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A note on k-quasi-$*$-paranormal operators

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Abstract

Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. In this paper we introduce a new class of operators satisfying

$$\left\|T^*T^kx\right\|^2 \leq \left\|T^{k+2}x\right\|\left\|T^kx\right\|$$

for all $x \in \mathcal{H}$, where k is a natural number. This class includes the classes of $*$-paranormal and k-quasi-$*$-class $\mathcal{A}$. We prove some of the properties of these operators.
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1 Introduction

Throughout this paper, let $\mathcal{H}$ be a complex separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{L}(\mathcal{H})$ denote the C^* algebra of all bounded operators on $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, we denote by $\ker T$ the null space and by $T(\mathcal{H})$ the range of T. We shall denote the set of all complex numbers and the complex conjugate of a complex number μ by $\mathbb{C}$ and $\bar{\mu}$, respectively. The closure of a set M will be denoted by $\overline{M}$, and we shall henceforth shorten $T - \mu I$ to $T - \mu$. We write $\alpha(T) = \dim\ker T$, $\beta(T) = \dim[\mathcal{H}/T(\mathcal{H})] = \dim\ker T^*$, and let $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote the spectrum, point spectrum and approximate point spectrum. Sets of isolated points and accumulation points of $\sigma(T)$ are denoted by $\operatorname{iso}\sigma(T)$ and $\operatorname{acc}\sigma(T)$, respectively. We write $r(T)$ for the spectral radius. It is well known that $r(T) \leq \|T\|$. The operator T is called *normaloid* if $r(T) = \|T\|$.

For an operator $T \in \mathcal{L}(\mathcal{H})$, as usual, $|T| = (T^*T)^{\frac{1}{2}}$ and $[T^*, T] = T^*T - TT^*$ (the self-commutator of T). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $[T^*, T]$ is zero, and T is said to be hyponormal if $[T^*, T]$ is nonnegative (equivalently if $|T| \geq |T^*|$). Furuta *et al.* [1] introduced a very interesting class of bounded linear Hilbert space operators: class $\mathcal{A}$ defined by $|T^2| \geq |T|^2$, which is called the absolute value of T, and they showed that the class A is a subclass of paranormal operators. Jeon and Kim [2] introduced quasi-class $\mathcal{A}$ (*i.e.*, $T^*|T^2|T \geq T^*|T|^2T$) operators as an extension of the notion of class $\mathcal{A}$ operators. Dugall *et al.* [3] introduced $*$-class $\mathcal{A}$ operator. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a $*$-class $\mathcal{A}$ operator if

$$\left|T^2\right| \geq \left|T^*\right|^2.$$

A $*$-class $\mathcal{A}$ operator is a generalization of a hyponormal operator [3, Theorem 1.2], and $*$-class $\mathcal{A}$ is a subclass of the class of $*$-paranormal operators [3, Theorem 1.3]. We denote the set of $*$-class $\mathcal{A}$ by $\mathcal{A}^*$. Shen *et al.* [4] introduced quasi-$*$-class $\mathcal{A}$ operator: An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a quasi-$*$-class $\mathcal{A}$ operator if

$$T^*\left|T^2\right|T \geq T^*\left|T^*\right|^2 T.$$

We denote the set of quasi-$*$-class $\mathcal{A}$ by $\mathcal{Q}(\mathcal{A}^*)$. Mecheri [5] introduced k-quasi-$*$-class $\mathcal{A}$ operator: An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a k-quasi-$*$-class $\mathcal{A}$ operator if

$$T^{*k}\left|T^2\right|T^k \geq T^{*k}\left|T^*\right|^2 T^k.$$

We denote the set of k-quasi-$*$-class $\mathcal{A}$ operator by $\mathcal{Q}(\mathcal{A}_k^*)$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be paranormal if $\|Tx\|^2 \leq \|T^2x\|$ for any unit vector x in $\mathcal{H}$. Further, T is said to be $*$-paranormal if $\|T^*x\|^2 \leq \|T^2x\|$ for any unit vector x in $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a quasi-paranormal operator if

$$\left\|T^2x\right\|^2 \leq \left\|T^3x\right\|\|Tx\|$$

for all $x \in \mathcal{H}$. Mecheri [6] introduced a new class of operators called k-quasi-paranormal operators. An operator T is called k-quasi-paranormal if

$$\left\|T^{k+1}x\right\|^2 \leq \left\|T^{k+2}x\right\|\left\|T^kx\right\|$$

for all $x \in \mathcal{H}$, where k is a natural number. Also, Mecheri [7] introduced a new class of operators called quasi-$*$-paranormal operators. An operator T is called quasi-$*$-paranormal if

$$\left\|T^*Tx\right\|^2 \leq \left\|T^3x\right\|\|Tx\|$$

for all $x \in \mathcal{H}$. In order to extend the class of paranormal and $*$-paranormal operators, we introduce the class of k-quasi-$*$-paranormal operators defined as follows.

Definition 1.1 An operator T is called k-quasi-$*$-paranormal if

$$\left\|T^*T^kx\right\|^2 \leq \left\|T^{k+2}x\right\|\left\|T^kx\right\|$$

for all $x \in \mathcal{H}$, where k is a natural number.

A 1-quasi-$*$-paranormal operator is quasi-$*$-paranormal.

2 Main results

It is well known that T is $*$-paranormal if and only if $T^{*2}T^2 - 2\lambda TT^* + \lambda^2 \geq 0$ for all $\lambda \in \mathbb{R}$ [8]. Similarly, we can prove the following proposition.

Proposition 2.1 *An operator $T \in \mathcal{L}(\mathcal{H})$ is k-quasi-$*$-paranormal if and only if*

$$T^{*k}\left(T^{*2}T^2 - 2\lambda TT^* + \lambda^2\right)T^k \geq 0 \quad \textit{for all } \lambda \in \mathbb{R}.$$

Proof Let us suppose that T is k-quasi-$*$-paranormal. Then it follows that the following relation holds:

$$\|T^*T^kx\|^2 \leq \|T^{k+2}x\|\|T^kx\|$$

for all $x \in \mathcal{H}$, where k is a natural number.

$$\begin{aligned}&\|T^*T^kx\|^2 \leq \|T^{k+2}x\|\|T^kx\| \quad\Leftrightarrow\quad 4\|T^*T^kx\|^2 - 4\|T^{k+2}x\|\|T^kx\| \leq 0\\ &\quad\Leftrightarrow\quad \|T^{k+2}x\|^2 - 2\lambda\|T^*T^kx\|^2 + \lambda^2\|T^kx\|^2 \geq 0.\end{aligned}$$

The last relation is equivalent to

$$T^{*k}\left(T^{*2}T^2 - 2\lambda TT^* + \lambda^2\right)T^k \geq 0$$

for every $\lambda \in \mathbb{R}$. □

Proposition 2.2 *Let M be a closed T-invariant subspace of $\mathcal{H}$. Then the restriction $T_{|M}$ of a k-quasi-$*$-paranormal operator T to M is a k-quasi-$*$-paranormal operator.*

Proof Let

$$T = \begin{pmatrix} A & B\\ 0 & C\end{pmatrix} \quad \text{on } \mathcal{H} = M \oplus M^\perp.$$

Since T is k-quasi-$*$-paranormal, we have

$$\begin{pmatrix} A & B\\ 0 & C\end{pmatrix}^{*k}\left\{\begin{pmatrix} A & B\\ 0 & C\end{pmatrix}^{*2}\begin{pmatrix} A & B\\ 0 & C\end{pmatrix}^{2} - 2\lambda\begin{pmatrix} A & B\\ 0 & C\end{pmatrix}\begin{pmatrix} A & B\\ 0 & C\end{pmatrix}^{*} + \lambda^2\right\}\begin{pmatrix} A & B\\ 0 & C\end{pmatrix}^{k} \geq 0$$

for all $\lambda \in \mathbb{R}$.

Therefore

$$\begin{pmatrix} A^{*k}(A^{*2}A^2 - 2\lambda(AA^* + BB^*) + \lambda^2)A^k & E\\ F & G\end{pmatrix} \geq 0$$

for some operators E, F and G.

Hence,

$$\left\langle\left(A^{*k}\left(A^{*2}A^2 - 2\lambda AA^* + \lambda^2\right)A^k\right)x, x\right\rangle \geq \left\langle\left(A^{*k}2\lambda BB^*A^k\right)x, x\right\rangle = 2|\lambda|\|B^*A^kx\|^2 \geq 0$$

for all $\lambda > 0$. This implies that $A = T_{|M}$ is a k-quasi-$*$-paranormal operator. □

Proposition 2.3 *Let $T \in \mathcal{L}(\mathcal{H})$, k-quasi-$*$-paranormal operator. If T^k has dense range, then T is a $*$-paranormal operator.*

Proof Since T^k has dense range, $\overline{T^k(\mathcal{H})} = \mathcal{H}$. Let $y \in \mathcal{H}$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ such that $T^k(x_n) \to y$ as $n \to \infty$. Since T is a k-quasi-$*$-paranormal operator, then

$$\langle (T^{*k}(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^k)x_n, x_n\rangle \geq 0 \quad \text{for all } \lambda \in \mathbb{R},$$
$$\langle (T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^k x_n, T^k x_n\rangle \geq 0 \quad \text{for all } \lambda \in \mathbb{R} \text{ and for all } n \in \mathbb{N}.$$

By the continuity of the inner product, we have

$$\langle (T^{*2}T^2 - 2\lambda TT^* + \lambda^2)y, y\rangle \geq 0 \quad \text{for all } \lambda \in \mathbb{R}.$$

Therefore T is a $*$-paranormal operator. □

Proposition 2.4 *Let $T \in \mathcal{L}(\mathcal{H})$ be a k-quasi-$*$-paranormal operator, let the range of T^k not be dense, and*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on } \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}.$$

Then A is $*$-paranormal on $\overline{T^k(\mathcal{H})}$, $C^k = 0$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof Since T is a k-quasi-$*$-paranormal operator and T^k does not have dense range, we can represent T as follows:

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on } \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}.$$

Since T is a k-quasi-$*$-paranormal operator, from Proposition 2.1 we have

$$T^{*k}(T^{*2}T^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0 \quad \text{for all } \lambda \in \mathbb{R}.$$

Therefore

$$\langle (T^{*2}T^2 - 2\lambda TT^* + \lambda^2)x, x\rangle = \langle (A^{*2}A^2 - 2\lambda AA^* + \lambda^2)x, x\rangle \geq 0$$

for all $\lambda \in \mathbb{R}$ and for all $x \in \overline{T^k(\mathcal{H})}$.

Hence, $A^{*2}A^2 - 2\lambda AA^* + \lambda^2 \geq 0$ for all $\lambda \in \mathbb{R}$. This shows that A is $*$-paranormal on $\overline{T^k(\mathcal{H})}$.

Let $x = \binom{x_1}{x_2} \in \mathcal{H} = \overline{T^k(\mathcal{H})} \oplus \ker T^{*k}$. Then

$$\langle C^k x_2, x_2\rangle = \langle T^k(I-P)x, (I-P)y\rangle = \langle (I-P)x, T^{*k}(I-P)y\rangle = 0.$$

Thus $T^{*k} = 0$.

Since $\sigma(A) \cup \sigma(C) = \sigma(T) \cup \vartheta$, where ϑ is the union of the holes in $\sigma(T)$, which happen to be a subset of $\sigma(A) \cap \sigma(C)$ by [9, Corollary 7]. Since $\sigma(A) \cap \sigma(C)$ has no interior points, then $\sigma(T) = \sigma(A) \cup \sigma(C) = \sigma(A) \cup \{0\}$ and $C^k = 0$. □

The converse of the above proposition is valid when $k = 1$.

Proposition 2.5 *If T is a quasi-$*$-paranormal operator, which commutes with an isometric operator S, then TS is a quasi-$*$-paranormal operator.*

Proof Let $A = TS$, $TS = ST$, $S^*T^* = T^*S^*$ and $S^*S = I$.

$$\begin{aligned} A^{*3}A^3 - 2\lambda\bigl(A^*A\bigr)^2 + \lambda^2 A^*A &= (TS)^{*3}(TS)^3 - 2\lambda\bigl((TS)^*(TS)\bigr)^2 + \lambda^2(TS)^*(TS) \\ &= T^{*3}T^3 - 2\lambda T^*TT^*T + \lambda^2 T^*T \geq 0, \end{aligned}$$

so that T is a quasi-$*$-paranormal operator. □

It is known that there exists a linear operator T, so that T^n is a compact operator for some $n \in \mathbb{N}$, but T itself is not compact. In this context, we will show that in cases where an operator T is k-quasi-$*$-paranormal and if its exponent T^n is compact, for some $n \in \mathbb{N}$, then T is compact too.

Proposition 2.6 *Let T be a k-quasi-$*$-paranormal operator such that T^n is compact for some $n \geq k+2$. Then T^k is compact if $k \geq 2$ and T is compact if $k = 0,1$.*

Proof To prove this proposition, it is enough to prove that T^{n-1} is compact. Let us consider the unit vector $\frac{T^{n-k-2}x}{\|T^{n-k-2}x\|} \in \mathcal{H}$ for $n \geq k+2$. Since T is k-quasi-$*$-paranormal, then

$$\left\| T^*T^k \frac{T^{n-k-2}x}{\|T^{n-k-2}x\|} \right\|^2 \leq \left\| T^{k+2} \frac{T^{n-k-2}x}{\|T^{n-k-2}x\|} \right\| \left\| T^k \frac{T^{n-k-2}x}{\|T^{n-k-2}x\|} \right\|,$$

hence

$$\bigl\| T^*T^{n-2}x \bigr\|^2 \leq \bigl\| T^n x \bigr\| \bigl\| T^{n-2}x \bigr\| \quad \text{for all } x \in \mathcal{H}. \tag{2.1}$$

Let (x_m) be any sequence in $\mathcal{H}$, satisfying $\|x_m\| = 1$ and $x_m \to 0$ weakly as $m \to \infty$. Now, by the compactness of T^n and from relation (2.1), we have

$$\bigl\| T^*T^{n-2}x_m \bigr\| \to 0 \quad \text{as } m \to \infty. \tag{2.2}$$

If $n = 2$, relation (2.2) implies the compactness of T^*, hence T is compact. If $n \geq 3$, relation (2.2) implies the compactness of T^*T^{n-2}, hence $T^{*(n-1)}T^{n-1} = T^{*(n-2)}T^*T^{n-2}T$ is compact. Then T^{n-1} is a compact operator. □

3 SVEP property

Let $\mathrm{Hol}(\sigma(T))$ be the space of all analytic functions in an open neighborhood of $\sigma(T)$. We say that $T \in \mathcal{L}(\mathcal{H})$ has the single-valued extension property (SVEP) at $\mu \in \mathbb{C}$ if for every open neighborhood U of μ, the only analytic function $f : U \to \mathbb{C}$ which satisfies the equation $(T-\mu)f(\mu) = 0$ is the constant function $f \equiv 0$. The operator T is said to have SVEP if T has SVEP at every $\mu \in \mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ has SVEP at every point of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$. Every operator T has SVEP at an isolated point of the spectrum.

Proposition 3.1 *Let T be a k-quasi-$*$-paranormal operator. If $\mu \neq 0$ and $(T-\mu)x = 0$, then $(T-\mu)^*x = 0$.*

Proof We may assume $x \neq 0$ and $(T-\mu)x = 0$. Since T is a k-quasi-$*$-paranormal operator, then $\|T^*T^kx\|^2 \le \|T^{k+2}x\|\|T^kx\|$ for all $x \in \mathcal{H}$. Hence, $\|T^*x\|^2 \le |\mu|^2\|x\|^2$. So,

$$\begin{aligned}\left\|(T-\mu)^*x\right\|^2 &= \langle\bar{\mu}x,\bar{\mu}x\rangle - \langle\bar{\mu}x,T^*x\rangle - \langle T^*x,\bar{\mu}x\rangle + \langle T^*x,T^*x\rangle \\ &\le |\mu|^2\|x\|^2 - |\mu|^2\|x\|^2 - |\mu|^2\|x\|^2 + |\mu|^2\|x\|^2 = 0.\end{aligned}$$

Therefore, $(T-\mu)^*x = 0$. □

For $T \in \mathcal{L}(\mathcal{H})$, the smallest nonnegative integer p such that $\ker T^p = \ker T^{p+1}$ is called the ascent of T and is denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. We say that $T \in \mathcal{L}(\mathcal{H})$ is of finite ascent (finitely ascensive) if $p(T-\mu) < \infty$ for all $\mu \in \mathbb{C}$.

Proposition 3.2 *Let $T \in \mathcal{L}(\mathcal{H})$ be a k-quasi-$*$-paranormal operator. Then $T-\mu$ has finite ascent for all $\mu \in \mathbb{C}$.*

Proof We have to tell that $\ker(T-\mu)^k = \ker(T-\mu)^{k+1}$. To do that, it is sufficient enough to show that $\ker(T-\mu)^{k+1} \subseteq \ker(T-\mu)^k$ since $\ker(T-\mu)^k \subseteq \ker(T-\mu)^{k+1}$ is clear.

Let $x \in \ker(T-\mu)^{k+1}$, then $(T-\mu)^{k+1}x = 0$. We consider two cases:

If $\mu \neq 0$, then from Proposition 3.1 we have $(T-\mu)^*(T-\mu)^kx = 0$. Hence,

$$\left\|(T-\mu)^k\right\|^2 = \left\langle (T-\mu)^*(T-\mu)^kx, (T-\mu)^{k-1}x\right\rangle = 0,$$

so we have $(T-\mu)^kx = 0$, which implies $\ker(T-\mu)^{k+1} \subseteq \ker(T-\mu)^k$.

If $\mu = 0$, then $T^{k+1}x = 0$, hence $T^{k+2}x = 0$.

Since T is a k-quasi-$*$-paranormal operator, then

$$\begin{aligned}\left\|T^kx\right\|^4 &= \left\langle T^kx, T^kx\right\rangle^2 = \left\langle T^*T^kx, T^{k-1}x\right\rangle^2 \\ &\le \left\|T^*T^kx\right\|^2\left\|T^{k-1}x\right\|^2 \le \left\|T^{k+2}x\right\|\left\|T^kx\right\|\left\|T^{k-1}x\right\|^2\end{aligned}$$

so

$$\left\|T^kx\right\|^3 \le \left\|T^{k+2}x\right\|\left\|T^{k-1}x\right\|^2.$$

Since $T^{k+2}x = 0$, then $T^kx = 0$, which implies $\ker T^{k+1} \subseteq \ker T^k$. □

Corollary 3.3 *Let $T \in \mathcal{L}(\mathcal{H})$ be a k-quasi-$*$-paranormal operator. Then T has the SVEP property.*

Proof The proof of the corollary follows directly from Proposition 3.2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors have given equal contribution in this paper.

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