{p}(u_h) - \mathbf{p}_h\right\|_{\mathrm{div}}^2 + \left\|y(u_h) - y_h\right\|_0^2 + \left\|\mathbf{q}(u_h) - \mathbf{q}_h\right\|_{\mathrm{div}}^2 \\ &\quad + \left\|z(u_h) - z_h\right\|_0^2 + \|u - u_h\|_U^2 \le C \sum_{i=1}^{3} \eta_i^2, \end{aligned} \tag{3.39}$$

where η_1*,* η_2*, and* η_3 *are defined in Lemma* 3.1*, Lemma* 3.2*, and Lemma* 3.4*, respectively.*

4 *A posteriori* error estimates

With the intermediate errors, we can decompose the errors as follows:

$$\begin{aligned} &\mathbf{p} - \mathbf{p}_h = \mathbf{p} - \mathbf{p}(u_h) + \mathbf{p}(u_h) - \mathbf{p}_h := \epsilon_1 + \varepsilon_1, \\ &y - y_h = y - y(u_h) + y(u_h) - y_h := r_1 + e_1, \\ &\mathbf{q} - \mathbf{q}_h = \mathbf{q} - \mathbf{q}(u_h) + \mathbf{q}(u_h) - \mathbf{q}_h := \epsilon_2 + \varepsilon_2, \\ &z - z_h = z - z(u_h) + z(u_h) - z_h := r_2 + e_2. \end{aligned}$$

By using the standard results of mixed finite element methods [26], we have the following results.

Lemma 4.1 *There is a positive constant* C *independent of* h *such that*

$$\|\epsilon_1\|_{\mathrm{div}} + \|r_1\|_0 \le C\|u - u_h\|_U, \tag{4.1}$$

$$\|\epsilon_2\|_{\mathrm{div}} + \|r_2\|_0 \le C\|u - u_h\|_U. \tag{4.2}$$

Proof It follows from (2.4)-(2.7) and (2.19)-(2.20) that we have the error equations:

$$\left(A^{-1}\epsilon_1, \mathbf{v}\right) - (r_1, \mathrm{div}\,\mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \tag{4.3}$$

$$(\mathrm{div}\,\epsilon_1, w) + \int_\Omega \int_\Omega G(s,t) r_1(s) w(t)\,ds\,dt = \left(B(u - u_h), w\right), \quad \forall w \in W, \tag{4.4}$$

$$\left(A^{-1}\epsilon_2, \mathbf{v}\right) - (r_2, \mathrm{div}\,\mathbf{v}) = -\left(g_1'(\mathbf{p}) - g_1'\left(\mathbf{p}(u_h)\right), \mathbf{v}\right), \quad \forall \mathbf{v} \in \mathbf{V}, \tag{4.5}$$

$$(\mathrm{div}\,\epsilon_2, w) + \int_\Omega \int_\Omega G(s,t) r_2(t) w(s)\,ds\,dt = \left(g_2'(y) - g_2'\left(y(u_h)\right), w\right), \quad \forall w \in W. \tag{4.6}$$

Choosing $\mathbf{v} = \epsilon_1$ and $w = r_1$ as the test functions and adding the two relations of (4.3)-(4.4), we have

$$\left(A^{-1}\epsilon_1, \epsilon_1\right) + \int_\Omega \int_\Omega G(s,t) r_1(s) r_1(t)\,ds\,dt = \left(B(u - u_h), r_1\right).$$

Then, using the assumption on A and (1.7), we obtain that

$$\|\epsilon_1\|_0^2 + \|r_1\|_0^2 \le C\|u - u_h\|_0^2 + \delta\|r_1\|_0^2. \tag{4.7}$$

Now we choose $\mathbf{v} = \operatorname{div}\epsilon_1$ in equation (4.4), then we obtain

$$(\operatorname{div}\epsilon_1, \operatorname{div}\epsilon_1) = \big(B(u-u_h), \operatorname{div}\epsilon_1\big) - \int_\Omega\int_\Omega G(s,t)r_1(s)\operatorname{div}\epsilon_1(t)\,ds\,dt. \tag{4.8}$$

Then, using the δ-Cauchy inequality, we can find an estimate as follows:

$$\begin{aligned}\|\operatorname{div}\epsilon_1\|^2 &\le C\left(\|u-u_h\|^2 + \left\|\int_\Omega G(s,t)r_1(s)\,ds\right\|^2\right) + \delta\|\operatorname{div}\epsilon_1\|^2 \\ &\le C\big(\|u-u_h\|^2 + \|r_1\|^2\big) + \delta\|\operatorname{div}\epsilon_1\|^2. \end{aligned}\tag{4.9}$$

Thus,

$$\|\operatorname{div}\epsilon_1\|^2 \le C\big(\|u-u_h\|^2 + \|r_1\|^2\big) \le C\|u-u_h\|^2. \tag{4.10}$$

This implies (4.1).

Similarly, we choose $\mathbf{v} = \epsilon_2$ and $w = r_2$ as the test functions and add the two relations of (4.5)-(4.6), then we have

$$\begin{aligned}&\big(A^{-1}\epsilon_2, \epsilon_2\big) + \int_\Omega\int_\Omega G(s,t)r_2(t)r_2(s)\,ds\,dt \\ &\quad = \big(g_2'(y) - g_2'\big(y(u_h)\big), r_2\big) - \big(g_1'(\mathbf{p}) - g_1'\big(\mathbf{p}(u_h)\big), \epsilon_2\big).\end{aligned}\tag{4.11}$$

Then, using the assumption on A and (1.7), we obtain that

$$\|\epsilon_2\|_0^2 + \|r_2\|_0^2 \le C\big(\|\epsilon_1\|_0^2 + \|r_1\|_0^2\big) + \delta\big(\|\epsilon_2\|_0^2 + \|r_2\|_0^2\big). \tag{4.12}$$

Hence, we derive that

$$\|\epsilon_2\|_0^2 + \|r_2\|_0^2 \le C\big(\|\epsilon_1\|_0^2 + \|r_1\|_0^2\big) \le C\|u-u_h\|^2. \tag{4.13}$$

Now we choose $\mathbf{v} = \operatorname{div}\epsilon_2$ in equation (4.6), then we obtain

$$(\operatorname{div}\epsilon_2, \operatorname{div}\epsilon_2) = \big(g_2'(y) - g_2'\big(y(u_h)\big), \operatorname{div}\epsilon_2\big) - \int_\Omega\int_\Omega G(s,t)r_2(t)\operatorname{div}\epsilon_2(s)\,ds\,dt. \tag{4.14}$$

Then, using the δ-Cauchy inequality, we can find an estimate as follows:

$$\|\operatorname{div}\epsilon_2\|^2 \le C\big(\|r_1\|^2 + \|r_2\|^2\big) + \delta\|\operatorname{div}\epsilon_2\|^2, \tag{4.15}$$

and hence,

$$\|\operatorname{div}\epsilon_2\|^2 \le C\|u-u_h\|^2. \tag{4.16}$$

Thus, (4.2) is proved by (4.13) and (4.16). □

Hence, we combine Theorem 3.1 and Lemma 4.1 and use the triangle inequality to conclude the following.

Theorem 4.1 *Let* $(\mathbf{p}, y, \mathbf{q}, z, u) \in (\mathbf{V} \times W)^2 \times U$ *and* $(\mathbf{p}_h, y_h, \mathbf{q}_h, z_h, u_h) \in (\mathbf{V}_h \times W_h)^2 \times U_h$ *be the solutions of* (2.4)-(2.8) *and* (2.14)-(2.18), *respectively. Assume that* $K_h \subset K$. *In addition, assume that* $(J_h'(u_h))|_s \in H^1(s)$, $\forall s \in \mathcal{T}_h(\Omega_U)$, *and that there is a* $v_h \in K_h$ *such that*

$$\left|J_h'(u_h), v_h - u\right| \leq C \sum_{s\in\mathcal{T}_h(\Omega_U)} h_s \left\|J_h'(u_h)\right\|_{H^1(s)} \|u - u_h\|_{L^2(s)}.$$

Then we have

$$\|\mathbf{p} - \mathbf{p}_h\|_{\mathrm{div}}^2 + \|y - y_h\|_0^2 + \|\mathbf{q} - \mathbf{q}_h\|_{\mathrm{div}}^2 + \|z - z_h\|_0^2 + \|u - u_h\|_U^2 \leq C \sum_{i=1}^{3} \eta_i^2, \tag{4.17}$$

where η_1, η_2, *and* η_3 *are defined in Lemma* 3.1, *Lemma* 3.2, *and Lemma* 3.4, *respectively.*

5 Some applications

In this section, we apply the previous results to two concrete optimal control problems.

Example 5.1 Consider the case $K = \{u \in U : u \geq 0\}$. Let $K_h = \{v \in U_h : v \geq 0\}$. Then it is easy to see that $K_h \subset K$. Let v_h in Lemma 3.1 be such that $v_h = \Pi_h u$, where

$$\Pi_h w|_{x\in s} = \int_s w/|s|, \quad \forall w \in L^2(\Omega_U),$$

where $|s|$ is the measure of the element s. Then $v_h = \Pi_h u \in K_h$, and

$$\begin{aligned}
\left|\left(j'(u_h) + B^* z_h, v_h - u\right)\right| &= \left|\left(j'(u_h) + B^* z_h, \Pi_h u - u\right)\right| \\
&= \left|\left(j'(u_h) + B^* z_h - \Pi_h\left(j'(u_h) + B^* z_h\right), \Pi_h(u - u_h) - (u - u_h)\right)\right| \\
&\leq \sum_{s\in\mathcal{T}_h(\Omega_U)} h_s \left\|j'(u_h) + B^* z_h\right\|_{H^1(s)} \|u - u_h\|_{L^2(s)}.
\end{aligned} \tag{5.1}$$

Hence, condition (3.4) in Lemma 3.1 is satisfied. If all the conditions in Theorem 4.1 hold, then

$$\|\mathbf{p} - \mathbf{p}_h\|_{\mathrm{div}}^2 + \|y - y_h\|_0^2 + \|\mathbf{q} - \mathbf{q}_h\|_{\mathrm{div}}^2 + \|z - z_h\|_0^2 + \|u - u_h\|_U^2 \leq C \sum_{i=1}^{3} \eta_i^2, \tag{5.2}$$

where η_1, η_2, and η_3 are defined in Lemma 3.1, Lemma 3.2, and Lemma 3.4, respectively.

Example 5.2 Consider the case $K = \{u \in U : \int_{\Omega_U} u \geq 0\}$. Let $K_h = \{v \in U_h : \int_{\Omega_U} v \geq 0\}$. Then it is easy to see that $K_h \subset K$. Let v_h in Lemma 3.1 be such that $v_h = \Pi_h u$, where Π_h is defined as in Example 5.1. Then $v_h = \Pi_h u \in K_h$, and similarly as in Example 5.1,

$$\left|\left(j'(u_h) + B^* z_h, v_h - u\right)\right| \leq \sum_{s\in\mathcal{T}_h(\Omega_U)} h_s \left\|j'(u_h) + B^* z_h\right\|_{H^1(s)} \|u - u_h\|_{L^2(s)}. \tag{5.3}$$

Hence, condition (3.4) in Lemma 3.1 is satisfied. If all the conditions in Theorem 4.1 hold, then

$$\|\mathbf{p}-\mathbf{p}_h\|_{\mathrm{div}}^2+\|y-y_h\|_0^2+\|\mathbf{q}-\mathbf{q}_h\|_{\mathrm{div}}^2+\|z-z_h\|_0^2+\|u-u_h\|_U^2\le C\sum_{i=1}^{3}\eta_i^2, \tag{5.4}$$

where η_1, η_2, and η_3 are defined in Lemma 3.1, Lemma 3.2, and Lemma 3.4, respectively.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZL participated in the sequence alignment and drafted the manuscript.

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References

1. Falk, FS: Approximation of a class of optimal control problems with order of convergence estimates. J. Math. Anal. Appl. **44**, 28-47 (1973)
2. Geveci, T: In the approximation of the solution of an optimal control problem governed by an elliptic equation. RAIRO. Anal. Numér. **13**, 313-328 (1979)
3. Li, R, Liu, W, Ma, H, Tang, T: Adaptive finite element approximation for distributed convex optimal control problems. SIAM J. Control Optim. **41**, 1321-1349 (2002)
4. Brunner, H, Yan, N: Finite element methods for optimal control problems governed by integral equations and integro-differential equations. Numer. Math. **101**, 1-27 (2005)
5. Lions, JL: Optimal Control of Systems Governed by Partial Differential Equations. Springer, Berlin (1971)
6. Liu, W, Yan, N: A posteriori error estimates for control problems governed by nonlinear elliptic equations. Appl. Numer. Math. **47**, 173-187 (2003)
7. Liu, W, Yan, N: A posteriori error estimates for control problems governed by Stokes equations. SIAM J. Numer. Anal. **40**, 1850-1869 (2002)
8. Brunner, H, Yan, N: On global superconvergence of iterated collocation solutions to linear second-kind Volterra integral equations. J. Comput. Appl. Math. **67**, 185-189 (2005)
9. Yan, N: Superconvergence analysis and a posteriori error estimates of a finite element method for an optimal control problem governed by integral equations. Appl. Math. **54**, 267-283 (2009)
10. Chen, Y: Superconvergence of mixed finite element methods for optimal control problems. Math. Comput. **77**, 1269-1291 (2008)
11. Chen, Y: Superconvergence of quadratic optimal control problems by triangular mixed finite elements. Int. J. Numer. Methods Eng. **75**, 881-898 (2008)
12. Chen, Y, Dai, L, Lu, Z: Superconvergence of quadratic optimal control problems by triangular mixed finite elements. Adv. Appl. Math. Mech. **75**, 881-898 (2009)
13. Lu, Z, Chen, Y: L^∞-error estimates of triangular mixed finite element methods for optimal control problem govern by semilinear elliptic equation. Numer. Anal. Appl. **12**, 74-86 (2009)
14. Xing, X Chen, Y: L^∞-error estimates for general optimal control problem by mixed finite element methods. Int. J. Numer. Anal. Model. **5**, 441-456 (2008)
15. Chen, Y, Liu, W: A posteriori error estimates for mixed finite element solutions of convex optimal control problems. J. Comput. Appl. Math. **211**, 76-89 (2008)
16. Lu, Z, Chen, Y: A posteriori error estimates of triangular mixed finite element methods for semilinear optimal control problems. Adv. Appl. Math. Mech. **1**, 242-256 (2009)
17. Kress, R: Linear Integral Equation, 2nd edn. Springer, New York (1999)
18. Zabreiko, PP, Koshelev, AI, Krasnosel'skii, MA, Mikhlin, SG, Rakovshchik, LS, Stet'senko, VY: Integral Equations-a Reference Text. Noordhoff, Groningen (1975)
19. Chen, Y, Liu, W: Posteriori error estimates for mixed finite elements of a quadratic optimal control problem. In: Recent Progress Comput. Appl. PDEs, vol. 2, pp. 123-134 (2002)

20. Raviart, PA, Thomas, JM: A mixed finite element method for 2nd order elliptic problems. In: Mathematical Aspects of the Finite Element Method. Lecture Notes in Math., vol. 606, pp. 292-315. Springer, Berlin (1977)
21. Bonnans, JF: Second-order analysis for control constrained optimal control problems of semilinear elliptic systems. Appl. Math. Optim. **38**, 303-325 (1998)
22. Babuska, I, Strouboulis, T: The Finite Element Method and Its Reliability. Oxford University Press, Oxford (2001)
23. Carstensen, C: A posteriori error estimate for the mixed finite element method. Math. Comput. **66**, 465-476 (1997)
24. Arada, N, Casas, E, Tröltzsch, F: Error estimates for the numerical approximation of a semilinear elliptic control problem. Comput. Optim. Appl. **23**, 201-229 (2002)
25. Verfürth, R: A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques. Wiley, New York (1996)
26. Brezzi, F, Fortin, M: Mixed and Hybrid Finite Element Methods. Springer, Berlin (1991)

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