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# Topological aspects of circular metric spaces and some observations on the KKM property towards quasi-equilibrium problems

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**Abstract**

The main purpose of this paper is to study some topological nature of circular metric spaces and deduce some fixed point theorems for maps satisfying the KKM property. We also investigate the solvability of a variant of a quasi-equilibrium problem as an application.

**MSC:** 54E35; 54H25

**Keywords:** circular metric space; KKM property; fixed point; optimization

## 1 Introduction

The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by Nakano [1, 2]. The attempt to generalize the notion of a modular to avoid its restriction on a linear space or on a space with additional algebraic structure resulted in defining and developing a new modular which works on an arbitrary set. This new modular is called a metric modular and was introduced by Chistyakov in [3]. He also pointed out the connection between a modular metric space, a metric space and a modular space. He even studied one of its applications in superposition operator theory. Recently, the concept of a metric modular has been involved in the development of fixed point theory (see [4–7]).

Amongst the flourishing growth of fixed point theory, it is the KKM maps that attracts the interests of most mathematicians. In [8, 9], the notion of an admissible hull is used to overcome the invalidity of a convex hull, and the class of KKM maps, generalized KKM maps and the KKM property were explored within metrizable spaces.

The purposes of this paper are to introduce and study the circular metric space, which generalizes the modular metric space, together with some elementary properties. Then, under the scope of circular metric space, we present a number of fixed point theorems for maps satisfying the KKM property. Examples of the contents are also provided alongside. In the final part of the paper, we employ the solvability of a variant of a quasi-equilibrium problem as an application of our fixed point results.

## 2 Circular metric spaces

In this section, we adopt the concept of circular metric space, which extends the notion of modular metric space. Some properties of the space are also deduced, emphasizing on those related to the compactness.

**Definition 2.1** Let $X$ be a nonempty set. A function $\delta : \mathbb{R}^+ \times X \times X \to [0,\infty]$ is said to be a *circular metric* if the following conditions are satisfied:

(C1) $\delta_\lambda(x,y) = 0$ for all $\lambda > 0$ if and only if $x = y$;
(C2) $\delta_\lambda(x,y) = \delta_\lambda(y,x)$ for all $(\lambda,x,y) \in \mathbb{R}^+ \times X \times X$;
(C3) For any $\lambda > 0$ and $x,y,z \in X$, we can find $\mu \in (0,\lambda)$ such that

$$\delta_\lambda(x,y) \leq \delta_\mu(x,z) + \delta_{\lambda-\mu}(z,y).$$

In this case, the pair $(X,\delta)$ is called a *circular metric space*.

**Remark 2.2** If the inequality in (C3) holds for all $\mu \in (0,\lambda)$, we say that $(X,\delta)$ is a modular metric space (see also [3]).

It is obvious that every modular metric space is in turn a circular metric space. It is natural to ask the converse, which is not true as we shall illustrate in the next example.

**Example 2.3** Let $X = \mathbb{R}$ and define a function $\gamma : \mathbb{R}^+ \times X \times X \to [0,\infty]$ by

$$\gamma_\lambda(x,y) := \begin{cases} 0 & \text{if } x = y, \\ \infty & \text{if } x \neq y \text{ and } 0 < \lambda \leq 1, \\ 1 & \text{if } x \neq y, \lambda > 1, \text{ and } x,y \in \mathbb{Q}, \\ \frac{1}{3} & \text{if } x \neq y, \lambda > 1, \text{ and at least one of } x \text{ and } y \text{ is not in } \mathbb{Q}, \end{cases}$$

for $(\lambda,x,y) \in \mathbb{R}^+ \times X \times X$. It is easy to verify that (C1)-(C3) hold. However, we may notice that

$$\gamma_4(0,1) > \gamma_2(0,\sqrt{2}) + \gamma_2(\sqrt{2},1),$$

which makes $(X,\gamma)$ fail to be a modular metric space.

Now, let us turn to some topological aspects of circular metric spaces.

**Definition 2.4** Let $(X,\delta)$ be a circular metric space. For $x_0 \in X$ and $\gamma > 0$, we define

$$\mathcal{B}_\delta(x_0,\gamma) := \Big\{ x \in X : \sup_{\lambda \in \mathbb{R}^+} \delta_\lambda(x_0,x) < \gamma \Big\}$$

to be an *open ball* centered at $x_0$ of radius $\gamma$. Analogously, we define

$$\overline{\mathcal{B}}_\delta(x_0,\gamma) := \Big\{ x \in X : \sup_{\lambda \in \mathbb{R}^+} \delta_\lambda(x_0,x) \leq \gamma \Big\}$$

to be a *closed ball* centered at $x_0$ of radius $\gamma$.

Let $\mathscr{B}$ be the family of all open balls in a circular metric space $(X,\delta)$. Then $\mathscr{B}$ determines, as a base, a unique topology $\tau$ on $X$. In what follows, a circular metric space will always be equipped with this topology. One may observe that $\tau$ is $T_2$-separable. Note also that a closed ball in $X$ is closed.

Next, we give some characterizations of a compact subset of a circular metric space, where the concept of convergence is embodied.

**Theorem 2.5** *A nonempty subset D of a circular metric space* $(X,\delta)$ *is sequentially compact if and only if every infinite subset of D has an accumulated point.*

*Proof* ($\Rightarrow$) Let $D$ be an infinite subset of $X$ which is sequentially compact, and let $A$ be an infinite subset of $D$. We may construct a sequence $\{a_n\}$ of distinct points in $A$. Since $\{a_n\}$ is also in $D$, we can find a subsequence $\{a_{n_k}\}$ which converges to some $a \in D$. Clearly, $a$ is an accumulated point of $A$.

($\Leftarrow$) Let $D$ be an infinite subset of $X$ such that every infinite subset of $D$ has an accumulated point $a$. Let $\{a_n\}$ be a sequence in $D$. Since $\{a_n\}$ is an infinite subset of $D$, there exists an accumulated point $a$ of $\{a_n\}$. Clearly, there is a sequence $\{b_n\}$ which converges to $a$. In fact, $\{b_n\}$ is a subsequence of $\{a_n\}$. Since $\{a_n\}$ is arbitrarily defined, the conclusion consequently holds. □

**Theorem 2.6** *A nonempty subset D of a circular metric space* $(X,\delta)$ *is compact if and only if D is sequentially compact.*

*Proof* ($\Rightarrow$) Equivalently, we may show that every infinite subset of a compact subset $D$ of $X$ has an accumulated point. So, let $A$ be an infinite subset of $D$. Assume that $A$ has no accumulated point. For each $x \in D$, we choose $\varepsilon_x > 0$ such that

$$\begin{cases} \mathcal{B}_\delta(x,\varepsilon_x) \cap A = \{x\} & \text{if } x \in A, \\ \mathcal{B}_\delta(x,\varepsilon_x) \cap A = \emptyset & \text{if } x \in D \setminus A. \end{cases}$$

Observe that these open balls form an infinite open cover for $D$. Since $D$ is compact, there exists a finite subcover. Since each of these open balls contains at most one point of $A$, $A$ is a finite set. This contradicts our hypothesis. Therefore, every infinite subset of $D$ has an accumulated point. That is, $D$ is sequentially compact.

($\Leftarrow$) Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an infinite open cover of $D$. For each $x \in D$, we define

$$\gamma_x := \sup\bigl\{\gamma > 0 : \mathcal{B}_\delta(x,\gamma) \subset U_\alpha \text{ for some } \alpha \in \Lambda\bigr\}.$$

We claim that $\inf_{x\in D}\gamma_x > 0$. Assume the contrary, so there exists a sequence $\{x_n\}$ in $D$ such that $\lim_{n\to\infty}\gamma_{x_n} = 0$. Since $D$ is sequentially compact, $\{x_n\}$ has a convergent subsequence, namely $\{x_{n_k}\}$, which converges to some point $x \in D$. Observe that $x \in U_\alpha$ for some $\alpha \in \Lambda$, so there exists $\gamma > 0$ such that $\mathcal{B}_\delta(x,\gamma) \subset U_\alpha$. For every $k \in \mathbb{N}$ large enough, we have $x_{n_k} \in \mathcal{B}_\delta(x,\frac{\gamma}{2})$ so that $\gamma_{x_{n_k}} \geq \frac{\gamma}{2} > 0$. This is a contradiction. So, we have proved our claim.

Let $\varepsilon := \inf_{x\in D}\gamma_x > 0$. Choose $z_1 \in D$ arbitrarily. Inductively, choose $z_n \in D$ such that

$$z_{n+1} \notin \bigcup_{k=1}^{n} \mathcal{B}_\delta\left(z_k, \frac{\varepsilon}{2}\right). \tag{2.1}$$

Assume that there are infinitely many points satisfying (2.1). This will result in the lack of existence of a convergent subsequence of $\{z_n\}$, which contradicts the sequential compactivity. So, the above process stops at some $N \in \mathbb{N}$. Clearly, $\bigcup_{k=1}^{N}\mathcal{B}_\delta(z_k,\frac{\varepsilon}{2}) \supset D$. Note that for each $\mathcal{B}_\delta(z_n, \frac{\varepsilon}{2})$, there exists $\alpha_n \in \Lambda$ such that $\mathcal{B}_\delta(z_n, \frac{\varepsilon}{2}) \subset U_{\alpha_n}$. Therefore, $\bigcup_{k=1}^N U_{\alpha_k} \supset D$. That is, the family $\{U_{\alpha_n} : n \in \mathbb{N} \text{ with } n \leq N\}$ is a finite subcover of $\{U_\alpha\}_{\alpha\in\Lambda}$. In other words, $D$ is compact. □

**Definition 2.7** A nonempty subset $D$ of a circular metric space $(X, \delta)$ is said to be *bounded* if

$$\sup_{x,y\in D} \sup_{\lambda>0} \delta_\lambda(x, y) < \infty.$$

Unlike in a metric space, a finite subset in a circular metric space needs not be bounded. We will provide an easy example of such a situation in the following.

**Example 2.8** Let $X = \mathbb{R}$ and $\delta_\lambda(x, y) := \frac{|x-y|}{\lambda}$ be a circular metric on $X$. Then any nonempty nonsingleton finite subset of $X$ is not bounded.

**Remark 2.9** Example 2.8 further implies that in a circular metric space, a compact set needs not be bounded.

**Definition 2.10** Let $D$ be a nonempty subset of a circular metric space $(X, \delta)$, and let $\varepsilon > 0$ be given. A nonempty subset $N_\varepsilon$ of $X$ is said to be an *$\varepsilon$-net for $D$* if for any $y \in D$ there exists $x \in N_\varepsilon$ such that $y \in \mathcal{B}_\delta(x, \varepsilon)$. If for any $\varepsilon > 0$ there is a bounded finite $\varepsilon$-net $N_\varepsilon$ for $D$, then $D$ is said to be *totally bounded*.

**Proposition 2.11** *In any circular metric space, a totally bounded set is bounded.*

*Proof* Let $(X, \delta)$ be a circular metric space and $D$ be a totally bounded subset of $X$. The conclusion is obvious if $D = \emptyset$. For the case $D \neq \emptyset$, let $\varepsilon > 0$ and let $N_\varepsilon$ be a bounded finite $\varepsilon$-net for $D$. So, for any $x, y \in D$, there exists $z_x, z_y \in N_\varepsilon$ such that $x \in \mathcal{B}_\delta(z_x, \varepsilon)$ and $y \in \mathcal{B}_\delta(z_y, \varepsilon)$. That is, $\sup_{\lambda>0} \delta_\lambda(z_x, x) < \varepsilon$ and $\sup_{\lambda>0} \delta_\lambda(z_y, y) < \varepsilon$. Now, observe that

$$\begin{aligned} \sup_{\lambda>0} \delta_\lambda(x, y) &\leq \sup_{\lambda>0} \bigl[\delta_{\lambda-\mu(\lambda)}(x, z_x) + \delta_{\mu(\lambda)-\mu(\mu(\lambda))}(z_x, z_y) + \delta_{\mu(\mu(\lambda))x}(z_y, y)\bigr] \\ &\leq \sup_{\lambda>0} \delta_\lambda(x, z_x) + \sup_{\lambda>0} \delta_\lambda(z_x, z_y) + \sup_{\lambda>0} \delta_\lambda(z_y, y) \\ &< \sup_{\lambda>0} \delta_\lambda(z_x, z_y) + 2\varepsilon. \end{aligned}$$

This further implies that

$$\sup_{x,y\in D} \sup_{\lambda>0} \delta_\lambda(x, y) \leq \sup_{x,y\in D} \sup_{\lambda>0} \delta_\lambda(z_x, z_y) + 2\varepsilon \leq \sup_{u,v\in N_\varepsilon} \sup_{\lambda>0} \delta_\lambda(u, v) + 2\varepsilon < \infty.$$

Therefore, we can conclude that $D$ is bounded. □

Note that the converse of this proposition is not generally true. In fact, the uniform boundedness is the necessary but not sufficient condition for the uniform total boundedness.

**Theorem 2.12** *Let $D$ be a nonempty subset of a circular metric space $(X, \delta)$. If $D$ is compact, then for each $\varepsilon > 0$ we can find an $\varepsilon$-net in $X$ for $D$.*

*Proof* Let $\varepsilon > 0$ be arbitrary. Observe that the family $U := \{\mathcal{B}_\delta(x,\varepsilon) : x \in D\}$ is an open cover for $D$. Since $D$ is compact, we can find a subcover $V := \{\mathcal{B}_\delta(x_n,\varepsilon)\} \subset U$. Therefore, $V$ is a finite $\varepsilon$-net for $D$. □

**Corollary 2.13** *Let $D$ be a nonempty subset of a circular metric space $(X,\delta)$. If $D$ is relatively compact, i.e., the closure $\overline{D}$ is compact, then for each $\varepsilon > 0$ we can find a finite $\varepsilon$-net in $X$ for $D$.*

*Proof* According to the proof of the previous theorem and since $D \subset \overline{D} \subset V$, we have our conclusion. □

The above two assertions lead us to the following consequences.

**Corollary 2.14** *A bounded compact subset in a circular metric space is totally bounded.*

**Corollary 2.15** *A bounded relatively compact subset in a circular metric space is totally bounded.*

## 3 The KKM property

We shall discuss now some fixed point theorems for maps that obey the KKM property. To begin with, we shall give some brief recollection of notions in multivalued analysis which will be used sooner or later in this paper.

**Definition 3.1** Suppose that $D$ is a bounded subset of a circular metric space $(X,\delta)$. We define

(i) $\operatorname{ad}(D) := \bigcap\{\mathcal{B} \subset X : \mathcal{B}$ is a closed ball in $X$ such that $\mathcal{B} \supset D\}$;

(ii) $\mathscr{A}(X) := \{D \subset X : D = \operatorname{ad}(D)\}$. If $D \in \mathscr{A}(X)$, we say that $D$ is *admissible*;

(iii) $D$ is said to be *subadmissible* if $A \in \langle D\rangle \Rightarrow \operatorname{ad}(A) \subset D$, where $\langle D\rangle$ denotes the family of all nonempty finite subsets of $D$.

**Definition 3.2** Let $X$ and $Y$ be two spaces which are $T_2$-separable, and let $F : X \multimap Y$ be a multivalued map with nonempty values. $F$ is said to be:

(i) *upper semicontinuous* if for each nonempty closed set $B \subset Y$, $F^-(B) := \{x \in X : F(x) \cap B \neq \emptyset\}$ is closed in $X$;

(ii) *lower semicontinuous* if for each nonempty open set $B \subset Y$, $F^-(B) := \{x \in X : F(x) \cap B \neq \emptyset\}$ is open in $X$;

(iii) *continuous* if it is both upper and lower semicontinuous;

(iv) *closed* if its graph $\Gamma_F := \{(x,y) \in X \times Y : y \in F(x)\}$ is closed;

(v) *firmly compact* if for each nonempty bounded set $A \subset X$, $F(A) := \bigcup_{x\in A} F(x)$ is bounded and relatively compact.

**Remark 3.3** It is well known that if the multivalued map $F : X \multimap Y$ is upper semicontinuous, then $F$ is closed. The converse holds when the space $Y$ is compact.

Now, let us introduce the class of maps satisfying the KKM property.

**Definition 3.4** Let $M$ be a circular metric space and $X$ be a subadmissible subset of $M$. A multivalued map $G : X \multimap M$ is said to be a *KKM map* if for each $A \in \langle X\rangle$ we have $\operatorname{ad}(A) \subset G(A)$.

**Definition 3.5** Let $M$ be a circular metric space, $X$ be a subadmissible subset of $M$ and $Y$ be a topological space. Let $F, G : X \multimap Y$ be two multivalued maps. If for each $A \in \langle X \rangle$ we have $F(\mathrm{ad}(A)) \subset G(A)$, then $G$ is said to be a *generalized KKM map with respect to F*.

**Definition 3.6** Let $M$ be a circular metric space, $X$ be a subadmissible subset of $M$ and $Y$ be a topological space. A multivalued map $F : X \multimap Y$ is said to satisfy the *KKM property* if for any generalized KKM map $G : X \multimap Y$ with respect to $F$, the family $\{\overline{G(x)} : x \in X\}$ has the finite intersection property.

In general, we write

$$\mathrm{KKM}(X, Y) := \{F : X \multimap Y : F \text{ satisfy the KKM property}\}.$$

**Definition 3.7** Let $(M, \delta)$ be a circular metric space and $X$ be a nonempty subset of $M$. A multivalued map $F : X \multimap M$ is said to have the approximate fixed point property if for any $\varepsilon > 0$, there exists $x_\varepsilon \in X$ such that $F(x_\varepsilon) \cap \overline{\mathcal{B}}_\delta(x_\varepsilon, \varepsilon) \neq \emptyset$. In other words, there exists $y \in F(x_\varepsilon)$ such that $\sup_{\lambda>0} \delta_\lambda(x_\varepsilon, y) < \varepsilon$.

Now, we are ready to give a fixed point theorem for such a class.

**Theorem 3.8** *Let $(M, \delta)$ be a circular metric space, $X$ be a nonempty subadmissible subset of $M$ and $F \in \mathrm{KKM}(X, X)$. If $\overline{F(X)}$ is totally bounded, then $F$ has the approximate fixed point property.*

*Proof* Let $Y := \overline{F(X)}$. By the uniform total boundedness of $Y$, for each $\varepsilon > 0$, there exists $A \in \langle X \rangle$ such that $Y \subset \bigcup_{x \in A} \mathcal{B}_\delta(x, \varepsilon)$. Now, define a multivalued map $G : X \multimap X$ by $G(x) = Y \setminus \mathcal{B}_\delta(x, \varepsilon)$ for all $x \in X$. So, $G(x)$ is closed for each $x \in X$ and $\bigcap_{x \in A} G(x) = \emptyset$. Therefore, $G$ is not a generalized KKM map with respect to $F$. This implies that there exists a finite subset $B = \{x_0, x_1, \dots, x_m\}$ of $X$ such that $F(\mathrm{ad}(B)) \not\subset G(B) = \bigcup_{i=0}^m G(x_i)$. Thus, there exists $x_\varepsilon \in F(\mathrm{ad}(B))$ such that $x_\varepsilon \notin \bigcup_{i=0}^m G(x_i)$. By the definition of $G$, it follows that $x_\varepsilon \in \bigcap_{i=0}^m \mathcal{B}_\delta(x_i, \varepsilon)$. Therefore, $x_i \in \mathcal{B}_\delta(x_\varepsilon, \varepsilon)$ for each $x_i \in B$. Hence, $B \subset \mathcal{B}_\delta(x_\varepsilon, \varepsilon) \subset \overline{\mathcal{B}}_\delta(x_\varepsilon, \varepsilon)$ so that $\mathrm{ad}(B) \subset \overline{\mathcal{B}}_\delta(x_\varepsilon, \varepsilon)$. Suppose that $x_\varepsilon \in F(x')_\varepsilon$ for some $x'_\varepsilon \in \mathrm{ad}(B)$, then we have $x'_\varepsilon \in \overline{\mathcal{B}}_\delta(x_\varepsilon, \varepsilon)$. This further implies that $x_\varepsilon \in \overline{\mathcal{B}}_\delta(x'_\varepsilon, \varepsilon)$. Therefore, $x_\varepsilon \in F(x')_\varepsilon \cap \overline{\mathcal{B}}_\delta(x'_\varepsilon, \varepsilon)$. That is, $F(x')_\varepsilon \cap \overline{\mathcal{B}}_\delta(x'_\varepsilon, \varepsilon) \neq \emptyset$. □

**Theorem 3.9** *Let $(M, \delta)$ be a circular metric space, $X$ be a nonempty subadmissible subset of $M$ and $F \in \mathrm{KKM}(X, X)$. If $F$ is closed and firmly compact, then $F$ has a fixed point.*

*Proof* By the firm compactness of $F$, we can see that $\overline{F(X)}$ is bounded and compact. According to Theorem 3.8, $F$ has the approximate fixed point property. Therefore, for each $n \in \mathbb{N}$, there exists $x_n, x'_n \in X$ such that $x'_n \in F(x_n) \cap \overline{\mathcal{B}}_\delta(x_n, \frac{1}{n})$. Now, since $Y := \overline{F(X)}$ is compact, we may assume that $\{x'_n\}$ converges to some $x_0 \in Y$. So, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sup_{\lambda>0} \delta_\lambda(x_0, x'_n) < \frac{\varepsilon}{2}$ whenever $n \in \mathbb{N}$ and $n \geq N$. Observe that for $n \in \mathbb{N}$ with $n > \max\{N, \frac{2}{\varepsilon}\}$, we have

$$\begin{aligned}\sup_{\lambda>0} \delta_\lambda(x_0, x_n) &\leq \sup_{\lambda>0} \bigl[\delta_{\lambda-\mu(\lambda)}\bigl(x_0, x'_n\bigr) + \delta_{\mu(\lambda)}\bigl(x'_n, x_n\bigr)\bigr] \\ &\leq \sup_{\lambda>0} \delta_\lambda\bigl(x_0, x'_n\bigr) + \sup_{\lambda>0} \delta_\lambda\bigl(x'_n, x_n\bigr)\end{aligned}$$

$$< \frac{\varepsilon}{2} + \frac{1}{n}$$
$$< \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we say that $\{x_n\}$ converges also to $x_0$. Since $x'_n \in F(x_n)$, we have $(x_n, x'_n) \in \Gamma_F$. Since $\{(x_n, x'_n)\}$ is a sequence in $\Gamma_F$ which converges to $(x_0, x_0)$ and $\Gamma_F$ is closed, we may conclude that $(x_0, x_0) \in \Gamma_F$. Therefore, $x_0 \in F(x_0)$. □

Before we stride on further results, we would like to give a simple example to visualize and support Theorem 3.9.

**Example 3.10** Assume that $M := \mathbb{R}$ is equipped with the circular metric $\delta : \mathbb{R}^+ \times M \times M \to [0, \infty]$ defined by

$$\delta_\lambda(x, y) := \frac{|x - y|}{1 + \lambda}$$

for $(\lambda, x, y) \in \mathbb{R}^+ \times M \times M$. Let $X = [0, 1]$ and define a map $F : X \multimap X$ by

$$F(x) := \begin{cases} [\frac{1}{3}, 1] & \text{if } x \in [0, \frac{1}{2}), \\ [0, \frac{2}{3}] & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Clearly, we have $X$ being subadmissible and $F$ being closed and firmly compact. Now, let $G : X \multimap X$ be a given KKM map w.r.t. $F$. It is clear that $G(x) \supset F(x)$ for all $x \in X$. Since $F$ has the finite intersection property, so does $G$. Therefore, we have $F \in \mathrm{KKM}(X, X)$. In view of Theorem 3.9, $F$ has a fixed point. To be precise, every point in $[\frac{1}{3}, \frac{2}{3}]$ is a fixed point of $F$.

Now, we give the following lemma which enables us to obtain the Shauder-type fixed point.

**Lemma 3.11** *Let $(M, \delta)$ be a circular metric space, $Y$ be a topological space and $X$ be a nonempty subadmissible subset of $M$. Suppose that $f : Y \to X$ is continuous. If $F \in \mathrm{KKM}(X, Y)$, then $fF \in \mathrm{KKM}(X, X)$.*

*Proof* Let $G : X \multimap Y$ be a generalized KKM map with respect to $fF$, and let $A = \{x_1, x_2, \dots, x_n\} \in \langle X \rangle$. So, $fF(\mathrm{ad}(A) \cap X) \subset \bigcup_{i=1}^n G(x_i)$. Note also that $F(\mathrm{ad}(A) \cap X) \subset f^{-1}fF(\mathrm{ad}(A) \cap X) \subset f^{-1}G(x_i)$. Since $A \subset \langle X \rangle$ is arbitrary, we have that $f^{-1}G$ is a generalized KKM map with respect to $F$. Now, since $F \in \mathrm{KKM}(X, Y)$, the family $\{\overline{f^{-1}G(x)} : x \in X\}$ has the finite intersection property, so does the family $\{\overline{G(x)} : x \in X\}$. This shows that $fF \in \mathrm{KKM}(X, X)$. □

**Corollary 3.12** *Let $(M, \delta)$ be a circular metric space, $Y$ be a topological space and $X$ be a nonempty subadmissible subset of $M$. Suppose that $f : X \to X$ is continuous and $\overline{f(A)}$ is bounded and compact for all nonempty bounded subset $A$ of $X$. If $I \in \mathrm{KKM}(X, X)$, then $f$ has a fixed point.*

*Proof* According to Lemma 3.11, we have $f \equiv fI \in \mathrm{KKM}(X, X)$. Since $f$ is continuous, $f$ is closed. So, we can apply Theorem 3.9 to obtain the desired result. □

We give some consequences of our results, some of which also appeared in [9], as follows.

**Theorem 3.13** *Let $M_\omega$ be a modular metric space, $X$ be a nonempty subadmissible subset of $M$ and $F \in \mathrm{KKM}(X, X)$. If $\overline{F(X)}$ is totally bounded, then $F$ has the approximate fixed point property.*

**Theorem 3.14** *Let $M_\omega$ be a modular metric space, $X$ be a nonempty subadmissible subset of $M$ and $F \in \mathrm{KKM}(X, X)$. If $F$ is closed and firmly compact, then $F$ has a fixed point.*

**Corollary 3.15** [9] *Let $(M, d)$ be a metric space, $X$ be a nonempty subadmissible subset of $M$ and $F \in \mathrm{KKM}(X, X)$. If $\overline{F(X)}$ is totally bounded, then $F$ has the approximate fixed point property.*

**Corollary 3.16** [9] *Let $(M, d)$ be a metric space, $X$ be a nonempty subadmissible subset of $M$ and $F \in \mathrm{KKM}(X, X)$. If $F$ is closed and compact, then $F$ has a fixed point.*

## 4 Subadmissible quasi-equilibrium problems

As an application of our previous results, we present in this section a form of quasi-equilibrium problem. Assume that $X$ is a nonempty subadmissible subset of a circular metric space $(M, \delta)$, and let $\varphi : X \times X \to \mathbb{R}$ and $H : X \multimap X$ be given. We consider the following problem:

$$\text{Find } \bar{x} \in X \text{ such that } \bar{x} \in H(\bar{x}) \text{ and } \varphi(\bar{x}, \bar{x}) = \min_{y \in H(\bar{x})} \varphi(\bar{x}, y). \qquad (\mathrm{QEP}_1)$$

In fact, this problem $(\mathrm{QEP}_1)$ is set in the context where the linear structure is absent. Hence, the convexity is as well not present. While most of the studies in optimization require the linearity, our next result will eventually overcome the situation when such notion is not valid.

**Theorem 4.1** *Suppose that $\varphi$ is l.s.c. and $H$ is firmly compact with compact values. Assume further that the following hold*:

(a) *the function $\Phi : X \to \mathbb{R}$ defined by*

$$\Phi(x) := \min_{y \in H(x)} \varphi(x, y)$$

*is u.s.c.*,

(b) *the map $F : X \multimap X$ defined by*

$$F(x) := \{ y \in H(x) : \varphi(x, y) = \Phi(x) \}$$

*is in the class* $\mathrm{KKM}(X, X)$.

*Then problem* $(\mathrm{QEP}_1)$ *has a solution*.

*Proof* Since $\varphi$ is l.s.c. and $H$ has compact values, the function $\Phi$ is defined and $F(x)$ is nonempty for all $x \in X$. We shall claim first that $F$ is closed. Consider the sequence $(x_n, y_n)$

in $\Gamma_F$ converging to a pair $(x,y) \in X \times X$. We have

$$\begin{aligned}\varphi(x,y) &\leq \liminf_{n\to\infty} \varphi(x_n,y_n) \\ &\leq \limsup_{n\to\infty} \varphi(x_n,y_n) \\ &= \limsup_{n\to\infty} \Phi(x_n) \\ &\leq \Phi(x).\end{aligned}$$

Thus, it must be the case that $\varphi(x,y) = \Phi(x)$. Since $H$ is closed and $y_n \in H(x_n)$ for all $n \in \mathbb{N}$, we conclude that $y \in H(x)$. Consequently, we have $(x,y) \in \Gamma_F$, and so $F$ is closed. Furthermore, since $H$ is firmly compact, $F$ is also firmly compact (by definition). Finally, Theorem 3.9 yields the existence of a fixed point $\bar{x} \in X$ of $F$. In turn, the point $\bar{x}$ is actually a solution of problem ($\mathrm{QEP}_1$). □

Next, we associate the above theorem with the following standard quasi-equilibrium problem:

$$\textit{Find } \bar{x} \in X \textit{ such that } \bar{x} \in H(\bar{x}) \textit{ and } \varphi(\bar{x},y) \geq 0 \textit{ for all } y \in H(\bar{x}). \qquad (\mathrm{QEP}_2)$$

The following solvability of ($\mathrm{QEP}_2$) resulted directly.

**Corollary 4.2** *According to Theorem* 4.1, *additionally assume that* $\varphi(x,x) \geq 0$ *for all* $x \in X$. *Then problem* ($\mathrm{QEP}_2$) *has a solution*.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
Both authors have carefully prepared, wrote, and checked this manuscript including its final appearance.

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