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Boundedness of the maximal operator in the local Morrey-Lorentz spaces

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Abstract

In this paper we define a new class of functions called local Morrey-Lorentz spaces $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$, $0 < p, q \le \infty$ and $0 \le \lambda \le 1$. These spaces generalize Lorentz spaces such that $M_{p,q;0}^{\mathrm{loc}}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$. We show that in the case $\lambda < 0$ or $\lambda > 1$, the space $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$ is trivial, and in the limiting case $\lambda = 1$, the space $M_{p,q;1}^{\mathrm{loc}}(\mathbb{R}^n)$ is the classical Lorentz space $\Lambda_{\infty, t^{\frac{1}{p}-\frac{1}{q}}}(\mathbb{R}^n)$. We show that for $0 < q \le p < \infty$ and $0 < \lambda \le \frac{q}{p}$, the local Morrey-Lorentz spaces $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$ are equal to weak Lebesgue spaces $WL_{\frac{1}{p}-\frac{\lambda}{q}}(\mathbb{R}^n)$. We get an embedding between local Morrey-Lorentz spaces and Lorentz-Morrey spaces. Furthermore, we obtain the boundedness of the maximal operator in the local Morrey-Lorentz spaces.

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Keywords: Morrey spaces; Lorentz spaces; Lorentz-Morrey spaces; local Morrey-Lorentz spaces; maximal operator

1 Introduction

The aim of this paper is to define a new class of functions called local Morrey-Lorentz spaces $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$ and to study the boundedness of the maximal operator in these spaces. Local Morrey-Lorentz spaces $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$ are generalizations of Lorentz spaces $L_{p,q}(\mathbb{R}^n)$. Lorentz spaces, introduced by Lorentz in the 1950s [1, 2], are generalizations of the more familiar L_p spaces. Lorentz spaces, which are Banach spaces, appear to be useful in the general interpolation theory by Calderón (see [3]). Peetre [4] identified Lorentz spaces as intermediate spaces for interpolation theory by Lions and Peetre (see [5]). Riviere and Sagher [6] generalized the results of Calderón contained in [3] to include Lorentz spaces having coefficients p and q greater than zero; similarly, Kree and Peetre generalized the results of Lions and Peetre obtained in [5].

For $x \in \mathbb{R}^n$ and $t > 0$, let $B(x,t)$ denote the open ball centered at x of radius t, and let $|B(x,t)|$ be the Lebesgue measure of the ball $B(x,t)$. Note that $|B(x,t)| = \omega_n t^n$, where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Let f be a locally integrable function on $\mathbb{R}^n$. The Hardy-Littlewood maximal function Mf of f is defined by

$$Mf(x) = \sup_{t>0} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y)| \, dy, \quad x \in \mathbb{R}^n.$$

Maximal operators play an important role in the differentiability properties of functions, singular integrals and partial differential equations. They often provide a deeper and more

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simplified approach to understanding problems in these areas than other methods. It is well known that for the classical Hardy-Littlewood maximal operator the rearrangement inequality

$$cf^{**}(t) \le (Mf)^*(t) \le Cf^{**}(t), \quad t \in (0, \infty),$$

holds, ([7], Chapter 3, Theorem 3.8), where $f^*(t)$ is the non-increasing rearrangement of f and

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(t) \, dt.$$

Mingione [8] defined the Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ as follows.

Definition 1 [8] The Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ is the set of all measurable functions f on $\mathbb{R}^n$: for $1 \le p < \infty$, $0 < q < \infty$ and $0 \le \lambda \le n$, $f \in \mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ iff

$$\|f\|_{\mathcal{L}_{p,q;\lambda}} = \sup_{x \in \mathbb{R}^n, t > 0} t^{-\frac{\lambda}{p}} \|\chi_{B(x,t)} f\|_{L_{p,q}} < \infty.$$

Here $\| \cdot \|_{L_{p,q}}$ denotes the Lorentz norm of a function (see preliminaries).

In [8], Section 4.1, Mingione studied the boundedness of the restricted fractional maximal operator $M_{\beta,B}$

$$M_{\beta,B} f(x) = \sup_{B(x,t) \subset B} |B(x,t)|^{\frac{\beta}{n} - 1} \int_{B(x,t)} |f(y)| \, dy, \quad x \in \mathbb{R}^n,$$

in the restricted Lorentz-Morrey spaces $\mathcal{L}_{p,q;\lambda}(B)$, where B is any ball. Mingione derived a general non-linear version, extending *a priori* estimates and regularity results for possibly degenerate non-linear elliptic problems to the various spaces of Lorentz and Lorentz-Morrey type considered in [9–11].

Ragusa [12] defined the Lorentz-Morrey spaces $L_{p,q;\lambda}(\mathbb{R}^n)$ and studied some embeddings between these spaces.

Definition 2 [12] The Lorentz-Morrey spaces $L_{p,q;\lambda}(\mathbb{R}^n)$ is the set of all measurable functions f on $\mathbb{R}^n$: for $1 \le p < \infty$, $0 < q < \infty$ and $0 \le \lambda \le n$, iff

$$\|f\|_{L_{p,q;\lambda}} = \sup_{x \in \mathbb{R}^n, t > 0} t^{-\frac{\lambda}{q}} \|\chi_{B(x,t)} f\|_{L_{p,q}} < \infty.$$

Accordingly, f belongs to

$$L_{p,\infty;\lambda}(\mathbb{R}^n) \equiv WL_{p,\lambda}(\mathbb{R}^n) \quad \text{iff} \quad \|f\|_{L_{p,\infty;\lambda}} = \|f\|_{WL_{p,\lambda}} < \infty.$$

Note that the spaces $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ and $L_{p,q;\lambda \frac{q}{p}}(\mathbb{R}^n)$ defined by Mingione and Ragusa respectively coincide, thus

$$\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n) = L_{p,q;\lambda \frac{q}{p}}(\mathbb{R}^n).$$

Recall that the local Morrey-type spaces $LM_{p\theta,w}$ were introduced and the boundedness in these spaces of the fractional integral operators and singular integral operators defined on homogeneous Lie groups were proved by Guliyev [13] in the doctoral thesis (see, also [14–16]). They are given by

$$\|f\|_{LM_{p\theta,w}} = \big\| w(r) \|f\|_{L_p(B(0,r))} \big\|_{L_\theta(0,\infty)},$$

where w is a positive measurable function defined on $(0,\infty)$. Some necessary and sufficient conditions for the boundedness of the maximal, fractional maximal, Riesz potential and singular integral operators in the local Morrey-type space $LM_{p\theta,w}$ were given in [17–21]. We should explain that the spaces $LM_{p\theta,w}$ are closely related to the B_σ spaces (see [22, 23]).

This paper is organized as follows. In Section 2 we give some notations and definitions of the Morrey, Lorentz and classical Lorentz spaces. In Section 3 we define a new class of functions called local Morrey-Lorentz spaces $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$, $0 < p, q \le \infty$ and $0 \le \lambda \le 1$. These spaces generalize Lorentz spaces such that $M_{p,q;0}^{\mathrm{loc}}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$. We show that in the case $\lambda < 0$ or $\lambda > 1$, the space $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$ is trivial, and in the limiting case $\lambda = 1$, the space $M_{p,q;1}^{\mathrm{loc}}(\mathbb{R}^n)$ is the classical Lorentz space $\Lambda_{\infty,t^{\frac{1}{p}-\frac{1}{q}}}(\mathbb{R}^n)$. We also show that for $0 < q \le p < \infty$ and $0 < \lambda \le \frac{q}{p}$, the local Morrey-Lorentz spaces $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$ are equal to weak Lebesgue spaces $WL_{\frac{1}{p}-\frac{\lambda}{q}}(\mathbb{R}^n)$. In Section 4 we prove the boundedness of the maximal operator in $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$.

Throughout the paper, we write $A \lesssim B$ if there exists a positive constant C, independent of appropriate quantities such as functions, satisfying $A \le CB$. If $p \in [1,\infty]$, the conjugate number p' is defined by $p' = \frac{p}{p-1}$ and if $p \in (0,1)$, the conjugate number p' is defined by $p' = \frac{p}{1-p}$.

2 Preliminaries

Let E be a measurable subset of $\mathbb{R}^n$ and $|E| = \int_E dx$. We denote by $L_p(E)$ the class of all measurable functions f defined on E for which

$$\|f\|_{L_p(E)} := \left(\int_E |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty, \quad 0 < p < \infty,$$

$$\|f\|_{L_\infty(E)} := \sup \big\{ \alpha : \big| \{ y \in E : |f(y)| \ge \alpha \} \big| > 0 \big\}.$$

We define rearrangement of f in decreasing order by

$$f^*(t) = \inf \big\{ \lambda > 0 : \mu_f(\lambda) \le t \big\}, \quad \forall t \in (0,\infty),$$

where $\mu_f(\lambda)$ denotes the distribution function of f given by

$$\mu_f(\lambda) = \big| \{ y \in \mathbb{R}^n : |f(y)| > \lambda \} \big|.$$

We denote by $WL_p(\mathbb{R}^n)$ the weak L_p space of all measurable functions f with the quasi-norm

$$\|f\|_{WL_p} = \sup_{t>0} t^{1/p} f^*(t), \quad 1 \le p < \infty.$$

Now we recall the definitions of Morrey spaces, Lorentz spaces and classical Lorentz spaces.

Morrey spaces were introduced by Morrey [24] in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations. Later, Morrey spaces found important applications to Navier-Stokes and Schrödinger equations, elliptic problems with discontinuous coefficients, and potential theory.

Definition 3 [24, 25] We denote by $L_{p,\lambda}(\mathbb{R}^n)$ the Morrey space for $0 \leq \lambda \leq n$, $1 \leq p < \infty$, $f \in L_{p,\lambda}(\mathbb{R}^n)$ if $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{L_{p,\lambda}} = \sup_{x \in \mathbb{R}^n, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_{p(B(x,t))}} < \infty.$$

If $\lambda = 0$, then $L_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$; if $\lambda = n$, then $L_{p,n}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$; if $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Also, by $WL_{p,\lambda}(\mathbb{R}^n)$ we denote the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WL_{p,\lambda}} = \sup_{x \in \mathbb{R}^n, t>0} t^{-\frac{\lambda}{p}} \|f\|_{WL_{p(B(x,t))}} < \infty.$$

Lorentz spaces were introduced by Lorentz in 1950. Lorentz spaces, which are Banach spaces and generalizations of the more familiar L_p spaces, appear to be useful in the general interpolation theory.

Definition 4 The Lorentz space $L_{p,q}(\mathbb{R}^n)$, $0 < p, q \leq \infty$, is the collection of all measurable functions f on $\mathbb{R}^n$ such the quantity

$$\|f\|_{L_{p,q}} = \left\| t^{\frac{1}{p} - \frac{1}{q}} f^*(t) \right\|_{L_q(0,\infty)} \tag{1}$$

is finite.

Note that $L_{p,\infty}(\mathbb{R}^n) = WL_p(\mathbb{R}^n)$ (see, for example, [26]).

If $p = q = \infty$, then the space $L_{\infty,\infty}$ is denoted by L_∞.

If $1 \leq q \leq p$ or $p = q = \infty$, then the functional $\|f\|_{p,q}$ is a norm.

For $0 < q \leq p \leq r \leq \infty$, we have, with continuous embeddings, that

$$L_{p,q} \subset L_p \subset L_{p,r} \subset WL_p.$$

The function $f^{**}: (0,\infty) \to [0,\infty]$ is defined as

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\,ds.$$

In the case $0 < p, q \leq \infty$, we give a functional $\|\cdot\|_{L_{p,q}}^*$ by

$$\|f\|_{L_{p,q}}^* := \|f\|_{L_{p,q}(0,\infty)}^* = \left\| t^{\frac{1}{p} - \frac{1}{q}} f^{**}(t) \right\|_{L_q(0,\infty)}$$

(with the usual modification if $0 < p \leq \infty$, $q = \infty$), which is a norm on $L_{p,q}(\mathbb{R}^n)$ for $1 < p < \infty$, $1 \leq q \leq \infty$ or $p = q = \infty$.

If $1 < p \leq \infty$, $1 \leq q \leq \infty$, then for $1 \leq r < p$ (see [27]),

$$\|f\|_{L_{p,q}} \leq \|f\|_{L_{p,q}}^* \leq \left(\frac{p}{p-r}\right)^{1/r} \|f\|_{L_{p,q}}. \tag{2}$$

We denote by $\mathfrak{M}(\mathbb{R}^n)$ the set of all extended real-valued measurable functions on $\mathbb{R}^n$ and by $\mathfrak{M}^+(0,\infty)$ the set of all non-negative measurable functions on $(0,\infty)$.

Definition 5 Let $0 < p \leq \infty$ and $\psi \in \mathfrak{M}^+(0,\infty)$. We denote by $\Lambda_{p,\psi}(\mathbb{R}^n)$ the classical Lorentz spaces, the spaces of all measurable functions with a finite quasi-norm

$$\Lambda_{p,\psi}(\mathbb{R}^n) := \left\{ f \in \mathfrak{M}(\mathbb{R}^n) : \|f\|_{\Lambda_{p,\psi}} := \left\| \psi f^* \right\|_{L_p(0,\infty)} \right\}.$$

Therefore, for $\psi(t) = t^{\frac{1}{p}-\frac{1}{q}}$, $0 < p, q \leq \infty$, we get $\Lambda_{p,t^{\frac{1}{p}-\frac{1}{q}}}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$ with equality of 'norms'.

Useful references for Lorentz spaces are, for instance, in [7, 28, 29].

Remark 1 Since $\|f\|_{L_{p,p}} = \|f\|_{L_p}$, it can be easily shown that $L_{p,p;\lambda} \equiv L_{p,\lambda}$.

3 Local Morrey-Lorentz spaces $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$

In this section we define the local Morrey-Lorentz spaces $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$, $0 < p, q \leq \infty$ and $0 \leq \lambda \leq 1$. These spaces generalize Lorentz spaces so that $M_{p,q;0}^{\mathrm{loc}}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$. We show that in the case $\lambda < 0$ or $\lambda > 1$, the space $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$ is trivial, and in the limiting case $\lambda = 1$, the space $M_{p,q;1}^{\mathrm{loc}}(\mathbb{R}^n)$ is the classical Lorentz space $\Lambda_{\infty,t^{\frac{1}{p}-\frac{1}{q}}}(\mathbb{R}^n)$. We also show that for $0 < q \leq p < \infty$ and $0 < \lambda \leq \frac{q}{p}$, the local Morrey-Lorentz spaces $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$ are equal to weak Lebesgue spaces $WL_{\frac{1}{p}-\frac{\lambda}{q}}(\mathbb{R}^n)$.

Definition 6 Let $0 < p, q \leq \infty$ and let $0 \leq \lambda \leq 1$. We denote by $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$ the local Morrey-Lorentz spaces, the spaces of all measurable functions with a finite quasi-norm

$$\|f\|_{M_{p,q;\lambda}^{\mathrm{loc}}} := \sup_{t>0} t^{-\frac{\lambda}{q}} \left\| s^{\frac{1}{p}-\frac{1}{q}} f^*(s) \right\|_{L_q(0,t)}.$$

If $\lambda < 0$ or $\lambda > 1$, then $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. Also, $M_{p,q;0}^{\mathrm{loc}}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$. In the case $q = p$, we denote the space $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$ by $M_{p,\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$.

Lemma 1 *Let* $0 < p, q \leq \infty$. *Then*

$$M_{p,q;1}^{\mathrm{loc}}(\mathbb{R}^n) = \Lambda_{\infty,t^{\frac{1}{p}-\frac{1}{q}}}(\mathbb{R}^n)$$

and

$$\|f\|_{M_{p,q;1}^{\mathrm{loc}}} = \|f\|_{\Lambda_{\infty,t^{\frac{1}{p}-\frac{1}{q}}}} \equiv \left\| \tau^{\frac{1}{p}-\frac{1}{q}} f^*(\tau) \right\|_{L_\infty(0,\infty)}.$$

Proof Let $f \in \Lambda_{\infty, t^{\frac{1}{p}-\frac{1}{q}}}(\mathbb{R}^n)$. Then

$$\left(t^{-1} \int_0^t \tau^{\frac{q}{p}-1} f^*(\tau)^q \, d\tau \right)^{1/q} \leq \|f\|_{\Lambda_{\infty, t^{\frac{1}{p}-\frac{1}{q}}}}.$$

Therefore $f \in M_{p,q;1}^{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{M_{p,q;1}^{\mathrm{loc}}} \leq \|f\|_{\Lambda_{\infty, t^{\frac{1}{p}-\frac{1}{q}}}}.$$

Let $f \in M_{p,q;1}^{\mathrm{loc}}(\mathbb{R}^n)$. By the Lebesgue theorem, we have

$$\lim_{s \to 0} \frac{1}{2s} \int_{t-s}^{t+s} \tau^{\frac{q}{p}-1} f^*(\tau)^q \, d\tau = t^{\frac{q}{p}-1} f^*(t)^q.$$

Then

$$\begin{aligned}
t^{\frac{1}{p}-\frac{1}{q}} f^*(t) &= \left(\lim_{s \to 0} \frac{1}{2s} \int_{t-s}^{t+s} \tau^{\frac{q}{p}-1} f^*(\tau)^q \, d\tau \right)^{1/q} \\
&\leq 2^{-\frac{1}{q}} \sup_{0 < s < t} s^{-\frac{1}{q}} \left\| \tau^{\frac{1}{p}-\frac{1}{q}} f^*(\tau) \right\|_{L_q(t-s, t+s)} \\
&= \left\| \tau^{\frac{1}{p}-\frac{1}{q}} f^*(\tau) \right\|_{L_\infty(0, \infty)} \\
&= \|f\|_{M_{p,q;1}^{\mathrm{loc}}}.
\end{aligned}$$

Therefore $f \in \Lambda_{\infty, t^{\frac{1}{p}-\frac{1}{q}}}(\mathbb{R}^n)$ and

$$\|f\|_{\Lambda_{\infty, t^{\frac{1}{p}-\frac{1}{q}}}} \leq \|f\|_{M_{p,q;1}^{\mathrm{loc}}}. \qquad \square$$

Corollary 1 *Let $0 < p \leq \infty$. Then*

$$M_{p,1}^{\mathrm{loc}}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n).$$

Lemma 2 $\| \cdot \|_{M_{p,q;\lambda}^{\mathrm{loc}}}$ *is a quasi-norm on $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof From the definition $f^* \geq 0$. This implies that $\|f\|_{M_{p,q;\lambda}^{\mathrm{loc}}} \geq 0$. Moreover, $\|f\|_{M_{p,q;\lambda}^{\mathrm{loc}}} = 0$ implies that $\|s^{\frac{1}{p}-\frac{1}{q}} f^*(s)\|_{L_q(0,t)} = 0$ for all $t > 0$. Hence, we get $f^* = 0$ a.e. Thus, $f = 0$ since f is representative of an equivalence class.

Now, let $a \neq 0$ be a real constant, $f \in M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$. Noting

$$(af)^*(t) = |a| f^*(t), \quad t > 0,$$

the homogeneity condition follows

$$\|af\|_{M_{p,q;\lambda}^{\mathrm{loc}}} = |a| \|f\|_{M_{p,q;\lambda}^{\mathrm{loc}}}.$$

Let $f, g \in M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$ and $0 < p, q < \infty$. Since $(f + g)^*(t) \le f^*(\frac{t}{2}) + g^*(\frac{t}{2})$, for any $t > 0$, we have

$$\int_0^t \left((f + g)^*(s) s^{1/p}\right)^q \frac{ds}{s} \le \max\{1, 2^{q-1}\} \left(\int_0^t \left(f^*\left(\frac{s}{2}\right) s^{1/p}\right)^q \frac{ds}{s} + \int_0^t \left(g^*\left(\frac{s}{2}\right) s^{1/p}\right)^q \frac{ds}{s}\right)$$

$$\le 2^{\frac{q}{p}} \max\{1, 2^{q-1}\} \left(\int_0^{\frac{t}{2}} \left(f^*(s) s^{1/p}\right)^q \frac{ds}{s} + \int_0^{\frac{t}{2}} \left(g^*(s) s^{1/p}\right)^q \frac{ds}{s}\right).$$

Therefore we get

$$\|f + g\|_{M_{p,q;\lambda}^{\mathrm{loc}}} \le 2^{\frac{1}{p}} \max\{1, 2^{1-\frac{1}{q}}\} \left(\|f\|_{M_{p,q;\lambda}^{\mathrm{loc}}} + \|g\|_{M_{p,q;\lambda}^{\mathrm{loc}}}\right).$$

Let $f, g \in M_{p,\infty;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$, $0 < p \le \infty$ and $q = \infty$. Then

$$\sup_{t>0} t^{1/p} (f + g)^*(t) \le \sup_{t>0} t^{1/p} \left(f^*\left(\frac{t}{2}\right) + g^*\left(\frac{t}{2}\right)\right)$$

$$= \sup_{u>0} (2u)^{1/p} \left(f^*(u) + g^*(u)\right)$$

$$\le 2^{1/p} \left(\sup_{u>0} u^{1/p} f^*(u) + \sup_{u>0} u^{1/p} g^*(u)\right)$$

$$= 2^{1/p} \left(\|f\|_{L_{p,\infty}} + \|g\|_{L_{p,\infty}}\right).$$

Therefore we get

$$\|f + g\|_{M_{p,\infty;\lambda}^{\mathrm{loc}}} \le 2^{\frac{1}{p}} \left(\|f\|_{M_{p,\infty;\lambda}^{\mathrm{loc}}} + \|g\|_{M_{p,\infty;\lambda}^{\mathrm{loc}}}\right). \qquad \square$$

The following theorem states that $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$ local Morrey-Lorentz spaces are equivalent to $WL_q(\mathbb{R}^n)$ weak Lebesgue spaces in the case $\frac{1}{q} = \frac{1}{p} - \frac{\lambda}{q}$ and $0 < \lambda \le \frac{q}{p}$.

Theorem 1 *Let* $0 < q \le p < \infty$, $\frac{1}{r} = \frac{1}{p} - \frac{\lambda}{q}$ *and* $0 < \lambda \le \frac{q}{p}$. *Then*

$$M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n) \equiv WL_r(\mathbb{R}^n)$$

and

$$\left(\frac{q}{p}\right)^{-\frac{1}{q}} \|f\|_{WL_r} \le \|f\|_{M_{p,q;\lambda}^{\mathrm{loc}}} \le \lambda^{-\frac{1}{q}} \|f\|_{WL_r}, \quad f \in WL_r(\mathbb{R}^n).$$

Proof Suppose $f \in M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$. Then, from the monotonicity of f^* on $(0, \infty)$ for all $t > 0$, we get

$$t^{\frac{1}{r}} f^*(t) = t^{-\frac{\lambda}{q} + \frac{1}{p}} f^*(t)$$

$$= \left(\frac{q}{p}\right)^{\frac{1}{q}} \left(t^{-\lambda} f^*(t)^q \int_0^t s^{\frac{q}{p}-1} \, ds\right)^{\frac{1}{q}}$$

$$\le \left(\frac{q}{p}\right)^{\frac{1}{q}} \left(t^{-\lambda} \int_0^t \left(f^*(s) s^{\frac{1}{p}}\right)^q \frac{ds}{s}\right)^{\frac{1}{q}} \le \left(\frac{q}{p}\right)^{\frac{1}{q}} \|f\|_{M_{p,q;\lambda}^{\mathrm{loc}}}.$$

Thus, $f \in WL_r(\mathbb{R}^n)$ and

$$\|f\|_{WL_r} := \sup_{t>0} t^{\frac{1}{r}} f^*(t) \le \left(\frac{q}{p}\right)^{\frac{1}{q}} \|f\|_{M^{\mathrm{loc}}_{p,q;\lambda}}.$$

Let $f \in WL_r(\mathbb{R}^n)$, where $\frac{1}{r} = \frac{1}{p} - \frac{\lambda}{q}$. Then, for all $t > 0$,

$$t^{\frac{1}{r}} f^*(t) = t^{-\frac{\lambda}{q} + \frac{1}{p}} f^*(t) \le \|f\|_{WL_r}.$$

This implies that

$$f^*(t) \le \|f\|_{WL_r} t^{\frac{\lambda}{q} - \frac{1}{p}}, \quad t > 0.$$

Therefore we get

$$\left(f^*(t) t^{\frac{1}{p}}\right)^q \le \|f\|_{WL_r} t^{\lambda}, \quad t > 0.$$

Consequently,

$$\|f\|_{M^{\mathrm{loc}}_{p,q;\lambda}} = \sup_{t>0} \left(t^{-\lambda} \int_0^t \left(f^*(s) s^{\frac{1}{p}}\right)^q \frac{ds}{s}\right)^{\frac{1}{q}}$$

$$\le \|f\|_{WL_r} \sup_{t>0} \left(t^{-\lambda} \int_0^t s^{\lambda - 1} ds\right)^{\frac{1}{q}} = \lambda^{-\frac{1}{q}} \|f\|_{WL_r}. \qquad \square$$

Corollary 2 *Let* $0 < q \le p < \infty$*. Then*

$$M^{\mathrm{loc}}_{p,q;\frac{q}{p}}(\mathbb{R}^n) \equiv L_\infty(\mathbb{R}^n).$$

The following embedding is valid.

Lemma 3 *Let* $0 < p, q < \infty$ *and* $0 \le \lambda \le 1$*. Then*

$$M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n) \hookrightarrow L_{p,q;n\lambda}(\mathbb{R}^n)$$

and for $f \in M^{\mathrm{loc}}_{p,q;\lambda}(\mathbb{R}^n)$*,*

$$\|f\|_{L_{p,q;n\lambda}} \lesssim \|f\|_{M^{\mathrm{loc}}_{p,q;\lambda}}.$$

Proof For $x \in \mathbb{R}^n$ and $t > 0$, from the monotonicity of f^* on $(0, \infty)$, we get

$$\|f \chi_{B(x,t)}\|_{L_{p,q}} = \left(\int_0^\infty s^{\frac{q}{p}-1} \left((f \chi_{B(x,t)})^*(s)\right)^q ds\right)^{1/q}$$

$$\le \left(\int_0^\infty s^{\frac{q}{p}-1} \left((f \chi_{B(x,t)})^{**}(s)\right)^q ds\right)^{1/q}$$

$$= \left(\int_0^\infty s^{\frac{q}{p}-1} \left(\frac{1}{s} \int_0^s (f \chi_{B(x,t)})^*(\tau) d\tau\right)^q ds\right)^{1/q}.$$

From the inequality $(fg)^*(s) \le f^*(s)g^*(s)$ and the equality $(\chi_B)^*(s) = \chi_{[0,|B|]}(s)$, we get

$$(f\chi_{B(x,t)})^*(s) \le f^*(s)\chi_{[0,\omega_n t^n]}(s).$$

For $x \in \mathbb{R}^n$ and $t > 0$, we have

$$\|f\chi_{B(x,t)}\|_{L_{p,q}}$$
$$\le \left(\int_0^\infty \tau^{\frac{q}{p}-1}\left(\frac{1}{\tau}\int_0^\tau f^*(s)\chi_{[0,\omega_n t^n]}(s)\,ds\right)^q d\tau\right)^{1/q}$$
$$\lesssim \left(\int_0^{\omega_n t^n} \tau^{\frac{q}{p}-1}\left(\frac{1}{\tau}\int_0^\tau f^*(s)\,ds\right)^q d\tau\right)^{1/q} + \int_0^{\omega_n t^n} f^*(s)\,ds\left(\int_{\omega_n t^n}^\infty \tau^{\frac{q}{p}-1-q}\,d\tau\right)^{1/q}$$
$$= \left(\int_0^{\omega_n t^n} \tau^{\frac{q}{p}-1}\left(\frac{1}{\tau}\int_0^\tau f^*(s)\,ds\right)^q d\tau\right)^{1/q} + \left(\frac{p'}{q}\right)^{\frac{1}{q}}\omega_n^{-\frac{1}{p'}}t^{-\frac{n}{p'}}\left(\int_0^{\omega_n t^n} f^*(s)\,ds\right)^{1/q}.$$

From the Hölder inequality we obtain

$$\int_0^{\omega_n t^n} f^*(s)\,ds \le \left(\int_0^{\omega_n t^n} f^*(s)^q s^{\frac{q}{p}-1}\,ds\right)^{1/q}\left(\int_0^{\omega_n t^n} s^{(\frac{1}{q}-\frac{1}{p})q'}\,ds\right)^{1/q'}$$
$$= \left(\left(\frac{1}{q}-\frac{1}{p}\right)q'+1\right)^{-\frac{1}{q'}}(\omega_n t^n)^{\frac{1}{p'}}\left(\int_0^{\omega_n t^n} f^*(s)^q s^{\frac{q}{p}-1}\,ds\right)^{1/q}.$$

Then

$$\left(\int_0^\infty s^{\frac{q}{p}-1}\left(\frac{1}{s}\int_0^{\min\{s,\omega_n t^n\}} f^*(\tau)\,d\tau\right)^q ds\right)^{1/q} \lesssim \left(\int_0^{\omega_n t^n} s^{\frac{q}{p}-1}f^*(s)^q\,ds\right)^{1/q}.$$

Therefore

$$\|f\chi_{B(x,t)}\|_{L_{p,q}} \lesssim \left\|s^{\frac{1}{p}-\frac{1}{q}}f^*(s)\right\|_{L_q(0,\omega_n t^n)}.$$

Consequently,

$$\|f\|_{L_{p,q;n\lambda}} = \sup_{x\in\mathbb{R}^n}\sup_{t>0} t^{-\frac{n\lambda}{q}}\|f\chi_{B(x,t)}\|_{L_{p,q}}$$
$$\lesssim \sup_{t>0} t^{-\frac{n\lambda}{q}}\left\|s^{\frac{1}{p}-\frac{1}{q}}f^*(s)\right\|_{L_q(0,\omega_n t^n)}$$
$$\approx \sup_{t>0} t^{-\frac{\lambda}{q}}\left\|s^{\frac{1}{p}-\frac{1}{q}}f^*(s)\right\|_{L_q(0,t)}$$
$$= \|f\|_{M_{p,q;\lambda}^{\mathrm{loc}}}. \qquad \square$$

4 Boundedness of the maximal operator in the local Morrey-Lorentz spaces

In this section, the boundedness of the maximal operator M in local Morrey-Lorentz spaces $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$ is proved.

Theorem 2 *Let $0 < p < \infty$, $1 \le q \le \infty$, $0 < \lambda \le 1$ or $1 < p < \infty$, $1 \le q \le \infty$, $\lambda = 0$. Then the maximal operator M is bounded on the local Morrey-Lorentz spaces $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof By the definition of a norm in local Morrey-Lorentz spaces,

$$\|Mf\|_{M_{p,q;\lambda}^{\mathrm{loc}}} = \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}} (Mf)^*(t) \right\|_{L_q(0,r)}$$

$$\leq C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t) \right\|_{L_q(0,r)}$$

$$= C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}-1} \int_0^t f^*(s)\, ds \right\|_{L_q(0,r)}.$$

Having applied the generalized Minkowski inequality, we get

$$\|Mf\|_{M_{p,q;\lambda}^{\mathrm{loc}}} \leq C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}-1} \int_0^t f^*(s)\, ds \right\|_{L_q(0,r)}$$

$$= C \sup_{r>0} r^{-\frac{\lambda}{q}} \left(\int_0^r \left(\int_0^t f^*(s)\, ds \right)^q t^{\frac{q}{p}-1-q}\, dt \right)^{\frac{1}{q}}$$

$$= C \sup_{r>0} r^{-\frac{\lambda}{q}} \left(\int_0^r \left(\int_0^1 f^*(ts)\, ds \right)^q t^{\frac{q}{p}-1}\, dt \right)^{\frac{1}{q}}$$

$$\leq C \sup_{r>0} r^{-\frac{\lambda}{q}} \int_0^1 \left(\int_0^r f^*(ts)^q t^{\frac{q}{p}-1}\, dt \right)^{\frac{1}{q}}\, ds$$

$$\leq C \int_0^1 \left(\sup_{r>0} r^{-\lambda} \int_0^r f^*(ts)^q t^{\frac{q}{p}-1}\, dt \right)^{\frac{1}{q}}\, ds$$

$$= C \int_0^1 \left(\sup_{r>0} r^{-\lambda} \int_0^{sr} f^*(t)^q t^{\frac{q}{p}-1}\, dt \right)^{\frac{1}{q}} s^{-\frac{1}{p}}\, ds$$

$$= C \int_0^1 \left(\sup_{r>0} r^{-\lambda} \int_0^r f^*(t)^q t^{\frac{q}{p}-1}\, dt \right)^{\frac{1}{q}} s^{-\frac{1}{p}+\frac{\lambda}{q}}\, ds$$

$$= \|f\|_{M_{p,q;\lambda}^{\mathrm{loc}}} C \int_0^1 s^{-\frac{1}{p}+\frac{\lambda}{q}}\, ds$$

$$= C \left(\frac{1}{p'} + \frac{\lambda}{q} \right)^{-1} \|f\|_{M_{p,q;\lambda}^{\mathrm{loc}}}$$

since $-\frac{1}{p} + \frac{\lambda}{q} > -1$.

Therefore, the maximal operator M is bounded on $M_{p,q;\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$. □

Corollary 3 *Let* $1 \leq p < \infty$, $0 < \lambda \leq 1$. *Then the maximal operator* M *is bounded in* $M_{p,\lambda}^{\mathrm{loc}}(\mathbb{R}^n)$.

Corollary 4 *Let* $1 < p < \infty$, $1 \leq q \leq \infty$. *Then the maximal operator* M *is bounded in* $L_{p,q}(\mathbb{R}^n)$.

Corollary 5 [29] *Let* $1 < q < \infty$. *Then the maximal operator* M *is bounded in* $WL_q(\mathbb{R}^n)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

This work was carried out in collaboration between all authors. VSG raised these interesting problems in the research. VSG, CA and AS proved the theorems, interpreted the results and wrote the article. All authors defined the research theme, read and approved the manuscript.

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