Journal of Inequalities and Applications
a SpringerOpen Journal

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Norm of an integral operator on some analytic function spaces on the unit disk

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Abstract

If f is an analytic function in the unit disc $\mathbb{D}$, a class of integral operators is defined as follows:

$$I_f(h)(z) = \int_0^z f(w)h'(w)\,dw, \qquad h \in H(\mathbb{D}), z \in \mathbb{D}.$$

The norm of I_f on some analytic function spaces is computed in this paper.

MSC: Primary 47B38; secondary 32A35

Keywords: norm; integral operator; analytic function space

1 Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk of a complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$ the class of functions analytic in $\mathbb{D}$. Let $d\sigma$ denote the normalized Lebesgue area measure in $\mathbb{D}$ and $g(a,z)$ the Green function with logarithmic singularity at a, *i.e.*, $g(a,z) = -\log|\varphi_a(z)|$, where $\varphi_a(z) = (a-z)/(1-\bar{a}z)$ is the Möbius transformation of $\mathbb{D}$.

For $0 < p < \infty$, the Q_p is the space of all functions $f \in H(\mathbb{D})$, for which

$$\|f\|_{Q_p}^2 = \left|f(0)\right|^2 + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^2 \left(1 - \left|\varphi_a(z)\right|^2\right)^p d\sigma(z) < \infty. \tag{1.1}$$

We know that $Q_1 = \mathrm{BMOA}$, the space of all analytic functions of bounded mean oscillation [1, 2]. For all $p > 1$, the space Q_p is the same and equal to the Bloch space $\mathfrak{B}$, consisting of analytic functions f in $\mathbb{D}$ such that

$$\|f\|_{\mathfrak{B}} = \left|f(0)\right| + \sup_{z\in\mathbb{D}} \left|f'(z)\right| \left(1 - |z|^2\right) < \infty. \tag{1.2}$$

See [3, 4] for the theory of Bloch functions.

For $\alpha > 0$, the α-Bloch space, denoted by $\mathfrak{B}^\alpha$, is the space of all functions f in $\mathbb{D}$, for which

$$\|f\|_{\mathfrak{B}^\alpha} = \left|f(0)\right| + \sup_{z\in\mathbb{D}} \left|f'(z)\right| \left(1 - |z|^2\right)^\alpha < \infty. \tag{1.3}$$

Obviously, $\mathfrak{B}^{\alpha_1} \subsetneqq \mathfrak{B} \subsetneqq \mathfrak{B}^{\alpha_2}$ for $0 < \alpha_1 < 1 < \alpha_2 < \infty$.

For any $f \in H(\mathbb{D})$, the next two integral operators on $H(\mathbb{D})$ are induced as follows:

$$I_f(h)(z) = \int_0^z h'(w)f(w)\,dw \quad \text{and} \quad J_f(h)(z) = \int_0^z h(w)f'(w)\,dw \quad (z \in \mathbb{D}).$$

Let M_f denote the multiplication operator, that is, $M_f(h) = fh$.

Let $f \in H(\mathbb{D})$. Then

$$(I_f + J_f)h = fh - f(0)h(0) = M_f(h) - f(0)h(0).$$

If f is a constant, then all results about I_f, J_f or M_f are trivial. In general, f is assumed to be non-constant. Both integral operators have been studied by many authors. See [5–21] and the references therein.

Norm of composition operator, weighted composition operator and some integral operators have been studied extensively by many authors, see [22–34] for example. Recently, Liu and Xiong discussed the norm of integral operators I_f and J_f on the Bloch space, Dirichlet space, BMOA space and so on in [35].

In this paper, we study the norm of integral operator I_f. The norm of I_f on several analytic function spaces is computed.

2 Main results

In this section, we state and prove our main results. In order to formulate our main results, we need an auxiliary result which is incorporated in the following lemma.

Lemma 2.1 *Let $0 < p < 1$. For any $z_0 \in \mathbb{D}$, the function*

$$g_{z_0}(z) = \frac{z_0 - z}{1 - \bar{z}_0 z} - z_0 \tag{2.1}$$

is analytic in $\mathbb{D}$ and $\|g_{z_0}\|_{Q_p} = 1/(p+1)^{1/2}$.

Proof By (1.1) and [1, Proposition 1, p.109], we have

$$\begin{aligned} \|g_{z_0}\|_{Q_p}^2 &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|g_{z_0}'(z)\right|^2 \left(1 - \left|\varphi_a(z)\right|^2\right)^p d\sigma(z) \\ &= \sup_{b\in\mathbb{D}} \int_{\mathbb{D}} \left(1 - \left|\varphi_b(z)\right|^2\right)^p d\sigma(z), \end{aligned}$$

where $b = \varphi_{z_0}(a)$. Taking $w = \varphi_b(z)$, we have

$$\|g_{z_0}\|_{Q_p}^2 = \sup_{b\in\mathbb{D}} \left(1 - |b|^2\right)^2 \int_{\mathbb{D}} \frac{(1 - |w|^2)^p}{|1 - \bar{b}w|^4}\, d\sigma(w).$$

Since

$$\frac{1}{(1 - \bar{b}w)^2} = \sum_{n=0}^{\infty} \frac{\Gamma(n+2)}{n!\Gamma(2)} \bar{b}^n w^n = \sum_{n=0}^{\infty} \frac{\Gamma(n+2)}{n!} \bar{b}^n w^n,$$

we have

$$\begin{aligned}\int_{\mathbb{D}} \frac{(1-|w|^2)^p}{|1-\bar{b}w|^4}\,d\sigma(w) &= \sum_{n=0}^{+\infty} \frac{\Gamma(n+2)^2}{(n!)^2}|b|^{2n}\int_{\mathbb{D}}\bigl(1-|w|^2\bigr)^p|w|^{2n}\,d\sigma(w)\\ &= \sum_{n=0}^{+\infty} \frac{\Gamma(n+2)^2}{(n!)^2}|b|^{2n}\int_0^1 (1-r)^p r^n\,dr\\ &= \sum_{n=0}^{+\infty} \frac{\Gamma(n+2)^2}{(n!)^2}\frac{\Gamma(p+1)\Gamma(n+1)}{\Gamma(n+p+2)}|b|^{2n}\\ &= \sum_{n=0}^{+\infty} \frac{\Gamma(p+1)\Gamma(n+2)^2}{n!\Gamma(n+p+2)}|b|^{2n}.\end{aligned}$$

A simple computation shows

$$\frac{\Gamma(p+1)\Gamma(n+2)^2}{n!\Gamma(n+p+2)} = \frac{(n+1)!(n+1)}{(p+1)(p+2)\cdots(p+n+1)}.$$

Also, it is easy to see

$$\frac{1}{p+1} \le \frac{n+1}{p+n+1} \le \frac{(n+1)!(n+1)}{(p+1)(p+2)\cdots(p+n+1)} \le \frac{n+1}{p+1}.$$

Thus,

$$\|g_{z_0}\|_{Q_p}^2 \le \sup_{b\in\mathbb{D}} \frac{(1-|b|^2)^2}{p+1}\sum_{n=0}^{+\infty}(n+1)|b|^{2n} = \sup_{b\in\mathbb{D}} \frac{(1-|b|^2)^2}{p+1}\frac{1}{(1-|b|^2)^2} = \frac{1}{p+1},$$

and

$$\|g_{z_0}\|_{Q_p}^2 \ge \sup_{b\in\mathbb{D}} \frac{(1-|b|^2)^2}{p+1}\sum_{n=0}^{+\infty}|b|^{2n} = \sup_{b\in\mathbb{D}} \frac{(1-|b|^2)^2}{p+1}\frac{1}{1-|b|^2} = \frac{1}{p+1}.$$

Then the proof is complete. □

First, we consider the norm of I_f on Q_p, $0 < p < 1$.

Theorem 2.2 *Let* $0 < p < 1$. *If* $f \in H(\mathbb{D})$, *then* I_f *is bounded on* Q_p *if and only if* $f \in H^\infty$. *Moreover,*

$$\|I_f\| = \|f\|_{H^\infty}.$$

Proof For any $h \in Q_p$ with $\|h\|_{Q_p} = 1$, it is trivial that $\|I_f\| \le \|f\|_{H^\infty}$. To prove the converse, define $c = \sup_{z\in\mathbb{D}}|f(z)|$. Given any $\epsilon > 0$, there exists $z_1 \in \mathbb{D}$ such that $|f(z_1)| > c - \epsilon$. Let $h(z) = g_{z_1}(z)/\|g_{z_1}\|_{Q_p}$, where

$$g_{z_1}(z) = \frac{z_1 - z}{1-\bar{z}_1 z} - z_1.$$

It is easy to see that

$$\|h\|_{Q_p} = 1, \qquad \bigl|h'(z_1)\bigr|\bigl(1-|z_1|^2\bigr) = 1/\|g_{z_1}\|_{Q_p}.$$

Henceforth,

$$\begin{aligned}\|I_f\|^2 \ge \|I_f h\|_{Q_p}^2 &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \bigl|h'(z)f(z)\bigr|^2\bigl(1-\bigl|\varphi_a(z)\bigr|^2\bigr)^p\,d\sigma(z)\\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \bigl|h'\bigl(\varphi_a(w)\bigr)f\bigl(\varphi_a(w)\bigr)\varphi_a'(w)\bigr|^2\bigl(1-|w|^2\bigr)^p\,d\sigma(w).\end{aligned}$$

Taking $w = re^{i\theta}$ and by the subharmonicity of $|h'(\varphi_a(w))f(\varphi_a(w))\varphi_a'(w)|^2$, we obtain

$$\begin{aligned}\|I_f\|^2 &\ge \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \bigl|h'(z)f(z)\bigr|^2\bigl(1-\bigl|\varphi_a(z)\bigr|^2\bigr)^p\,d\sigma(z)\\ &= \sup_{a\in\mathbb{D}} \int_0^1 \frac{1}{\pi}\int_0^{2\pi} \bigl|h'\bigl(\varphi_a\bigl(re^{i\theta}\bigr)\bigr)f\bigl(\varphi_a\bigl(re^{i\theta}\bigr)\bigr)\varphi_a'\bigl(re^{i\theta}\bigr)\bigr|^2\bigl(1-r^2\bigr)^p r\,dr\,d\theta\\ &\ge \sup_{a\in\mathbb{D}}\bigl|h'(a)f(a)\bigr|^2\bigl(1-|a|^2\bigr)^2 2\int_0^1\bigl(1-r^2\bigr)^p r\,dr\\ &= \frac{1}{p+1}\sup_{a\in\mathbb{D}}\bigl|h'(a)f(a)\bigr|^2\bigl(1-|a|^2\bigr)^2 \ge \frac{1}{p+1}\bigl|h'(z_1)f(z_1)\bigr|^2\bigl(1-|z_1|^2\bigr)^2\\ &\ge \frac{1}{p+1}\frac{|f(z_1)|^2}{\|g_{z_1}\|_{Q_p}^2}.\end{aligned} \tag{2.2}$$

By Lemma 2.1 we have

$$\|I_f\| \ge \bigl|f(z_1)\bigr| > c-\epsilon.$$

Since ϵ is arbitrary, we have $\|I_f\| \ge \sup_{z\in\mathbb{D}}|f(z)|$ and the proof is complete. □

Next, we consider the norm of I_f from Q_p $(0<p<1)$ to $\mathfrak{B}$.

Theorem 2.3 *Let $0<p<1$. If $f\in H(\mathbb{D})$, then I_f is bounded from Q_p space to $\mathfrak{B}$ space if and only if $f\in H^\infty$. Moreover, we have*

$$\|I_f\| = (p+1)^{1/2}\|f\|_{H^\infty}.$$

Proof If $f\in H^\infty$, then (1.2) gives

$$\|I_f h\|_{\mathfrak{B}} = \sup_{z\in\mathbb{D}}\bigl|f(z)h'(z)\bigr|\bigl(1-|z|^2\bigr) \le \|f\|_{H^\infty}\sup_{z\in\mathbb{D}}\bigl|h'(z)\bigr|\bigl(1-|z|^2\bigr).$$

From a part of the proof of estimate (2.2) for $f\equiv 1$, we see that

$$\sup_{z\in\mathbb{D}}\bigl|h'(z)\bigr|\bigl(1-|z|^2\bigr) \le (p+1)^{1/2}\|h\|_{Q_p},$$

and so

$$\|I_f h\|_{\mathfrak{B}} \le \|f\|_{H^\infty}(p+1)^{1/2}\|h\|_{Q_p}.$$

This leads to

$$\|I_f\| \le (p+1)^{1/2}\|f\|_{H^\infty}.$$

On the other hand, define $c = \sup_{z\in\mathbb{D}} |f(z)|$. Given any $\epsilon > 0$, there exists $z_1 \in \mathbb{D}$ such that $|f(z_1)| > c - \epsilon$. Let $h(z) = g_{z_1}(z)/\|g_{z_1}\|_{Q_p}$, where

$$g_{z_1}(z) = \frac{z_1 - z}{1 - \bar{z}_1 z} - z_1.$$

This together with Lemma 2.1 gives the following:

$$\begin{aligned}\|I_f\| \ge \|I_f h\|_{\mathfrak{B}} &= \sup_{z\in\mathbb{D}}\bigl|f(z)h'(z)\bigr|\bigl(1-|z|^2\bigr) \ge \bigl|f(z_1)h'(z_1)\bigr|\bigl(1-|z_1|^2\bigr)\\ &= \bigl|f(z_1)\bigr|/\|g_{z_1}\|_{Q_p} > (p+1)^{1/2}(c-\epsilon).\end{aligned}$$

Since ϵ is arbitrary, we have

$$\|I_f\| \ge (p+1)^{1/2}\sup_{z\in\mathbb{D}}\bigl|f(z)\bigr| = (p+1)^{1/2}\|f\|_{H^\infty}.$$

The proof is complete. □

Finally, we consider the norm of the integral operator I_f on $\mathfrak{B}^\alpha$, $0 < \alpha < 1$.

Theorem 2.4 *Let $0 < \alpha < 1$ and $f \in H(\mathbb{D})$. Then the integral operator I_f is bounded on $\mathfrak{B}^\alpha$ if and only if $f \in H^\infty$. Moreover,*

$$\|I_f\| = \|f\|_{H^\infty}.$$

Proof For any $h \in \mathfrak{B}^\alpha$ with $\|h\|_{\mathfrak{B}^\alpha} = 1$, by (1.3) we have

$$\|I_f h\|_{\mathfrak{B}^\alpha} = \sup_{z\in\mathbb{D}}\bigl(1-|z|^2\bigr)^\alpha\bigl|f(z)\bigr|\bigl|h'(z)\bigr| \le \|h\|_{\mathfrak{B}^\alpha}\cdot\|f\|_{H^\infty}.$$

This implies $\|I_f\| \le \|f\|_{H^\infty}$.

Now we need to show the reverse inequality. Define $c = \sup_{z\in\mathbb{D}} |f(z)|$. Given any $\epsilon > 0$, there exists $z_1 \in \mathbb{D}$ such that $|f(z_1)| > c - \epsilon$. Put

$$h(z) = \int_{\Gamma(z)} \frac{(1-|z_1|^2)^\alpha}{(1-\bar{z}_1\zeta)^{2\alpha}}\,d\zeta, \tag{2.3}$$

where $\Gamma(z)$ is any path in $\mathbb{D}$ from 0 to z, and a single-valued analytic branch is specified. By Theorem 13.11 in [36, p.274], we know h is an analytic function in $\mathbb{D}$ and $h'(z) = (1 -$

$|z_1|^2)^\alpha/(1-\bar{z}_1 z)^{2\alpha}$. Also, it is easy to check $\|h\|_{\mathfrak{B}^\alpha}=1$. In fact,

$$\begin{aligned}\|h\|_{\mathfrak{B}^\alpha} &= \sup_{z\in\mathbb{D}}\big|h'(z)\big|\big(1-|z|^2\big)^\alpha = \sup_{z\in\mathbb{D}}\frac{(1-|z_1|^2)^\alpha}{|1-\bar{z}_1 z|^{2\alpha}}\big(1-|z|^2\big)^\alpha \\ &\le \sup_{z\in\mathbb{D}}\frac{(1-|z_1|^2)^\alpha(1-|z|^2)^\alpha}{(1-|z_1||z|)^{2\alpha}} \le 1. \end{aligned} \tag{2.4}$$

On the other hand, we have

$$\begin{aligned}\|h\|_{\mathfrak{B}^\alpha} &= \sup_{z\in\mathbb{D}}\big|h'(z)\big|\big(1-|z|^2\big)^\alpha = \sup_{z\in\mathbb{D}}\frac{(1-|z_1|^2)^\alpha}{|1-\bar{z}_1 z|^{2\alpha}}\big(1-|z|^2\big)^\alpha \\ &\ge \frac{(1-|z_1|^2)^\alpha}{(1-|z_1|^2)^{2\alpha}}\big(1-|z_1|^2\big)^\alpha = 1. \end{aligned} \tag{2.5}$$

Hence, the assertion follows by (2.4) and (2.5). Thus

$$\begin{aligned}\|I_f\| &\ge \|I_f h\|_{\mathfrak{B}^\alpha} = \sup_{z\in\mathbb{D}}\big|f(z)h'(z)\big|\big(1-|z|^2\big)^\alpha \ge \big|f(z_1)h'(z_1)\big|\big(1-|z_1|^2\big)^\alpha \\ &\ge \big|f(z_1)\big|\frac{(1-|z_1|^2)^\alpha}{(1-|z_1|^2)^{2\alpha}}\big(1-|z_1|^2\big)^\alpha = \big|f(z_1)\big| > c-\epsilon. \end{aligned}$$

Since the ϵ is arbitrary, the proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed to the writing of the present article. They also read and approved the final manuscript.

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Acknowledgements
The first author is supported by the National Natural Science Foundation of China (No. 11126284). The second author is supported by the project of Department of Education of Guangdong Province (No. 2012KJCX0096).

Received: 21 April 2013 Accepted: 10 July 2013 Published: 25 July 2013

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doi:10.1186/1029-242X-2013-342
Cite this article as: Li and Li: Norm of an integral operator on some analytic function spaces on the unit disk. *Journal of Inequalities and Applications* 2013 **2013**:342.