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Boundedness of Toeplitz-type operator associated to general integral operator on L^p spaces with variable exponent

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Abstract

In this paper, the boundedness for some Toeplitz-type operator related to some general integral operator on L^p spaces with variable exponent is obtained by using a sharp estimate of the operator. The operator includes Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

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Keywords: Toeplitz-type operator; Littlewood-Paley operator; Marcinkiewicz operator; Bochner-Riesz operator; variable L^p space; BMO

1 Introduction

As the development of singular integral operators (see [1, 2]), their commutators have been well studied. In [3–5], the authors proved that the commutators generated by the singular integral operators and *BMO* functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. In [6–8], some Toeplitz-type operators associated to the singular integral operators and strongly singular integral operators were introduced, and the boundedness for the operators was obtained. In the last years, the theory of L^p spaces with variable exponent was developed because of its connections with some questions in fluid dynamics, calculus of variations, differential equations and elasticity (see [9–13] and their references). Karlovich and Lerner studied the boundedness of the commutators of singular integral operators on L^p spaces with variable exponent (see [12]). Motivated by these papers, in this paper, we have the purpose to introduce some Toeplitz-type operator related to some integral operator and *BMO* functions, and prove the boundedness for the operator on L^p spaces with variable exponent by using a sharp estimate of the operator. The operators include Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

2 Preliminaries and results

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f and $\delta > 0$, the sharp function of f is defined by

$$f_\delta^\#(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q \left| f(y) - f_Q \right|^\delta dy \right)^{1/\delta},$$

where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)\,dx$. It is well known that (see [1, 2])

$$f_\delta^\#(x) \approx \sup_{Q\ni x}\inf_{c\in C}\left(\frac{1}{|Q|}\int_Q \left|f(y)-c\right|^\delta dy\right)^{1/\delta}.$$

We write that $f^\# = f_\delta^\#$ if $\delta = 1$. We say that f belongs to $BMO(R^n)$ if $f^\#$ belongs to $L^\infty(R^n)$ and define $\|f\|_{BMO} = \|f^\#\|_{L^\infty}$. Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q\ni x}|Q|^{-1}\int_Q \left|f(y)\right|dy.$$

For $k \in N$, we denote by M^k the operator M iterated k times, *i.e.*, $M^1(f)(x) = M(f)(x)$ and

$$M^k(f)(x) = M\bigl(M^{k-1}(f)\bigr)(x) \quad \text{when } k \geq 2.$$

Let Φ be a Young function and let $\tilde{\Phi}$ be the complementary associated to Φ. For a function f, we denote the Φ-average by

$$\|f\|_{\Phi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right)dy \leq 1\right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{Q\ni x}\|f\|_{\Phi,Q}.$$

The Young functions to be used in this paper are $\Phi(t) = t(1+\log t)^r$ and $\tilde{\Phi}(t) = \exp(t^{1/r})$, the corresponding average and maximal functions are denoted by $\|\cdot\|_{L(\log L)^r,Q}$, $M_{L(\log L)^r}$ and $\|\cdot\|_{\exp L^{1/r},Q}$, $M_{\exp L^{1/r}}$. Following [4, 5], we know the generalized Hölder inequality

$$\frac{1}{|Q|}\int_Q \left|f(y)g(y)\right|dy \leq \|f\|_{\Phi,Q}\|g\|_{\tilde{\Phi},Q}$$

and the following inequality, for $r, r_j \geq 1$, $j = 1,\ldots,l$ with $1/r = 1/r_1 + \cdots + 1/r_l$, and any $x \in R^n$, $b \in BMO(R^n)$,

$$\|f\|_{L(\log L)^{1/r},Q} \leq M_{L(\log L)^{1/r}}(f) \leq CM_{L(\log L)^l}(f) \leq CM^{l+1}(f),$$
$$\|f - f_Q\|_{\exp L^r,Q} \leq C\|f\|_{BMO},$$
$$|f_{2^{k+1}Q} - f_{2Q}| \leq Ck\|f\|_{BMO}.$$

The non-increasing rearrangement of a measurable function f on R^n is defined by

$$f^*(t) = \inf\bigl\{\lambda > 0 : \bigl|\bigl\{x \in R^n : \left|f(x)\right| > \lambda\bigr\}\bigr| \leq t\bigr\} \quad (0 < t < \infty).$$

For $\lambda \in (0,1)$ and the measurable function f on R^n, the local sharp maximal function of f is defined by

$$M_\lambda^\#(f)(x) = \sup_{Q\ni x}\inf_{c\in C}\bigl((f-c)\chi_Q\bigr)^*\bigl(\lambda|Q|\bigr).$$

Let $p : R^n \to [1, \infty)$ be a measurable function. Denote by $L^{p(\cdot)}(R^n)$ the sets of all Lebesgue measurable functions f on R^n such that $m(\lambda f, p) < \infty$ for some $\lambda = \lambda(f) > 0$, where

$$m(f,p) = \int_{R^n} |f(x)|^{p(x)}\,dx.$$

The sets become Banach spaces with respect to the following norm:

$$\|f\|_{L^{p(\cdot)}} = \inf\{\lambda > 0 : m(f/\lambda, p) \le 1\}.$$

Denote by $M(R^n)$ the sets of all measurable functions $p : R^n \to [1, \infty)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(R^n)$ and the following hold:

$$1 < p_- = \operatorname{ess}\inf_{x\in R^n} p(x), \qquad \operatorname{ess}\sup_{x\in R^n} p(x) = p_+ < \infty. \tag{1}$$

In this paper, we study some integral operators as follows (see [14, 15]).

Definition Let $F_t(x, y)$ be defined on $R^n \times R^n \times [0, +\infty)$, set

$$F_t(f)(x) = \int_{R^n} F_t(x,y) f(y)\,dy$$

for every bounded and compactly supported function f. F_t satisfies: for fixed $\delta > 0$,

$$\|F_t(x-y)\| \le C|x-y|^{-n}$$

and

$$\|F_t(y-x) - F_t(z-x)\| \le C|y-z|^{\delta}|x-z|^{-n-\delta}$$

if $2|y - z| \le |x - z|$. We define that $T(f)(x) = \|F_t(f)(x)\|$.

Let H be the Banach space $H = \{h : \|h\| < \infty\}$. For each fixed $x \in R^n$, we view $F_t(f)(x)$ and $F_t^b(f)(x)$ as the mappings from $[0, +\infty)$ to H. Set

$$T(f)(x) = \|F_t(f)(x)\|.$$

Moreover, let b be a locally integrable function on R^n. The Toeplitz-type operator related to T is defined by

$$T^b(f) = \|F_t^b(f)\|,$$

where

$$F_t^b(f) = \sum_{k=1}^{m} F_t^{k,1} M_b F_t^{k,2}(f),$$

$F_t^{k,1}(f)$ are $F_t(f)$ or $\pm I$ (the identity operator), $T^{k,2}(f) = \|F_t^{k,2}(f)\|$ are the operators for $k = 1, \dots, m$ and $M_b(f) = bf$.

Note that the commutator is a particular operator of the Toeplitz-type operator T^b. The Toeplitz-type operators T^b are the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis, and they have been widely studied by many authors (see [4, 5]). In recent years, the boundedness of classical operators on spaces $L^{p(\cdot)}(R^n)$ have attracted great attention (see [9–13] and their references). The main purpose of this paper is twofold. First, we establish a sharp estimate for the operator T^b; and second, we prove the boundedness for the operator on L^p spaces with variable exponent by using the sharp estimate. In Section 4, we give some applications of the theorems in this paper.

We shall prove the following theorems.

Theorem 1 *Let T be the integral operator as defined in Definition, $0 < \delta < 1$ and $b \in BMO(R^n)$. If $F_t^1(g) = 0$ for any $g \in L^u(R^n)$ $(1 < u < \infty)$, then there exists a constant $C > 0$ such that for any $f \in L_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$\big(T^b(f)\big)_\delta^\#(\tilde{x}) \le C\|b\|_{BMO}\sum_{k=1}^m M^2\big(T^{k,2}(f)\big)(\tilde{x}).$$

Theorem 2 *Let T be the integral operator as defined in Definition, $p(\cdot) \in M(R^n)$ and $b \in BMO(R^n)$. If $F_t^1(g) = 0$ for any $g \in L^u(R^n)$ $(1 < u < \infty)$ and $T^{k,2}$ are the bounded operators on $L^{p(\cdot)}(R^n)$ for $k = 1, \dots, m$, then T^b is bounded on $L^{p(\cdot)}(R^n)$, that is,*

$$\big\|T^b(f)\big\|_{L^{p(\cdot)}} \le C\|b\|_{BMO}\|f\|_{L^{p(\cdot)}}.$$

Corollary *Let $[b,T](f) = bT(f) - T(bf)$ be the commutator generated by the integral operator T and b. Then Theorems 1 and 2 hold for $[b,T]$.*

3 Proofs of theorems

To prove the theorems, we need the following lemmas.

Lemma 1 [5, p.485] *Let $0 < p < q < \infty$. We define that for any function $f \ge 0$ and $1/r = 1/p - 1/q$,*

$$\|f\|_{WL^q} = \sup_{\lambda>0}\lambda\big|\{x \in R^n : f(x) > \lambda\}\big|^{1/q}, \qquad N_{p,q}(f) = \sup_E \|f\chi_E\|_{L^p}/\|\chi_E\|_{L^r},$$

where the sup *is taken for all measurable sets E with $0 < |E| < \infty$. Then*

$$\|f\|_{WL^q} \le N_{p,q}(f) \le \big(q/(q-p)\big)^{1/p}\|f\|_{WL^q}.$$

Lemma 2 [4] *Let $r_j \ge 1$ for $j = 1, \dots, l$, we denote that $1/r = 1/r_1 + \cdots + 1/r_l$. Then*

$$\frac{1}{|Q|}\int_Q \big|f_1(x)\cdots f_l(x)g(x)\big|\,dx \le \|f\|_{\exp L^{r_1},Q}\cdots\|f\|_{\exp L^{r_l},Q}\|g\|_{L(\log L)^{1/r},Q}.$$

Lemma 3 [14] *Let T be the integral operator as defined in Definition. Then T is bounded from $L^1(R^n)$ to $WL^1(R^n)$.*

Lemma 4 [12] *Let $p : R^n \to [1,\infty)$ be a measurable function satisfying* (1). *Then $L_0^\infty(R^n)$ is dense in $L^{p(\cdot)}(R^n)$.*

Lemma 5 [12] *Let $f \in L^1_{\mathrm{loc}}(R^n)$ and g be a measurable function satisfying*

$$\left|\left\{x \in R^n : \left|g(x)\right| > \alpha\right\}\right| < \infty \quad \text{for all } \alpha > 0.$$

Then

$$\int_{R^n} \left|f(x)g(x)\right| dx \le C_n \int_{R^n} M^{\#}_{\lambda_n}(f)(x)M(g)(x)\,dx.$$

Lemma 6 [12, 16] *Let $\delta > 0$, $0 < \lambda < 1$ and $f \in L^\delta_{\mathrm{loc}}(R^n)$. Then*

$$M^{\#}_\lambda(f)(x) \le (1/\lambda)^{1/\delta} f^{\#}_\delta(x).$$

Lemma 7 [17] *Let $p : R^n \to [1,\infty)$ be a measurable function satisfying* (1). *If $f \in L^{p(\cdot)}(R^n)$ and $g \in L^{p'(\cdot)}(R^n)$ with $p'(x) = p(x)/(p(x) - 1)$, then fg is integrable on R^n and*

$$\int_{R^n} \left|f(x)g(x)\right| dx \le C\|f\|_{L^{p(\cdot)}}\|g\|_{L^{p'(\cdot)}}.$$

Lemma 8 [12] *Let $p : R^n \to [1,\infty)$ be a measurable function satisfying* (1). *Set*

$$\|f\|'_{L^{p(\cdot)}} = \sup\left\{\int_{R^n} \left|f(x)g(x)\right| dx : f \in L^{p(\cdot)}\left(R^n\right), g \in L^{p'(\cdot)}\left(R^n\right)\right\}.$$

Then $\|f\|_{L^{p(\cdot)}} \le \|f\|'_{L^{p(\cdot)}} \le C\|f\|_{L^{p(\cdot)}}$.

Proof of Theorem 1 It suffices to prove, for $f \in C_0^\infty(R^n)$ and some constant C_0, that the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_0\right|^\delta dx\right)^{1/\delta} \le C\|b\|_{BMO}\sum_{k=1}^m M^2\left(T^{k,2}(f)\right)(\tilde{x}).$$

Without loss of generality, we may assume that $T^{k,1}$ are T $(k = 1,\dots,m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We write, by $F_t^1(g) = 0$,

$$F_t^b(f)(x) = F_t^{b-b_{2Q}}(f)(x) = F_t^{(b-b_{2Q})\chi_{2Q}}(f)(x) + F_t^{(b-b_{2Q})\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x).$$

Then

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - \left\|f_2(x_0)\right\|\right|^\delta dx\right)^{1/\delta} \\
&= \left(\frac{1}{|Q|}\int_Q \left|\left\|F_t^b(f)(x)\right\| - \left\|f_2(x_0)\right\|\right|^\delta dx\right)^{1/\delta} \le \left(\frac{1}{|Q|}\int_Q \left\|F_t^b(f)(x) - f_2(x_0)\right\|^\delta dx\right)^{1/\delta} \\
&\le C\left(\frac{1}{|Q|}\int_Q \left\|f_1(x)\right\|^\delta dx\right)^{1/\delta} + C\left(\frac{1}{|Q|}\int_Q \left\|f_2(x) - f_2(x_0)\right\|^\delta dx\right)^{1/\delta} = I_1 + I_2.
\end{aligned}$$

For I_1, by the weak (L^1, L^1) boundedness of T (see Lemma 3) and Kolmogorov's inequality (see Lemma 1), we obtain

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left\| F_t^{k,1} M_{(b-b_{2Q})\chi_{2Q}} F_t^{k,2}(f)(x)\right\|^\delta dx\right)^{1/\delta} \\
&= \left(\frac{1}{|Q|}\int_Q \left| T^{k,1} M_{(b-b_{2Q})\chi_{2Q}} T^{k,2}(f)(x)\right|^\delta dx\right)^{1/\delta} \\
&\le \frac{|Q|^{1/\delta-1}}{|Q|^{1/\delta}} \frac{\| T^{k,1} M_{(b-b_{2Q})\chi_{2Q}} T^{k,2}(f)\chi_Q\|_{L^\delta}}{\|\chi_Q\|_{L^{\delta/(1-\delta)}}} \\
&\le \frac{C}{|Q|}\left\| T^{k,1} M_{(b-b_{2Q})\chi_{2Q}} T^{k,2}(f)\right\|_{WL^1} \\
&\le \frac{C}{|Q|}\int_{R^n} \left| M_{(b-b_{2Q})\chi_{2Q}} T^{k,2}(f)(x)\right| dx \\
&\le C|Q|^{-1}\left\| M_{(b-b_{2Q})\chi_{2Q}} T^{k,2}(f)\right\|_{L^1} \\
&\le C|Q|^{-1}\int_{2Q} \left|b(x)-b_{2Q}\right|\left|T^{k,2}(f)(x)\right| dx \\
&\le C\|b-b_Q\|_{\exp L,2Q}\left\| T^{k,2}(f)\right\|_{L(\log L),2Q} \\
&\le C\|b\|_{BMO} M^2\big(T^{k,2}(f)\big)(\tilde{x}).
\end{aligned}$$

Thus

$$\begin{aligned}
I_1 &\le C\sum_{k=1}^m \left(\frac{1}{|Q|}\int_Q \left\| F_t^{k,1} M_{(b-b_{2Q})\chi_{2Q}} F_t^{k,2}(f)(x)\right\|^\delta dx\right)^{1/\delta} \\
&\le C\|b\|_{BMO}\sum_{k=1}^m M^2\big(T^{k,2}(f)\big)(\tilde{x}).
\end{aligned}$$

For I_2, we get, for $x \in Q$,

$$\begin{aligned}
&\left\| F_t^{k,1} M_{(b-b_{2Q})\chi_{(2Q)^c}} F_t^{k,2}(f)(x) - F_t^{k,1} M_{(b-b_{2Q})\chi_{(2Q)^c}} F_t^{k,2}(f)(x_0)\right\| \\
&\le \int_{(2Q)^c} \left|b(y)-b_{2Q}\right| \left\| F_t(x,y) - F_t(x_0,y)\right\| \left|T^{k,2}(f)(y)\right| dy \\
&\le C\sum_{j=1}^\infty \int_{2^j d\le|y-x_0|<2^{j+1}d} \left|b(y)-b_{2Q}\right| \frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta}} \left|T^{k,2}(f)(y)\right| dy \\
&\le C\sum_{j=1}^\infty 2^{-j\delta}\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q} \left|b(y)-b_{2Q}\right|\left|T^{k,2}(f)(y)\right| dy \\
&\le C\sum_{j=1}^\infty 2^{-j\delta}\|b-b_{2Q}\|_{\exp L,2^{j+1}Q}\left\| T^{k,2}(f)\right\|_{L(\log L),2^{j+1}Q} \\
&\le C\sum_{j=1}^\infty j2^{-j\delta}\|b\|_{BMO} M^2\big(T^{k,2}(f)\big)(\tilde{x}) \\
&\le C\|b\|_{BMO} M^2\big(T^{k,2}(f)\big)(\tilde{x}).
\end{aligned}$$

Thus

$$I_2 \le \frac{C}{|Q|}\int_Q \sum_{k=1}^m \left\| F_t^{k,1} M_{(b-b_{2Q})\chi_{(2Q)^c}} F_t^{k,2}(f)(x) - F_t^{k,1} M_{(b-b_{2Q})\chi_{(2Q)^c}} F_t^{k,2}(f)(x_0)\right\| dx$$
$$\le C\|b\|_{BMO}\sum_{k=1}^m M^2\big(T^{k,2}(f)\big)(\tilde{x}).$$

These complete the proof of Theorem 1. □

Proof of Theorem 2 By Lemmas 4-7, we get, for $f \in L_0^\infty(R^n)$ and $g \in L^{p'(\cdot)}(R^n)$,

$$\begin{aligned}
\int_{R^n} \big|T^b(f)(x)g(x)\big|\,dx &\le C\int_{R^n} M^\#_{\lambda_n}\big(T^b(f)\big)(x)M(g)(x)\,dx \\
&\le C\int_{R^n} \big(T^b(f)\big)^\#_\delta(x)M(g)(x)\,dx \\
&\le C\|b\|_{BMO}\sum_{k=1}^m \int_{R^n} M^2\big(T^{k,2}(f)\big)(x)M(g)(x)\,dx \\
&\le C\|b\|_{BMO}\sum_{k=1}^m \big\|M^2\big(T^{k,2}(f)\big)\big\|_{L^{p(\cdot)}}\big\|M(g)\big\|_{L^{p'(\cdot)}} \\
&\le C\|b\|_{BMO}\sum_{k=1}^m \big\|T^{k,2}(f)\big\|_{L^{p(\cdot)}}\big\|M(g)\big\|_{L^{p'(\cdot)}} \\
&\le C\|b\|_{BMO}\|f\|_{L^{p(\cdot)}}\|g\|_{L^{p'(\cdot)}}.
\end{aligned}$$

Thus, by Lemma 8,

$$\big\|T^b(f)\big\|_{L^{p(\cdot)}} \le \|f\|_{L^{p(\cdot)}}.$$

This completes the proof of Theorem 2. □

4 Applications

In this section we shall apply Theorems 1 and 2 of the paper to some particular operators such as Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

Application 1 Littlewood-Paley operator.

Fixed $\varepsilon > 0$. Let ψ be a fixed function which satisfies:

(1) $\int_{R^n}\psi(x)\,dx = 0$,

(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y)-\psi(x)| \le C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

Let $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$ and $F_t(f)(x) = \int_{R^n} f(y)\psi_t(x-y)\,dy$. The Littlewood-Paley operator is defined by (see [18])

$$g_\psi(f)(x) = \left(\int_0^\infty \big|F_t(f)(x)\big|^2\frac{dt}{t}\right)^{1/2}.$$

Set H to be the space

$$H = \left\{ h : \|h\| = \left(\int_0^\infty |h(t)|^2 \, dt/t \right)^{1/2} < \infty \right\}.$$

Let b be a locally integrable function on R^n. The Toeplitz-type operator related to the Littlewood-Paley operator is defined by

$$g_\psi^b(f)(x) = \left(\int_0^\infty |F_t^b(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where

$$F_t^b = \sum_{k=1}^m F_t^{k,1} M_b F_t^{k,2},$$

$F_t^{k,1}$ are F_t or $\pm I$ (the identity operator), $T^{k,2} = \|F_t^{k,2}\|$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$ and $k = 1, \dots, m$, $M_b(f) = bf$. Then, for each fixed $x \in R^n$, $F_t^b(f)(x)$ may be viewed as the mapping from $[0, +\infty)$ to H, and it is clear that

$$g_\psi^b(f)(x) = \|F_t^b(f)(x)\|, \qquad g_\psi(f)(x) = \|F_t(f)(x)\|.$$

It is easy to see that g_ψ^b satisfies the conditions of Theorems 1 and 2 (see [14, 15, 19]), thus Theorems 1 and 2 hold for g_ψ^b.

Application 2 Marcinkiewicz operator.

Fixed $0 < \gamma \le 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0$. Assume that $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$. Set $F_t(f)(x) = \int_{|x-y| \le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\, dy$. The Marcinkiewicz operator is defined by (see [20])

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}.$$

Set H to be the space

$$H = \left\{ h : \|h\| = \left(\int_0^\infty |h(t)|^2 \, dt/t^3 \right)^{1/2} < \infty \right\}.$$

Let b be a locally integrable function on R^n. The Toeplitz-type operator related to the Marcinkiewicz operator is defined by

$$\mu_\Omega^b(f)(x) = \left(\int_0^\infty |F_t^b(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_t^b = \sum_{k=1}^m F_t^{k,1} M_b F_t^{k,2},$$

$F_t^{k,1}$ are F_t or $\pm I$ (the identity operator), $T^{k,2} = \|F_t^{k,2}\|$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$ and $k = 1, \ldots, m$, $M_b(f) = bf$. Then it is clear that

$$\mu_\Omega^b(f)(x) = \left\|F_t^b(f)(x)\right\|, \qquad \mu_\Omega(f)(x) = \left\|F_t(f)(x)\right\|.$$

It is easy to see that μ_Ω^b satisfies the conditions of Theorems 1 and 2 (see [14, 15, 20]), thus Theorems 1 and 2 hold for μ_Ω^b.

Application 3 Bochner-Riesz operator.

Let $\delta > (n-1)/2$, $F_t^\delta(f)\hat{}(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. The maximal Bochner-Riesz operator is defined by (see [17])

$$B_{\delta,*}(f)(x) = \sup_{t>0}\left|F_t^\delta(f)(x)\right|.$$

Set H to be the space $H = \{h : \|h\| = \sup_{t>0} |h(t)| < \infty\}$. Let b be a locally integrable function on R^n. The Toeplitz-type operator related to the maximal Bochner-Riesz operator is defined by

$$B_{\delta,*}^b(f)(x) = \sup_{t>0}\left|B_{\delta,t}^b(f)(x)\right|,$$

where

$$B_{\delta,t}^b = \sum_{k=1}^m F_t^{k,1} M_b F_t^{k,2},$$

$F_t^{k,1}$ are F_t or $\pm I$ (the identity operator), $T^{k,2} = \|F_t^{k,2}\|$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$ and $k = 1, \ldots, m$, $M_b(f) = bf$. Then

$$B_{\delta,*}^b(f)(x) = \left\|B_{\delta,t}^b(f)(x)\right\|, \qquad B_*^\delta(f)(x) = \left\|B_t^\delta(f)(x)\right\|.$$

It is easy to see that $B_{\delta,*}^b$ satisfies the conditions of Theorems 1 and 2 (see [14, 15]), thus Theorems 1 and 2 hold for $B_{\delta,*}^b$.

Competing interests
The authors declare that they have no competing interests.

Authors' information
All authors read and approved the final manuscript.

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