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# On two-valued measure and double statistical convergence in 2-normed spaces

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## Abstract

In this paper we introduce some new double difference lacunary sequence spaces using Orlicz functions, generalized double difference sequences and a two-valued measure  $\mu$  in 2-normed spaces, and we also examine some of their properties.

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**Keywords:** statistical convergence;  $\mu$ -statistical convergence; 2-normed space; paranormed space; sequence spaces; double lacunary sequence

## 1 Introduction

The notion of summability of single sequences with respect to a two-valued measure was introduced by Connor [1, 2] as a very interesting generalization of statistical convergence which was defined by Fast [3]. Over the years, and under different names, statistical convergence was discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence spaces point of view and linked with summability theory by Fridy [4], Salat [5]. The notion of statistical convergence was further extended to double sequences independently by Móricz [6] and Mursaleen *et al.* [7]. Savaş [8] studied statistical convergence in random 2-normed space. For more recent developments on double sequences one can consult the papers (see [9–16]) where more references can be found. In particular, very recently Das and Bhunia investigated the summability of double sequences of real numbers with respect to a two-valued measure and made many interesting observations [17]. In [18], Das and Savaş *et al.* introduced some generalized double difference sequence spaces using summability with respect to a two-valued measure and an Orlicz function in 2-normed spaces which have a unique non-linear structure. The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Lindenstrauss and Tzafriri [19] investigated Orlicz sequence spaces in more detail and they proved that every Orlicz sequence space  $l_M$  contains a subspace isomorphic to  $l_p$  ( $1 \leq p < \infty$ ). The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [20]. Orlicz spaces find a number of useful applications in the theory of nonlinear integral equations. Whereas the Orlicz sequence spaces are the generalization of  $l_p$ -spaces, the  $l_p$ -spaces find themselves enveloped in Orlicz spaces [21].

The concept of 2-normed spaces was initially introduced by Gahler [22, 23] as a very interesting non-linear extension of the idea of usual normed linear spaces. Some initial studies on this structure can be seen in [22–24]. Recently a lot of interesting developments have occurred in 2-normed spaces in summability theory and related topics (see [18, 25–30]).

In this paper, in a natural way, we first define statistical convergence for double sequences in 2-normed spaces using a two-valued measure and also prove some interesting theorems. Furthermore, we introduce some new sequence spaces in 2-normed spaces using Orlicz functions, generalized double difference sequences and a two-valued measure  $\mu$ .

## 2 Preliminaries

Throughout the paper  $\mathbb{N}$  denotes the set of all natural numbers,  $\chi_A$  represents the characteristic function of  $A \subseteq \mathbb{N}$  and  $\mathbb{R}$  represents the set of all real numbers.

Recall that a set  $A \subseteq \mathbb{N}$  is said to have the asymptotic density  $d(A)$  if

$$d(A) = \lim_{n \rightarrow \infty} \left( \frac{1}{n} \right) \sum_{j=1}^n \chi_A(j)$$

exists.

**Definition 2.1** [3] A sequence  $\{x_n\}_{n \in \mathbb{N}}$  of real numbers is said to be statistically convergent to  $\xi \in \mathbb{R}$  if for any  $\epsilon > 0$  we have  $d(A(\epsilon)) = 0$ , where  $A(\epsilon) = \{n \in \mathbb{N} : |x_n - \xi| \geq \epsilon\}$ .

In [31] the notion of convergence for double sequences was presented by Pringsheim.

A double sequence  $x = \{x_{kl}\}$  of real numbers is said to be convergent to  $L \in \mathbb{R}$  if, for any  $\epsilon > 0$ , there exists  $N_\epsilon \in \mathbb{N}$  such that  $|x_{kl} - L| < \epsilon$  whenever  $k, l \geq N_\epsilon$ . In this case, we write  $\lim_{k, l \rightarrow \infty} x_{kl} = L$ .

A double sequence  $x = \{x_{kl}\}$  of real numbers is said to be bounded if there exists a positive real number  $M$  such that  $|x_{kl}| < M$  for all  $k, l \in \mathbb{N}$ . That is,  $\|x\|_{(\infty, 2)} = \sup_{k, l \in \mathbb{N}} |x_{kl}| < \infty$ .

Let  $K \subseteq \mathbb{N} \times \mathbb{N}$  and let  $K(k, l)$  be the cardinality of the set  $\{(m, n) \in K : m \leq k, n \leq l\}$ . If the sequence  $\{\frac{K(k, l)}{k \cdot l}\}_{k, l \in \mathbb{N}}$  has a limit in the Pringsheim's sense, then we say that  $K$  has double natural density and is denoted by  $d_2(K) = \lim_{k, l \rightarrow \infty} \frac{K(k, l)}{k \cdot l}$ .

A statistically convergent double sequence of elements of a metric space  $(X, \rho)$  is defined essentially in the same way ( $\rho(x_{kl}, \xi) \geq \epsilon$  instead of  $|x_{kl} - \xi| \geq \epsilon$ ).

Throughout the paper  $\mu$  will denote a complete  $\{0, 1\}$ -valued finite additive measure defined on algebra  $\Gamma$  of subsets of  $\mathbb{N} \times \mathbb{N}$  that contains all subsets of  $\mathbb{N} \times \mathbb{N}$  that are contained in the union of a finite number of rows and columns of  $\mathbb{N} \times \mathbb{N}$  and  $\mu(A) = 0$  if  $A$  is contained in the union of a finite number of rows and columns of  $\mathbb{N} \times \mathbb{N}$  (see [18]).

**Definition 2.2** [18] A double sequence  $x = \{x_{kl}\}$  of real numbers is said to be  $\mu$ -statistically convergent to  $L \in \mathbb{R}$  if and only if for any  $\epsilon > 0$ ,  $\mu(\{(k, l) \in \mathbb{N} \times \mathbb{N} : |x_{kl} - L| \geq \epsilon\}) = 0$ .

**Definition 2.3** [18] A double sequence  $x = \{x_{kl}\}$  of real numbers is said to be convergent to  $L \in \mathbb{R}$  in  $\mu$ -density if there exists an  $A \in \Gamma$  with  $\mu(A) = 1$  such that  $\{x_{kl}\}_{(k, l) \in A}$  is convergent to  $L$ .

**Definition 2.4** [23] Let  $X$  be a real vector space of dimension  $d$ , where  $2 \leq d < \infty$ . A 2-norm on  $X$  is a function  $\|\cdot, \cdot\| : X \times X \rightarrow \mathbb{R}$  which satisfies (i)  $\|x, y\| = 0$  if and only if  $x$  and  $y$  are linearly independent; (ii)  $\|x, y\| = \|y, x\|$ ; (iii)  $\|\alpha x, y\| = |\alpha| \|x, y\|$ ,  $\alpha \in \mathbb{R}$ ; (iv)  $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ . The ordered pair  $(X, \|\cdot, \cdot\|)$  is then called a 2-normed space.

Let  $(X, \|\cdot, \cdot\|)$  be a finite dimensional 2-normed space and let  $u = \{u_1, u_2, \dots, u_d\}$  be the basis of  $X$ . We can define the norm  $\|\cdot, \cdot\|_\infty$  on  $X$  by  $\|\cdot, \cdot\|_\infty = \max\{\|x, u_i\| : i = 1, \dots, d\}$ . Also,  $\|x - y\|_\infty = \max\{\|x - y, u_j\| : j = 1, \dots, d\}$ .

Let  $(X, \|\cdot, \cdot\|)$  be any 2-normed space and let  $S''(2 - X)$  be the set of all double sequences defined over the 2-normed space  $(X, \|\cdot, \cdot\|)$ . Clearly  $S''(2 - X)$  is a linear space under addition and scalar multiplication.

Recall in [20] that an Orlicz function  $M : [0, \infty) \rightarrow [0, \infty)$  is a continuous, convex and non-decreasing function such that  $M(0) = 0$  and  $M(x) > 0$  for  $x > 0$ , and  $M(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

Note that if  $M$  is an Orlicz function, then  $M(\lambda x) \leq \lambda$  for all  $\lambda$  with  $0 < \lambda < 1$ .

In the later stage, different classes of Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [32], Savaş [28–30, 33–38] and many others.

**Definition 2.5** [18] A double sequence  $x = \{x_{kl}\}$  in a 2-normed space  $(X, \|\cdot, \cdot\|)$  is said to be convergent to  $L$  in  $(X, \|\cdot, \cdot\|)$  if for each  $\epsilon > 0$  and each  $z \in X$ , there exists  $n_\epsilon \in \mathbb{N}$  such that  $\|x_{kl} - L, z\| < \epsilon$  for all  $i, j \geq n_\epsilon$ .

**Definition 2.6** [18] Let  $\mu$  be a two-valued measure on  $\mathbb{N} \times \mathbb{N}$ . A double sequence  $\{x_{kl}\}$  in a 2-normed space  $(X, \|\cdot, \cdot\|)$  is said to be  $\mu$ -statistically convergent to some point  $x$  in  $X$  if for each pre-assigned  $\epsilon > 0$  and for each  $z \in X$ ,  $\mu(A(z, \epsilon)) = 0$ , where  $A(z, \epsilon) = \{(k, l) \in \mathbb{N} \times \mathbb{N} : \|x_{kl} - x, z\| \geq \epsilon\}$ .

**Definition 2.7** Let  $\mu$  be a two-valued measure on  $\mathbb{N} \times \mathbb{N}$ . A double sequence  $\{x_{kl}\}$  in a 2-normed space  $(X, \|\cdot, \cdot\|)$  is said to be  $\mu$ -statistically Cauchy if for each pre-assigned  $\epsilon > 0$  and for each  $z \in X$ , there exist integers  $N = N(\epsilon, z)$  and  $M = M(\epsilon, z)$  such that  $\mu(A(z, \epsilon)) = 0$ , where  $A(z, \epsilon) = \{(k, l) \in \mathbb{N} \times \mathbb{N} : \|x_{kl} - x_{N(\epsilon, z)M(\epsilon, z)}, z\| \geq \epsilon\}$ .

We first give the following theorem.

**Theorem 2.1** Let  $\mu$  be a two-valued measure on  $\mathbb{N} \times \mathbb{N}$  and two sequences  $(x_{kl})$  and  $(y_{kl})$  in 2-normed space  $(X, \|\cdot, \cdot\|)$ . If  $(y_{kl})$  is a  $\mu$ -convergent sequence such that  $x_{kl} = y_{kl}$  a.a.  $(k, l)$ , then  $(x_{kl})$  is  $\mu$ -statistically convergent.

*Proof* Suppose  $\mu(\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{kl} \neq y_{kl}\}) = 0$  and  $\lim_{m, n \rightarrow \infty} \|y_{kl}, z\| = \|L, z\|$ . Then, for every  $\epsilon > 0$  and  $z \in X$ ,

$$\begin{aligned} \{(k, l) \in \mathbb{N} \times \mathbb{N} : \|x_{kl} - L, z\| \geq \epsilon\} &\subseteq \{(k, l) \in \mathbb{N} \times \mathbb{N} : \|y_{kl} - L, z\| \geq \epsilon\} \\ &\cup \{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{kl} \neq y_{kl}\}. \end{aligned}$$

Therefore

$$\begin{aligned} \mu\{(k, l) \in \mathbb{N} \times \mathbb{N} : \|x_{kl} - L, z\| \geq \epsilon\} &\leq \mu\{(k, l) \in \mathbb{N} \times \mathbb{N} : \|y_{kl} - L, z\| \geq \epsilon\} \\ &+ \mu\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{kl} \neq y_{kl}\}. \end{aligned}$$

Since

$$\lim_{m \rightarrow \infty} \|y_{kl}, z\| = \|L, z\|$$

for every  $z \in X$ , the set

$$\{(k, l) \in \mathbb{N} \times \mathbb{N} : \|y_{kl} - L, z\| \geq \varepsilon\}$$

contains a finite number of integers. Hence,

$$\mu\{(k, l) \in \mathbb{N} \times \mathbb{N} : \|y_{kl} - L, z\| \geq \varepsilon\} = 0.$$

We get

$$\mu\{(k, l) \in \mathbb{N} \times \mathbb{N} : \|x_{kl} - L, z\| \geq \varepsilon\} = 0$$

for every  $\varepsilon > 0$  and  $z \in X$ . Consequently,  $\mu$ -st- $\lim_{k,l \rightarrow \infty} \|x_{kl}, z\| = \|L, z\|$ . This completes the proof.  $\square$

**Theorem 2.2** *Let  $\mu$  be a two-valued measure on  $\mathbb{N} \times \mathbb{N}$  and let  $\{x_{kl}\}_{k,l \geq 1}$  be a  $\mu$ -statistically Cauchy sequence in a finite dimensional 2-normed space  $(X, \|\cdot, \cdot\|)$ . Then there exists a  $\mu$ -convergent sequence  $\{y_{kl}\}_{k,l \geq 1}$  in  $(X, \|\cdot, \cdot\|)$  such that  $x_{kl} = y_{kl}$  for a.a.  $(k, l)$ .*

*Proof* First suppose that  $\{x_{kl}\}_{k,l \geq 1}$  is a statistically Cauchy sequence in  $(X, \|\cdot, \cdot\|_\infty)$ . Choose natural numbers  $M_1$  and  $N_1$  such that the closed ball  $B_u^{1,1} = B_u(x_{M_1 N_1}, 1)$  contains  $x_{kl}$  for a.a.  $(k, l)$ . Then choose natural numbers  $M_2$  and  $N_2$  such that the closed ball  $B_{2,2} = B_u(x_{M_2 N_2}, \frac{1}{(2.2)})$  contains  $x_{kl}$  for a.a.  $(k, l)$ . Note that  $B_u^{2,2} = B_u^{1,1} \cup B_{2,2}$  also contains  $x_{kl}$  for a.a.  $(k, l)$ . Thus, by continuing this process, we can obtain a sequence  $\{B_u^{p,q}\}_{p,q \geq 1}$  of nested closed balls such that  $\text{diam}\{B_u^{p,q}\} \leq \frac{1}{2^{p2^q}}$ . Therefore  $\bigcap_{p,q=1,1}^{\infty, \infty} B_u^{p,q} = \{A\}$ . Since each  $B_u^{p,q}$  contains  $x_{kl}$  for a.a.  $(k, l)$ , we can choose a sequence of strictly increasing natural numbers  $\{T_{p,q}\}_{p,q \geq 1}$  such that

$$\frac{1}{kl} |\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{kl} \notin B_u^{p,q}\}| < \frac{1}{p, q} \quad \text{if } k, l > T_{p,q}.$$

Hence we have

$$\mu\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{kl} \notin B_u^{p,q}\} = 0 \quad \text{if } k, l > T_{p,q}.$$

Put  $W_{p,q} = \{(k, l) \in \mathbb{N} \times \mathbb{N} : (k, l) > T_{p,q}, x_{kl} \notin B_u^{p,q}\}$  for all  $p, q \geq 1$ , and  $W = \bigcup_{p,q=1,1}^{\infty, \infty} W_{p,q}$ . Now define the sequence  $\{y_{kl}\}_{k,l \geq 1}$  as follows:

$$y_{kl} = \begin{cases} A & \text{if } (k, l) \in W, \\ x_{kl} & \text{otherwise.} \end{cases}$$

Note that  $\lim_{k,l \rightarrow \infty, \infty} y_{kl} = A$ . In fact, for each  $\varepsilon > 0$ , choose a natural number  $p, q$  such that  $\varepsilon > \frac{1}{2^{p2^q}} > 0$ . Then, for each  $kl > T_{p,q}$ , or  $y_{kl} = A$  or  $y_{kl} = x_{kl} \in B_u^{p,q}$  and so in each case

$\|y_{kl} - A\|_\infty \leq \text{diam}\{B_u^{p,q}\} \leq \frac{1}{2^{p-1}2^{q-1}}$ . Since  $\{(k, l) \in \mathbb{N} \times \mathbb{N} : y_{kl} \neq x_{kl}\} \subseteq \{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{kl} \notin B_u^{p,q}\}$ , we have

$$\mu\{(k, l) \in \mathbb{N} \times \mathbb{N} : y_{kl} \neq x_{kl}\} \subseteq \mu\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{kl} \notin B_u^{p,q}\}.$$

Hence  $\mu(\{(k, l) \in \mathbb{N} \times \mathbb{N} : y_{kl} \neq x_{kl}\}) = 0$ . Thus, in the space  $(X, \|\cdot, \cdot\|_\infty)$ ,  $y_{kl} = x_{kl}$  for a.a.  $(k, l)$ . Suppose that  $\{u_1, u_2, \dots, u_d\}$  is the basis for  $(X, \|\cdot, \cdot\|)$ . Since  $\lim_{m \rightarrow \infty} \|y_{kl} - A\|_\infty = 0$  and  $\|y_{kl} - A, u_i\| \leq \|y_{kl} - A\|_\infty$  for all  $1 \leq i \leq d$ ,  $\lim_{m \rightarrow \infty} \|y_{kl} - A, z\|_\infty = 0$  for every  $z \in X$ . This completes the proof.  $\square$

In order to prove the equivalence of Definitions 2.5 and 2.6, we shall find it helpful to use Theorems 2.1 and 2.2.

**Theorem 2.3** *Let  $\mu$  be a two-valued measure on  $\mathbb{N} \times \mathbb{N}$  and let  $\{x_{kl}\}_{kl \geq 1}$  be a sequence in a 2-normed space  $(X, \|\cdot, \cdot\|)$ . The sequence  $\{x_{kl}\}$  is  $\mu$ -statistically convergent if and only if  $\{x_{kl}\}$  is a  $\mu$ -statistically Cauchy sequence.*

### 3 New double sequence spaces

We first state an inequality which will be used throughout this paper : If  $\{p_{kl}\}$  is a bounded double sequence of non-negative real numbers and  $\sup_{k,l \in \mathbb{N}} p_{kl} = H$  and  $D = \text{Max}\{1, 2^{H-1}\}$ , then

$$|a_{kl} + b_{kl}|^{p_{kl}} \leq D\{|a_{kl}|^{p_{kl}} + |b_{kl}|^{p_{kl}}\}$$

for all  $k, l$  and  $a_{kl}, b_{kl} \in \mathbb{C}$ , the set of all complex numbers. Also,

$$|a|^{p_{kl}} \leq \text{Max}\{1, |a|^H\}$$

for all  $a \in \mathbb{C}$ .

By a lacunary sequence  $\theta = (k_r); r = 0, 1, 2, \dots$ , where  $k_0 = 0$ , we shall mean an increasing sequence of non-negative integers with  $k_r - k_{r-1} \rightarrow \infty$  as  $r \rightarrow \infty$ . The intervals determined by  $\theta$  will be denoted by  $I_r = (k_{r-1}, k_r]$  and  $h_r = k_r - k_{r-1}$ . The ratio  $\frac{k_r}{k_{r-1}}$  will be denoted by  $q_r$ .

**Definition 3.1** The double sequence  $\theta_{r,s} = \{(k_r, l_s)\}$  is called *double lacunary* if there exist two increasing sequences of integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \rightarrow \infty \quad \text{as } r \rightarrow \infty$$

and

$$l_0 = 0, \quad \bar{h}_s = l_s - l_{s-1} \rightarrow \infty \quad \text{as } s \rightarrow \infty.$$

Let us denote  $k_{r,s} = k_r l_s$ ,  $h_{r,s} = h_r \bar{h}_s$  and  $\theta_{r,s}$  is determined by  $I_{r,s} = \{(k, l) : k_{r-1} < k \leq k_r \text{ \& } l_{s-1} < l \leq l_s\}$ ,  $q_r = \frac{k_r}{k_{r-1}}$ ,  $\bar{q}_s = \frac{l_s}{l_{s-1}}$ , and  $q_{r,s} = q_r \bar{q}_s$ .

**Definition 3.2** Suppose that as before  $\mu$  is a two-valued measure on  $\mathbb{N} \times \mathbb{N}$  and let  $M$  be an Orlicz function and  $(X, \|\cdot, \cdot\|)$  be a 2-normed space. Further, let  $p = \{p_{kl}\}$  be a bounded

sequence of positive real numbers. Now we introduce the following different types of sequence spaces, for all  $\epsilon > 0$ ,

$$W^\mu(\theta^2, M, \Delta^m, p, \|\cdot, \cdot\|) = \left\{ x \in S''(2-X) : \mu \left( (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^m x_{kl} - L}{\rho}, z \right\| \right) \right]^{p_{kl}} \geq \epsilon \right) = 0 \right. \\ \left. \text{for some } \rho > 0 \text{ and } L \in X \text{ and each } z \in X \right\},$$

$$W_0^\mu(\theta^2, M, \Delta^m, p, \|\cdot, \cdot\|) = \left\{ x \in S''(2-X) : \mu \left( (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} \geq \epsilon \right) = 0 \right. \\ \left. \text{for some } \rho > 0 \text{ and each } z \in X \right\},$$

$$W_\infty^\mu(\theta^2, M, \Delta^m, p, \|\cdot, \cdot\|) = \left\{ x \in S''(2-X) : \exists k > 0, \mu \left( \left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} \geq k \right\} \right) = 0 \text{ for some } \rho > 0 \text{ and each } z \in X \right\},$$

where  $\Delta^m x_{kl} = \Delta^{m-1} x_{kl} - \Delta^{m-1} x_{k+1,l} - \Delta^{m-1} x_{k,l+1} + \Delta^{m-1} x_{k+1,l+1}$ .

We now prove the following theorem.

**Theorem 3.1**  $W^\mu(\theta, M, \Delta^m, p, \|\cdot, \cdot\|)$ ,  $W_0^\mu(\theta^2, M, \Delta^m, p, \|\cdot, \cdot\|)$  and  $W_\infty^\mu(\theta^2, M, \Delta^m, p, \|\cdot, \cdot\|)$  are linear spaces.

*Proof* We shall prove the theorem for  $W_0^\mu(\theta^2, M, \Delta^m, p, \|\cdot, \cdot\|)$  and others can be proved similarly. Let  $\epsilon > 0$  be given. Assume that  $x, y \in W_0^\mu(\theta, M, \Delta^m, p, \|\cdot, \cdot\|)$  and  $\alpha, \beta \in \mathbb{R}$ , where  $x = \{x_{kl}\}$  and  $y = \{y_{kl}\}$ . Further let  $z \in X$ . Then

$$\mu \left( \left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^m x_{kl}}{\rho_1}, z \right\| \right) \right]^{p_{kl}} \geq \epsilon \right\} \right) = 0 \tag{3.1}$$

for some  $\rho_1 > 0$  and

$$\mu \left( \left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^m y_{kl}}{\rho_2}, z \right\| \right) \right]^{p_{kl}} \geq \epsilon \right\} \right) = 0 \tag{3.2}$$

for some  $\rho_2 > 0$ .

Since  $\|\cdot, \cdot\|$  is 2-normed,  $\Delta^m$  is linear, therefore the following inequality holds:

$$\frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^m (\alpha x_{kl} + \beta y_{kl})}{|\alpha|\rho_1 + |\beta|\rho_2}, z \right\| \right) \right]^{p_{kl}} \\ \leq D \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ \frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2} M \left( \left\| \frac{\Delta^m x_{kl}}{\rho_1}, z \right\| \right) \right]^{p_{st}}$$

$$\begin{aligned}
 &+ D \frac{1}{h_{rs}} \sum_{(k,l) \in I_{kl}} \left[ \frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2} M \left( \left\| \frac{\Delta^m y_{kl}}{\rho_2}, z \right\| \right) \right]^{p_{kl}} \\
 &\leq DF \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^m x_{kl}}{\rho_1}, z \right\| \right) \right]^{p_{kl}} + DF \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^m y_{kl}}{\rho_2}, z \right\| \right) \right]^{p_{kl}},
 \end{aligned}$$

where

$$F = \text{Max} \left\{ 1, \left[ \frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2} \right]^H, \left[ \frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2} \right]^H \right\}.$$

From the above inequality we get

$$\begin{aligned}
 &\left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^m (\alpha x_{kl} + \beta y_{kl})}{|\alpha|\rho_1 + |\beta|\rho_2}, z \right\| \right) \right]^{p_{kl}} \geq \epsilon \right\} \\
 &\subseteq \left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : DF \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^m x_{kl}}{\rho_1}, z \right\| \right) \right]^{p_{kl}} \geq \frac{\epsilon}{2} \right\} \\
 &\cup \left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : DF \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^m y_{kl}}{\rho_2}, z \right\| \right) \right]^{p_{kl}} \geq \frac{\epsilon}{2} \right\}.
 \end{aligned}$$

Hence from (3.1) and (3.2) the required result is proved.  $\square$

**Theorem 3.2** For any fixed  $(k, l) \in \mathbb{N} \times \mathbb{N}$ ,  $W_\infty^\mu(\theta^2, M, \Delta^m, p, \|\cdot, \cdot\|)$  is a paranormed space with respect to the paranorm  $g_{kl} : X \rightarrow \mathbb{R}$ , defined by

$$\begin{aligned}
 g_{kl}(x) &= \inf_{z \in X} \sum_{(k,l) \in I_{rs}} \|x_{kl}, z\| \\
 &+ \inf \left\{ \rho^{p_{kl}/H} : \rho > 0 \text{ s.t. } \sup_{(k,l) \in \mathbb{N} \times \mathbb{N}} \left[ M \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} \leq 1, \forall z \in X \right\}.
 \end{aligned}$$

*Proof* This can be proved by using the techniques similar to those used in Theorem 4.3 in [18].  $\square$

**Theorem 3.3** Let  $M, M_1, M_2$  be Orlicz functions. Then

- (i)  $W_0^\mu(\theta^2, M_1, \Delta^m, p, \|\cdot, \cdot\|) \subseteq W_0^\mu(\theta^2, M \circ M_1, \Delta^m, p, \|\cdot, \cdot\|)$ , provided  $\{p_{kl}\}_{k,l \in \mathbb{N} \times \mathbb{N}}$  are such that  $H_0 = \inf p_{kl} > 0$ ;
- (ii)  $W_0^\mu(\theta^2, M_1, \Delta^m, p, \|\cdot, \cdot\|) \cap W_0^\mu(\theta^2, M_2, \Delta^m, p, \|\cdot, \cdot\|) \subseteq W_0^\mu(\theta^2, M_1 + M_2, \Delta^m, p, \|\cdot, \cdot\|)$ .

*Proof* (i) Let  $\epsilon > 0$  be given. Choose  $\epsilon_0 > 0$  such that  $\max\{\epsilon_0^H, \epsilon_0^{H_0}\} < \epsilon$ . Now, using the continuity of  $M$ , choose  $0 < \delta < 1$  such that  $0 < t < \delta$  implies that  $M(t) < \epsilon_0$ . Let  $\{x_{kl}\} \in W_0^\mu(\theta^2, M_1, \Delta^m, p, \|\cdot, \cdot\|)$ . Now, from the definition  $\mu(A(\delta)) = 0$ , where

$$A(\delta) = \left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M_1 \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} \geq \delta^H \right\}.$$

Thus if  $(k, l) \notin A(\delta)$ , then

$$\frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M_1 \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} < \delta^H,$$

i.e.,

$$\sum_{(r,s) \in I_{rs}} \left[ M_1 \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} < h_{rs} \delta^H,$$

i.e.,

$$\left[ M_1 \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} < \delta^H$$

for all  $(k, l) \in I_{rs}$ . Hence

$$\left[ M_1 \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right] < \delta$$

for all  $(k, l) \in I_{rs}$ .

Hence, from the above, using the continuity of  $M$ , we must have

$$M \left( \left[ M_1 \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right] \right) < \epsilon_0$$

for all  $(k, l) \in I_{rs}$ . This implies that

$$\left[ \text{Mo} M_1 \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} < \max \{ \epsilon_0^H, \epsilon_0^H \}$$

for all  $(k, l) \in I_{rs}$ , i.e.,

$$\sum_{(k,l) \in I_{rs}} \left[ \text{Mo} M_1 \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} < h_{rs} \max \{ \epsilon_0^H, \epsilon_0^H \} < h_{rs} \epsilon,$$

which again implies that

$$\frac{1}{h_{rs}} \sum_{(r,s) \in I_{rs}} \left[ \text{Mo} M_1 \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} < \epsilon.$$

This shows that

$$\left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ \text{Mo} M_1 \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} \geq \epsilon \right\} \subseteq A(\delta).$$

Therefore

$$\mu \left( \left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ \text{Mo} M_1 \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{p_{kl}} \geq \epsilon \right\} \right) = 0.$$

Thus

$$\{x_{kl}\} \in W_0^\mu(\theta^2, M_1, \Delta^m, p, \|\cdot, \cdot\|).$$



(ii) Let  $\{x_{kl}\} \in W_0^\mu(\theta^2, M_1, \Delta^m, p, \|\cdot, \cdot\|) \cap W_0^\mu(\theta^2, M_2, \Delta^m, p, \|\cdot, \cdot\|)$ . Then the fact that

$$\begin{aligned} & \frac{1}{h_{rs}} \left[ (M_1 + M_2) \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{pkl} \\ & \leq \frac{D}{h_{rs}} \left[ M_1 \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{pkl} + \frac{D}{h_{rs}} \left[ M_2 \left( \left\| \frac{\Delta^m x_{kl}}{\rho}, z \right\| \right) \right]^{pkl} \end{aligned}$$

gives us the result. Hence this completes the proof of the theorem.  $\square$

Finally we conclude this paper by stating the following theorem.

**Theorem 3.4** *Let  $X(\Delta^{m-1})$ ,  $m \geq 1$ , stand for  $W^\mu(\theta^2, M, \Delta^{m-1}, p, \|\cdot, \cdot\|)$  or  $W_0^\mu(\theta^2, M, \Delta^{m-1}, p, \|\cdot, \cdot\|)$  or  $W_\infty^\mu(\theta^2, M, \Delta^{m-1}, p, \|\cdot, \cdot\|)$ . Then  $X(\Delta^{m-1}) \subsetneq X(\Delta^m)$ . In general,  $X(\Delta^i) \subsetneq X(\Delta^m)$  for all  $i = 1, 2, 3, \dots, m - 1$ .*

*Proof* We shall give the proof for  $W_0^\mu(\theta^2, M, \Delta^{m-1}, p, \|\cdot, \cdot\|)$  only. It can be proved in a similar way for  $W^\mu(\theta^2, M, \Delta^{m-1}, p, \|\cdot, \cdot\|)$  and  $W_\infty^\mu(\theta^2, M, \Delta^{m-1}, p, \|\cdot, \cdot\|)$ .

Let  $x = \{x_{kl}\}_{k,l \in \mathbb{N}} \in W_0^\mu(\theta^2, M, \Delta^{m-1}, p, \|\cdot, \cdot\|)$ . Let also  $\epsilon > 0$  be given. Then

$$\mu \left( \left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^{m-1} x_{kl}}{\rho}, z \right\| \right) \right]^{pkl} \geq \epsilon \right\} \right) = 0 \tag{3.3}$$

for some  $\rho > 0$ . Since  $M$  is non-decreasing and convex, it follows that

$$\begin{aligned} & \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^m x_{kl}}{4\rho}, z \right\| \right) \right]^{pkl} \\ & = \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^{m-1} x_{k+1,l+1} - \Delta^{m-1} x_{k+1,l} - \Delta^{m-1} x_{k,l+1} + \Delta^{m-1} x_{kl}}{4\rho}, z \right\| \right) \right]^{pkl} \\ & \leq \frac{D^2}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left( \left[ \frac{1}{4} M \left( \left\| \frac{\Delta^{m-1} x_{k+1,l+1}}{\rho}, z \right\| \right) \right]^{pkl} + \left[ \frac{1}{4} M \left( \left\| \frac{\Delta^{m-1} x_{k+1,l}}{\rho}, z \right\| \right) \right]^{pkl} \right. \\ & \quad \left. + \left[ \frac{1}{4} M \left( \left\| \frac{\Delta^{m-1} x_{k,l+1}}{\rho}, z \right\| \right) \right]^{pkl} + \left[ \frac{1}{4} M \left( \left\| \frac{\Delta^{m-1} x_{k,l}}{\rho}, z \right\| \right) \right]^{pkl} \right) \\ & \leq \frac{D^2 G}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left( \left[ M \left( \left\| \frac{\Delta^{m-1} x_{k+1,l+1}}{\rho}, z \right\| \right) \right]^{pkl} + \left[ M \left( \left\| \frac{\Delta^{m-1} x_{k+1,l}}{\rho}, z \right\| \right) \right]^{pkl} \right. \\ & \quad \left. + \left[ M \left( \left\| \frac{\Delta^{m-1} x_{k,l+1}}{\rho}, z \right\| \right) \right]^{pkl} + \left[ M \left( \left\| \frac{\Delta^{m-1} x_{k,l}}{\rho}, z \right\| \right) \right]^{pkl} \right), \end{aligned}$$

where  $G = \text{Max}\{1, (\frac{1}{4})^H\}$ . Hence we have

$$\begin{aligned} & \left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^m x_{kl}}{4\rho}, z \right\| \right) \right]^{pkl} \geq \epsilon \right\} \\ & \subseteq \left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{D^2 G}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^{m-1} x_{k+1,l+1}}{\rho}, z \right\| \right) \right]^{pkl} \geq \frac{\epsilon}{4} \right\} \\ & \cup \left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{D^2 G}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^{m-1} x_{k+1,l}}{\rho}, z \right\| \right) \right]^{pkl} \geq \frac{\epsilon}{4} \right\} \end{aligned}$$

$$\cup \left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{D^2 G}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^{m-1} x_{k,l+1}}{\rho}, z \right\| \right) \right]^{pkl} \geq \frac{\epsilon}{4} \right\}$$

$$\cup \left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{D^2 G}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^{m-1} x_{k,l}}{\rho}, z \right\| \right) \right]^{pkl} \geq \frac{\epsilon}{4} \right\}.$$

Using (3.3) we get

$$\mu \left( \left\{ (k, l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{(k,l) \in I_{rs}} \left[ M \left( \left\| \frac{\Delta^m x_{kl}}{4\rho}, z \right\| \right) \right]^{pkl} \geq \epsilon \right\} \right) = 0.$$

Therefore  $x = \{x_{kl}\} \in W_0^\mu(\theta^2, M, \Delta^m, p, \|\cdot, \cdot\|)$ . This completes the proof.  $\square$

#### Competing interests

The author declares that they have no competing interests.

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