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# The boundedness of multilinear operators on generalized Morrey spaces over the quasi-metric space of non-homogeneous type

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**Abstract**

In this paper we study the multilinear fractional integral operators, the multilinear Calderón-Zygmund operators and the multi-sublinear maximal operators defined on the quasi-metric space with non-doubling measure. We obtain the boundedness of these operators on the generalized Morrey spaces over the quasi-metric space of non-homogeneous type.

**MSC:** 42B20; 42B25; 42B35

**Keywords:** multilinear Calderón-Zygmund operator; multilinear fractional integral; multi-sublinear maximal function; generalized Morrey spaces; non-doubling measure

## 1 Introduction and main results

The boundedness of fractional integral operators on the classical Morrey spaces was studied by Adams [1], Chiarenza and Frasca *et al.* [2]. In [2], by establishing a pointwise estimate of fractional integrals in terms of the Hardy-Littlewood maximal function, they showed the boundedness of fractional integral operators on the Morrey spaces. In 2005, Sawano and Tanaka [3] gave a natural definition of Morrey spaces for Radon measures which might be non-doubling but satisfied the growth condition, and they investigated the boundedness in these spaces of some classical operators in harmonic analysis. Later on, Sawano [4] defined the generalized Morrey spaces on $\mathbb{R}^n$ for non-doubling measure and showed the properties of maximal operators, fractional integral operators and singular operators in this space.

Simultaneously, in 1999, Kenig and Stein [5] gave the boundedness for multilinear fractional integrals on Lebesgue spaces. In 2002, Grafakos and Torres [6] obtained the boundedness for multilinear Calderón-Zygmund operators on Lebesgue spaces. From then on, the theory on multilinear integral operators has attracted much attention as a rapidly developing field in harmonic analysis. Recently, the authors have studied the boundedness of multilinear fractional integrals on Herz-Morrey spaces in [7–10] and the boundedness of multilinear Calderón-Zygmund operators on the Morrey-type spaces in [11–14]. Particularly, the authors [7, 11, 13, 15] established the boundedness for the multilinear operators on Morrey spaces over $\mathbb{R}^n$ with non-doubling measures. In this paper, we focus on the multilinear operators on generalized Morrey spaces over quasi-metric space $(X,\rho,\mu)$ of non-homogeneous type and extend the works in [3–7, 11, 16].

Let $(X,\rho)$ be a quasi-metric space with the quasi-metric function $\rho : X \to [0,\infty)$ satisfying the conditions:

(1) $\rho(x,y) > 0$ for all $x \neq y$, and $\rho(x,x) = 0$ for all $x \in X$.
(2) There exists a constant $a_0 \geq 1$ such that $\rho(x,y) \leq a_0\rho(y,x)$ for all $x,y \in X$.
(3) There exists a constant $a_1 \geq 1$ such that

$$\rho(x,y) \leq a_1\bigl(\rho(x,z) + \rho(z,y)\bigr) \tag{1.1}$$

for all $x,y,z \in X$.

Here we point out that there is no notion of dyadic cubes on the quasi-metric space and so the method for $\mathbb{R}^n$ used in [15] does not work on $(X,\rho)$. Recently, Hytönen [17] introduced the notion of geometrically doubling space.

**Definition 1.1** The quasi-metric space $(X,\rho)$ is called geometrically doubling if there exists some $N_0 \in \mathbb{N}$ such that any ball $B(x,r) \subset X$, where $B(x,r) := \{y \in X : \rho(x,y) < r\}$ with the center $x$ and the radius $r$, can be covered by at most $N_0$ balls $B(x_i, r/2)$ with $x_i \in B(x,r)$.

**Remark 1.2** Similarly as Hytönen showed in Lemma 2.3 in [17], one can deduce that if the quasi-metric space $(X,\rho)$ is geometrically doubling, then, for any $\delta \in (0,1)$, any ball $B(x,r) \subset X$ can be covered by at most $N_0\delta^{-n}$ balls $B(x_i,\delta r)$ with $x_i \in B(x,r)$, where $n = \log_2 N_0$.

Given a Borel measure $\mu$ on the quasi-metric space $(X,\rho)$ such that $\mu$ is finite on bounded sets, and let $(X,\rho)$ be geometrically doubling, then continuous, boundedly supported functions are dense in $L^p(X,\mu)$ for $p \in [1,\infty)$. See Proposition 3.4 in [17] for details.

The above triple $(X,\rho,\mu)$ will be called a quasi-metric space of non-homogeneous type if the measure $\mu$ satisfies the following growth condition,

$$\mu\bigl(B(x,r)\bigr) \leq C_0 r \tag{1.2}$$

with the constant $C_0$ independent of the ball $B(x,r) \subset X$. The set of all balls $B \subseteq X$ satisfying $\mu(B) > 0$ is denoted by $\mathscr{B}(\mu)$. We know that the analysis on non-homogeneous spaces plays important roles in solving the Painlevé problem as well as the Vitushkin conjecture [18, 19]. For motives of developing analysis on non-homogeneous spaces and more examples, one could see [20].

Now we give the definition of the generalized Morrey space over $(X,\rho,\mu)$, which is a generalization of the classical Morrey space. Here we remark that Morrey spaces play important roles in the study of partial differential equations.

**Definition 1.3** Let $1 \leq p < \infty$ and a function $\phi : (0,\infty) \to (0,\infty)$ be such that $r^{\frac{1}{p}}\phi(r)$ is non-decreasing. The generalized Morrey space $L^{p,\phi}(X,k,\mu)$ over $X$, where $k > a_1$, is defined as

$$L^{p,\phi}(X,k,\mu) := \bigl\{f \in L^p_{\mathrm{loc}}(\mu) : \|f\|_{L^{p,\phi}(X,k,\mu)} < \infty\bigr\}$$

with the norm $\|f\|_{L^{p,\phi}(X,k,\mu)}$ given by

$$\|f\|_{L^{p,\phi}(X,k,\mu)} := \sup_{B\in\mathscr{B}(\mu)} \frac{1}{\phi(\mu(kB))}\left(\frac{1}{\mu(kB)}\int_B |f(x)|^p\,d\mu(x)\right)^{\frac{1}{p}},$$

where $kB$ is the ball with the same center and $k$ times radius of the ball $B$.

In case $\phi(r) = r^{-\frac{1}{q}}$, $1 \le p < q < \infty$, the space $L^{p,\phi}(X,k,\mu)$ becomes the classical Morrey space $L^{p,q}(X,k,\mu)$ over $X$. Particularly, $L^{p,p}(X,k,\mu) = L^p(X,\mu)$.

**Remark 1.4** It is worth to point out that if $k_1, k_2 > a_1$, then $L^{p,\phi}(X,k_1,\mu)$ and $L^{p,\phi}(X,k_2,\mu)$ coincide as a set and their norms are mutually equivalent. This can be observed by the same arguments used in [4]. For the sake of convenience, we provide the detail. Let $a_1 < k_1 \le k_2$. Then the inclusion $L^{p,\phi}(X,k_1,\mu) \subseteq L^{p,\phi}(X,k_2,\mu)$ is obvious by that fact that $r^{\frac{1}{p}}\phi(r)$ is non-decreasing. To see the reverse inclusion, let $f \in L^{p,\phi}(X,k_2,\mu)$ and $B(x,r) \in \mathscr{B}(\mu)$ be fixed. It is sufficient to estimate

$$I = \frac{1}{\phi(\mu(B(x,k_1r)))}\left(\frac{1}{\mu(B(x,k_1r))}\int_{B(x,r)} |f(x)|^p\,d\mu(x)\right)^{\frac{1}{p}}.$$

The geometrically doubling condition shows that the ball $B(x,r)$ can be covered by at most $N = N_0\delta^{-n}$ balls $B(x_i,\delta r)$ with $x_i \in B(x,r)$ for any $\delta \in (0,1)$. Moreover, by the quasi-triangle inequality (1.1), we can see that $B(x_i,k_2\delta r) \subseteq B(x,k_1r)$ if we choose $0 < \delta < (k_1 - a_1)/(a_1k_2)$. Thus,

$$\begin{aligned} I^p &\le \sum_{i=1}^{N} \frac{1}{\phi(\mu(B(x,k_1r)))^p\mu(B(x,k_1r))}\int_{B(x_i,\delta r)} |f(x)|^p\,d\mu(x) \\ &\le \sum_{i:B(x_i,\delta r)\in\mathscr{B}(\mu)} \frac{1}{\phi(\mu(B(x_i,k_2\delta r)))^p\mu(B(x_i,k_2\delta r))}\int_{B(x_i,\delta r)} |f(x)|^p\,d\mu(x) \\ &\le N\big(\|f\|_{L^{p,\phi}(X,k_2,\mu)}\big)^p, \end{aligned}$$

which implies that $L^{p,\phi}(X,k_1,\mu) = L^{p,\phi}(X,k_2,\mu)$ for any $k_1, k_2 > a_1$. With this fact in mind, we sometimes omit parameter $k$ in $L^{p,\phi}(X,k,\mu)$, *i.e.*, write it by $L^{p,\phi}(X,\mu)$.

In this article, we consider the multilinear fractional integral operator, the multilinear Calderón-Zygmund operator and the multi-sublinear maximal operator. The multilinear fractional integral is defined by

$$\mathcal{I}_{\alpha,m}(f_1,\dots,f_m)(x) = \int_{X^m} \frac{f_1(y_1)\cdots f_m(y_m)}{(\rho(x,y_1)+\cdots+\rho(x,y_m))^{m-\alpha}}\,d\mu(y_1)\cdots d\mu(y_m),$$

where $0 < \alpha < m$. When $m = 1$, we denote $\mathcal{I}_{\alpha,m}$ by $\mathcal{I}_\alpha$.

Let $\mathcal{T}$ be a multilinear operator initially defined on the $m$-fold product of Schwartz spaces and taking values into the space of tempered distributions. Following [6], we say that $\mathcal{T}$ is an $m$-linear Calderón-Zygmund operator if it extends to a bounded multilinear operator from $L^{p_1}(X,\mu) \times L^{p_2}(X,\mu) \times \cdots \times L^{p_m}(X,\mu)$ to $L^p(X,\mu)$ for some $1 \le p_1,\dots,p_m <$

$\infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$, and if there exists a kernel function $\mathcal{K}$, the so-called multilinear Calderón-Zygmund kernel, defined away from the diagonal $x = y_1 = \cdots = y_m$ in $X^{m+1}$, satisfying

$$\mathcal{T}(f_1,\dots,f_m)(x) = \int_{X^m} \mathcal{K}(x,y_1,\dots,y_m)f_1(y_1)\cdots f_m(y_m)\,d\mu(y_1)\cdots d\mu(y_m)$$

for all $x \notin \bigcap_{i=1}^m \operatorname{supp} f_i$, where $f_i$'s are smooth functions with compact support; and the kernel function $\mathcal{K}$ satisfies the size condition

$$\bigl|\mathcal{K}(x,y_1,y_2,\dots,y_m)\bigr| \le C\left(\sum_{i=1}^m \rho(x,y_i)\right)^{-m} \tag{1.3}$$

and some smoothness conditions; see [6, 16] for details. In fact, as for the $m$-linear Calderón-Zygmund operator $\mathcal{T}$, we assume that, by a similar argument as that in [6, 16] for the case $X = \mathbb{R}^n$, if $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$, then the $m$-linear Calderón-Zygmund operator $\mathcal{T}$ satisfies

$$\mathcal{T} : L^{p_1}(X,\mu) \times L^{p_2}(X,\mu) \times \cdots \times L^{p_m}(X,\mu) \to L^p(X,\mu)$$

for any $1 < p_1, p_2, \dots, p_m < \infty$.

We will also consider the multi-sublinear maximal operator $\mathcal{M}_\kappa$, for $\kappa > a_1^2$, defined by

$$\mathcal{M}_\kappa(f_1,\dots,f_m)(x) = \sup_{x\in B\in\mathscr{B}(\mu)} \prod_{i=1}^m \frac{1}{\mu(\kappa B)} \int_B \bigl|f_i(y_i)\bigr|\,d\mu(y_i).$$

In case $m = 1$, we denote it by $M_\kappa$.

The main result of this paper can be stated as follows.

**Theorem 1.5** *Let $0 < \alpha < m$ and $1 < p_i < \infty$, and let $\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \alpha > 0$. For each $i = 1,\dots,m$, let $\phi_i : (0,\infty) \to (0,\infty)$ satisfy*

$$\frac{1}{C_1} \le \frac{\phi_i(t)}{\phi_i(r)} \le C_1 \quad \text{if } 1 \le \frac{t}{r} \le 2, \tag{1.4}$$

*and*

$$\int_r^\infty t^{\alpha/m-1}\phi_i(t)\,dt \le C_2 r^{\alpha/m}\phi_i(r) \tag{1.5}$$

*with positive constants $C_1$ and $C_2$ independent of $r > 0$. Then there exists a constant $C$ independent of any admissible $f_i$ such that*

$$\bigl\|\mathcal{I}_{\alpha,m}(f_1,\dots,f_m)\bigr\|_{L^{q,\psi}(X,\mu)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i,\phi_i}(X,\mu)},$$

*where $\psi(t) = t^\alpha\phi_1(t)\phi_2(t)\cdots\phi_m(t)$.*

If we take $\phi_i(t) = t^{-\frac{1}{l_i}}$ and $0 < l_i < \frac{m}{\alpha}$, then $\phi_i$ satisfies conditions (1.4) and (1.5). We remark that if condition (1.4) is replaced by

$$\phi_i(u) \le C_3\phi_i(v) \quad \text{for } u \ge v \tag{1.6}$$

with the constant $C_3 > 0$, then the theorem is also valid. This can be seen from the proof of the theorem in the next section. Theorem 1.5 yields the following corollary.

**Corollary 1.6** *Let* $0 < \alpha < m$ *and* $1 < p_i < \infty$, *and let* $\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \alpha > 0$. *Let* $0 < l_i < \frac{m}{\alpha}$ *and* $\frac{1}{h} = \frac{1}{l_1} + \cdots + \frac{1}{l_m} - \alpha$. *Then there exists a constant C independent of* $f_i$ *such that*

$$\left\|\mathcal{I}_{\alpha,m}(f_1,\ldots,f_m)\right\|_{L^{q,h}(X,\mu)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i,l_i}(X,\mu)}.$$

**Theorem 1.7** *Let* $1 < p_i < \infty$ *and* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} < 1$. *If the functions* $\phi_i : (0,\infty) \to (0,\infty)$ *satisfy condition* (1.4) *or* (1.6), *and satisfy*

$$\int_r^\infty \phi_i(t)^{\frac{p_i}{p}} \frac{dt}{t} \le C_4\phi_i(r)^{\frac{p_i}{p}} \tag{1.7}$$

*with the constant* $C_4$ *independent of* $r > 0$, *then there exists a constant C independent of any admissible* $f_i$ *such that*

$$\left\|\mathcal{T}(f_1,\ldots,f_m)\right\|_{L^{p,\phi}(X,\mu)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i,\phi_i}(X,\mu)}, \tag{1.8}$$

*where* $\phi(t) = \phi_1(t)\phi_2(t)\cdots\phi_m(t)$.

Here we point out that if each $f_i \in L^{p_i}(X,\mu) \cap L^{p_i,\phi_i}(X,\mu)$, then the multilinear Calderón-Zygmund operator $\mathcal{T}(f_1,\ldots,f_m)$ is well defined, and we will prove estimate (1.8) with the absolute constant $C$ independent of these admissible functions. More remarks on the admissibility for $f_i \in L^{p_i,\phi_i}(X,\mu)$ will be given in Remark 3.1 in Section 3.

Observe that $\phi_i(t) = t^{-\frac{1}{l_i}}$, for any $0 < l_i < \infty$, satisfies the conditions in the theorem, thus the corollary follows.

**Corollary 1.8** *Let* $1 < p_i < \infty$ *and* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} < 1$. *Let* $0 < l_i < \infty$ *and* $\frac{1}{l} = \frac{1}{l_1} + \cdots + \frac{1}{l_m}$. *Then there exists a constant C independent of* $f_i$ *such that*

$$\left\|\mathcal{T}(f_1,\ldots,f_m)\right\|_{L^{p,l}(X,\mu)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i,l_i}(X,\mu)}.$$

**Theorem 1.9** *Assume that* $\mathcal{M}_\kappa$ *is a multi-sublinear maximal operator. Let* $1 < p_i < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} < 1$, *and* $\phi_i$ *satisfy condition* (1.6). *Then there exists a constant C independent of any admissible* $f_i$ *such that*

$$\left\|\mathcal{M}_\kappa(f_1,\ldots,f_m)\right\|_{L^{p,\phi}(X,\mu)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i,\phi_i}(X,\mu)},$$

*where* $\phi(t) = \phi_1(t)\cdots\phi_m(t)$.

We notice that the results above are new even for the case of Euclidean spaces. Throughout this paper, the letter $C$ always denotes a positive constant that may vary at each occurrence but is independent of the essential variable.

## 2 Proof of Theorem 1.5

Let us first give some requisite theorems and lemmas.

**Theorem 2.1** [21] *Let $0 < \alpha < 1$, $1 < p < \frac{1}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$, then the operator $\mathcal{I}_\alpha$ is bounded from $L^p(X,\mu)$ into $L^q(X,\mu)$ if and only if $\mu(B(x,r)) \le Cr$, where the constant $C$ is independent of $x$ and $r$.*

**Lemma 2.2** [13] *Suppose that $\mu$ is a Borel measure on $X$ with the growth condition* (1.2). *Let $\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \alpha > 0$ with $0 < \alpha < m$ and $1 \le p_j \le \infty$. Then*

(a) *if each $p_j > 1$,*

$$\left\|\mathcal{I}_{\alpha,m}(f_1,\dots,f_m)\right\|_{L^q(X,\mu)} \le C\prod_{j=1}^m \|f_j\|_{L^{p_j}(X,\mu)};$$

(b) *if $p_j = 1$ for some $j$,*

$$\left\|\mathcal{I}_{\alpha,m}(f_1,\dots,f_m)\right\|_{L^{q,\infty}(X,\mu)} \le C\prod_{j=1}^m \|f_j\|_{L^{p_j}(X,\mu)}.$$

*Proof* This lemma can follow the same argument that, for the classical setting, was given by Kenig and Stein [5]. We may assume that all $1 \le p_i < \infty$. One can find $0 < \alpha_i < 1/p_i$ such that $\alpha = \sum_{i=1}^m \alpha_i$. Set $1/q_i = 1/p_i - \alpha_i$, since $1/q = \sum_{i=1}^m 1/q_i$, $0 < \alpha_i \le 1$, $1 < q_i < \infty$, and

$$\rho(x,y_1)^{1-\alpha_1}\rho(x,y_2)^{1-\alpha_2}\cdots\rho(x,y_m)^{1-\alpha_m} \le \left(\rho(x,y_1) + \cdots + \rho(x,y_m)\right)^{m-\alpha}.$$

It follows that

$$\mathcal{I}_{\alpha,m}(f_1,\dots,f_m)(x) \le \prod_{i=1}^m \mathcal{I}_{\alpha_i}(f_i)(x).$$

Then, by the Hölder inequality and Theorem 2.1, we obtain the lemma. In fact, one could also get the lemma from [13] (or Remark 1.3, p.290 in [13]). □

Now we give the proof of Theorem 1.5.

*Proof of Theorem* 1.5 Let $B = B(x_0, r)$ be the ball in $B \in \mathscr{B}(\mu)$, with center $x_0 \in X$ and radius $r > 0$, and let $B^* = B(x_0, 2a_1 r)$. For $f_i \in L^{p_i,\phi_i}(X,\mu)$, we split it as $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i\chi_{B^*}$ for $i = 1,\dots,m$. Using this decomposition, we get

$$\left|\mathcal{I}_{\alpha,m}(f_1,\dots,f_m)(x)\right| \le \left|\mathcal{I}_{\alpha,m}\left(f_1^0,\dots,f_m^0\right)(x)\right| + \sum{}' \left|\mathcal{I}_{\alpha,m}\left(f_1^{\tau_1},\dots,f_m^{\tau_m}\right)(x)\right|,$$

where each term in $\sum'$ contains at least one $\tau_i = \infty$.

Then it suffices to show

$$\frac{1}{\psi(\mu(kB^*))}\left(\frac{1}{\mu(kB^*)}\int_B\left|\mathcal{I}_{\alpha,m}(f_1^{\tau_1},\ldots,f_m^{\tau_m})\right|^q d\mu\right)^{\frac{1}{q}} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i,\phi_i}(X,\mu)} \tag{2.1}$$

for some $k > a_1$ and for each $\tau_i \in \{0,\infty\}$.

Let us first estimate for the case $\tau_1 = \cdots = \tau_m = 0$. From the definition of $L^{p_i,\phi_i}(X,\mu)$ we have

$$\left(\int_{B^*}\left|f_i(x)\right|^{p_i} d\mu(x)\right)^{\frac{1}{p_i}} \le C\|f_i\|_{L^{p_i,\phi_i}(X,\mu)}\phi_i\big(\mu\big(kB^*\big)\big)\mu\big(kB^*\big)^{\frac{1}{p_i}}. \tag{2.2}$$

From this and by the $L^{p_1}(X,\mu)\times\cdots\times L^{p_m}(X,\mu)\to L^q(X,\mu)$ boundedness of $\mathcal{I}_{\alpha,m}$, Lemma 2.2, we have

$$\begin{aligned}
&\left(\frac{1}{\mu(kB^*)}\int_B\left|\mathcal{I}_{\alpha,m}(f_1^0,\ldots,f_m^0)(x)\right|^q d\mu(x)\right)^{\frac{1}{q}}\\
&\quad\le \frac{C}{\mu(kB^*)^{\frac{1}{q}}}\prod_{i=1}^m\left\|f_i^0\right\|_{L^{p_i}(X,\mu)} \le C\mu\big(kB^*\big)^{\alpha}\phi\big(\mu\big(kB^*\big)\big)\prod_{i=1}^m\|f_i\|_{L^{p_i,\phi_i}(X,\mu)}\\
&\quad\le C\psi\big(\mu\big(kB^*\big)\big)\prod_{i=1}^m\|f_i\|_{L^{p_i,\phi_i}(X,\mu)},
\end{aligned}$$

which implies that in the case all $\tau_i = 0$, inequality (2.1) holds.

To estimate (2.1) for the case $\tau_1 = \cdots = \tau_m = \infty$, let $x \in B$, then

$$\left|\mathcal{I}_{\alpha,m}(f_1^\infty,\ldots,f_m^\infty)(x)\right| \le \int_{(X\setminus B^*)^m}\frac{|f_1(y_1)\cdots f_m(y_m)|\,d\mu(y_1)\cdots d\mu(y_m)}{(\rho(x,y_1)+\cdots+\rho(x,y_m))^{m-\alpha}}.$$

Note, for $x \in B$ and $y_i \in X\setminus B^*$, we get by (1.1) that

$$2a_1r < \rho(x_0,y_i) \le a_1\big(\rho(x_0,x)+\rho(x,y_i)\big) \le a_1\big(r+\rho(x,y_i)\big),$$

hence $\rho(x,y_i)\ge r$. This and condition (1.2) of the measure $\mu$ imply that

$$\rho(x,y_i)\ge r \ge (2a_1C_0k)^{-1}\mu\big(kB^*\big) := r^*$$

and so

$$\big(X\setminus B^*\big)^m \subseteq \prod_{i=1}^m\big\{y_i:\rho(x,y_i)\ge r^*\big\} \subseteq \left\{(y_1,y_2,\ldots,y_m):\sum_{i=1}^m\rho(x,y_i)\ge r^*\right\}.$$

Hence we can derive that

$$\begin{aligned}
&\left|\mathcal{I}_{\alpha,m}(f_1^\infty,\ldots,f_m^\infty)(x)\right|\\
&\quad\le \int_{\sum_{i=1}^m\rho(x,y_i)\ge r^*}\frac{|f_1(y_1)\cdots f_m(y_m)|\,d\mu(y_1)\cdots d\mu(y_m)}{(\rho(x,y_1)+\cdots+\rho(x,y_m))^{m-\alpha}}
\end{aligned}$$

$$= \sum_{j=0}^{\infty} \int_{2^j r^* \le \sum_{i=1}^m \rho(x,y_i) < 2^{j+1} r^*} \frac{|f_1(y_1)\cdots f_m(y_m)|\, d\mu(y_1)\cdots d\mu(y_m)}{(\rho(x,y_1)+\cdots+\rho(x,y_m))^{m-\alpha}}$$
$$\le \sum_{j=0}^{\infty} \frac{1}{(2^j r^*)^{m-\alpha}} \prod_{i=1}^{m} \int_{\rho(x,y_i)<2^{j+1}r^*} \left|f_i(y_i)\right| d\mu(y_i).$$

Using the Hölder inequality and the inequality similar to (2.2), we can see that the inequality above can be controlled by

$$C\sum_{j=0}^{\infty} \frac{1}{(2^j r^*)^{m-\alpha}} \prod_{i=1}^{m} \left( \int_{B(x,2^{j+1}r^*)} \left|f_i(y_i)\right|^{p_i} d\mu(y_i) \right)^{\frac{1}{p_i}} \left(\mu\left(B\left(x,2^{j+1}r^*\right)\right)\right)^{1-\frac{1}{p_i}}$$
$$\le C\sum_{j=0}^{\infty} \frac{1}{(2^j r^*)^{m-\alpha}} \prod_{i=1}^{m} \phi_i\left(\mu\left(B\left(x,k2^{j+1}r^*\right)\right)\right)\mu\left(B\left(x,k2^{j+1}r^*\right)\right) \|f_i\|_{L^{p_i,\phi_i}(X,\mu)}$$
$$\le C\sum_{j=0}^{\infty} \frac{1}{(2^j r^*)^{m-\alpha}} \prod_{i=1}^{m} \phi_i\left(2^j\mu\left(kB^*\right)\right) 2^j \mu\left(kB^*\right) \|f_i\|_{L^{p_i,\phi_i}(X,\mu)}$$
$$\le C\prod_{i=1}^{m} \left[ \sum_{j=0}^{\infty} \left(2^j\mu\left(kB^*\right)\right)^{\alpha/m} \phi_i\left(2^j\mu\left(kB^*\right)\right) \right] \|f_i\|_{L^{p_i,\phi_i}(X,\mu)},$$

where, in the second inequality, we have utilized the non-decreasing of function $r^{\frac{1}{p_i}}\phi_i(r)$ and the fact $\mu(B(x,k2^{j+1}r^*)) \le C_0 k2^{j+1}r^* \le 2^j\mu(kB^*)$. Recall conditions (1.4) (or (1.6)) and (1.5) for the function $\phi_i$, one sees that

$$\sum_{j=0}^{\infty} \left(2^j\mu\left(kB^*\right)\right)^{\alpha/m} \phi_i\left(2^j\mu\left(kB^*\right)\right) \le C\sum_{j=0}^{\infty} \int_{2^j\mu(kB^*)}^{2^{j+1}\mu(kB^*)} t^{\alpha/m-1}\phi_i(t)\,dt$$
$$\le C\int_{\mu(B^*)}^{\infty} t^{\alpha/m-1}\phi_i(t)\,dt \le C\mu\left(kB^*\right)^{\alpha/m}\phi_i\left(\mu\left(kB^*\right)\right).$$

Hence we obtain the pointwise estimate

$$\left|\mathcal{I}_{\alpha,m}\left(f_1^{\infty},\ldots,f_m^{\infty}\right)(x)\right| \le C\psi\left(\mu\left(kB^*\right)\right) \prod_{i=1}^{m} \|f_i\|_{L^{p_i,\phi_i}(X,\mu)}$$

for $x \in B$, which follows from inequality (2.1) for the case all $\tau_i = \infty$.

It is left to consider the case $\tau_{i_1} = \cdots = \tau_{i_l} = 0$ for some $\{i_1,\ldots,i_l\} \subset \{1,\ldots,m\}$ and $1 \le l < m$. For this case, we can write for $x \in B$ that

$$\left|\mathcal{I}_{\alpha,m}\left(f_1^{\tau_1},\ldots,f_m^{\tau_m}\right)(x)\right| \le \int_{X^m} \frac{|f_1(y_1)\cdots f_m(y_m)|\, d\mu(y_1)\cdots d\mu(y_m)}{(\rho(x,y_1)+\cdots+\rho(x,y_m))^{m-\alpha}}$$
$$\le \int_{(B^*)^l} \prod_{i\in\{i_1,\ldots,i_l\}} \left|f_i(y_i)\right| d\mu(y_i) \cdot \int_{(X\setminus B^*)^{m-l}} \frac{\prod_{i\notin\{i_1,\ldots,i_l\}} |f_i(y_i)|\, d\mu(y_i)}{(\sum_{i\notin\{i_1,\ldots,i_l\}} \rho(x,y_i))^{m-\alpha}}$$
$$:= A_1(x)\cdot A_2(x).$$

To estimate $A_1(x)$, we use the Hölder inequality to give that, for any $x \in B$,

$$\begin{aligned} A_1(x) &= \prod_{i\in\{i_1,\dots,i_l\}} \int_{B^*} |f_i(y_i)|\,d\mu(y_i) \\ &\le C \prod_{i\in\{i_1,\dots,i_l\}} \left(\int_{B^*} |f_i|^{p_i}\,d\mu(y_i)\right)^{\frac{1}{p_i}} \mu(B^*)^{1-\frac{1}{p_i}} \\ &\le C\mu(kB^*)^l \prod_{i\in\{i_1,\dots,i_l\}} \left(\phi_i(\mu(kB^*))\|f_i\|_{L^{p_i,\phi_i}(X,\mu)}\right). \end{aligned}$$

Estimating $A_2(x)$, by the same idea used for the case all $\tau_i = \infty$ above, we get for any $x \in B$ that

$$\begin{aligned} A_2(x) &\le \sum_{j=0}^{\infty} \frac{1}{(2^j r^*)^{m-\alpha}} \prod_{i\notin\{i_1,\dots,i_l\}} \int_{B(x,2^{j+1}r^*)} |f_i(y_i)|\,d\mu(y_i) \\ &\le C\sum_{j=0}^{\infty} \frac{1}{(2^j r^*)^{m-\alpha}} \prod_{i\notin\{i_1,\dots,i_l\}} \phi_i(2^j\mu(kB^*))2^j\mu(kB^*)\|f_i\|_{L^{p_i,\phi_i}(X,\mu)} \\ &\le C\sum_{j=0}^{\infty} \frac{1}{(2^j\mu(kB^*))^{l-\alpha l/m}} \prod_{i\notin\{i_1,\dots,i_l\}} \phi_i(2^j\mu(kB^*))(2^j\mu(kB^*))^{\alpha/m}\|f_i\|_{L^{p_i,\phi_i}(X,\mu)}. \end{aligned}$$

Noting $l - l\alpha/m > 0$ and using condition (1.5), we have

$$\begin{aligned} A_2(x) &\le C(\mu(kB^*))^{\alpha l/m-l} \prod_{i\notin\{i_1,\dots,i_l\}} \phi_i(\mu(kB^*))\mu(kB^*)^{\alpha/m}\|f_i\|_{L^{p_i,\phi_i}(X,\mu)} \\ &\le C(\mu(kB^*))^{\alpha-l} \prod_{i\notin\{i_1,\dots,i_l\}} \phi_i(\mu(kB^*))\|f_i\|_{L^{p_i,\phi_i}(X,\mu)}. \end{aligned}$$

Therefore, for $x \in B$, we have

$$\begin{aligned} \left|\mathcal{I}_{\alpha,m}(f_1^{\tau_1},\dots,f_m^{\tau_m})(x)\right| &\le A_1(x)\cdot A_2(x) \\ &\le C\mu(kB^*)^{\alpha} \prod_{i=1}^{m} \phi_i(\mu(kB^*))\|f_i\|_{L^{p_i,\phi_i}(X,\mu)} \\ &\le C\psi(\mu(kB^*)) \prod_{i=1}^{m} \|f_i\|_{L^{p_i,\phi_i}(X,\mu)}. \end{aligned}$$

Hence we obtain the desired inequality (2.1) for any cases. The proof of the theorem is complete. □

## 3 Proof of Theorems 1.7 and 1.9

In this section we first investigate the boundedness of the $m$-linear Calderón-Zygmund operator $\mathcal{T}$ on the product of spaces $L^{p_i,\phi_i}(X,\mu)$ for $i = 1, 2, \dots, m$.

*Proof of Theorem* 1.7 We also let $B = B(x_0, r)$ be the ball in $\mathscr{B}(\mu)$, with center $x_0 \in X$ and radius $r > 0$, and let $B^* = B(x_0, 2a_1r)$. For the admissible $f_i \in L^{p_i,\phi_i}(X,\mu)$, without loss of generality, we may initially assume that $f_i$ are all smooth boundedly supported functions, which are dense in $L^{p_i}(X,\mu)$, and let $f_i \in L^{p_i}(X,\mu) \cap L^{p_i,\phi_i}(X,\mu)$, then $\mathcal{T}(f_1,\ldots,f_m)$ is a well-defined function belonging to $L^p(X,\mu)$. If we split each $f_i$ as $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i\chi_{B^*}$ for $i = 1,\ldots,m$, and utilize the multi-linearity of $\mathcal{T}$, we have the following decomposition,

$$\left|\mathcal{T}(f_1,\ldots,f_m)(x)\right| \le \left|\mathcal{T}\left(f_1^0,\ldots,f_m^0\right)(x)\right| + {\sum}' \left|\mathcal{T}\left(f_1^{\tau_1},\ldots,f_m^{\tau_m}\right)(x)\right|,$$

where each term in $\sum'$ contains at least one $\tau_i = \infty$.

Noting that $\mathcal{T}$ is bounded from $L^{p_1}(X,\mu) \times \cdots \times L^{p_m}(X,\mu) \to L^p(X,\mu)$, we have

$$\begin{aligned}
&\left(\frac{1}{\mu(kB^*)}\int_B \left|\mathcal{T}\left(f_1^0,\ldots,f_m^0\right)(x)\right|^p \, d\mu(x)\right)^{\frac{1}{p}} \\
&\quad \le \frac{C}{\mu(kB^*)^{\frac{1}{p}}} \prod_{i=1}^m \|f_i^0\|_{L^{p_i}(X,\mu)} \\
&\quad \le C\phi\left(\mu\left(kB^*\right)\right) \prod_{i=1}^m \|f_i\|_{L^{p_i,\phi_i}(X,\mu)}.
\end{aligned} \tag{3.1}$$

For the case $\tau_1 = \cdots = \tau_m = \infty$, we note that for $x \in B$ and $y_i \in X \setminus B^*$, one can deduce from the properties of the quasi-metric $\rho$ that

$$\frac{1}{2a_1}\rho(x_0,y_i) \le \rho(x,y_i) \le \left(\frac{a_0}{2} + a_1\right)\rho(x_0,y_i).$$

Thus we can observe that

$$\frac{1}{(\sum_{i=1}^m \rho(x,y_i))^m} \simeq \frac{1}{(\sum_{i=1}^m \rho(x_0,y_i))^m} = m\int_{\rho(x_0,y_1)+\cdots+\rho(x_0,y_m)}^\infty \frac{dl}{l^{m+1}}.$$

This, together with the Fubini theorem, we have, for $x \in B$,

$$\begin{aligned}
\left|\mathcal{T}\left(f_1^\infty,\ldots,f_m^\infty\right)(x)\right| &\le \int_{(X\setminus B^*)^m} \frac{\prod_{i=1}^m |f_i(y_i)| \, d\mu(y_i)}{(\sum_{i=1}^m \rho(x,y_i))^m} \\
&\le C\int_{\sum_{i=1}^m \rho(x_0,y_i)\ge 2a_1 r} \left(\int_{\rho(x_0,y_1)+\cdots+\rho(x_0,y_m)}^\infty \frac{dl}{l^{m+1}}\right) \prod_{i=1}^m \left|f_i(y_i)\right| d\mu(y_i) \\
&\le C\int_{2a_1 r}^\infty \frac{1}{l^{m+1}} \left(\int_{\sum_{i=1}^m \rho(x_0,y_i)<l} \prod_{i=1}^m \left|f_i(y_i)\right| d\mu(y_i)\right) dl \\
&\le C\int_{2a_1 r}^\infty \frac{1}{l^{m+1}} \left(\prod_{i=1}^m \int_{\rho(x_0,y_i)<l} \left|f_i(y_i)\right| d\mu(y_i)\right) dl.
\end{aligned}$$

Noting $\mu(kB^*) \le C_0 2ka_1 r$ and applying the Hölder inequality, we see that the inequality above is bounded by

$$\begin{aligned}
&C\int_{\mu(kB^*)/C_0k}^\infty \frac{1}{l^{m+1}} \prod_{i=1}^m \left(\int_{B(x_0,l)} \left|f_i(y_i)\right|^{p_i} d\mu(y_i)\right)^{\frac{1}{p_i}} \mu\left(B(x_0,l)\right)^{1-\frac{1}{p_i}} \, dl \\
&\quad \le C\int_{\mu(kB^*)/C_0k}^\infty \frac{1}{l^{m+1}} \prod_{i=1}^m \left(\|f_i\|_{L^{p_i,\phi_i}(X,\mu)} \phi_i\left(\mu\left(kB(x_0,l)\right)\right)\mu\left(kB(x_0,l)\right)\right) dl
\end{aligned}$$

which, by using the non-decreasing of function $r\phi_i(r)$, is controlled by

$$\begin{aligned}
&C\int_{\mu(kB^*)}^{\infty}\left(\prod_{i=1}^{m}\phi_i(l)\right)\frac{dl}{l}\cdot\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\phi_i}(X,\mu)}\\
&\quad\le C\left(\prod_{i=1}^{m}\int_{\mu(kB^*)}^{\infty}\phi_i(l)^{\frac{p_i}{p}}\frac{dl}{l}\right)^{\frac{p}{p_i}}\cdot\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\phi_i}(X,\mu)}\\
&\quad\le C\phi\big(\mu\big(kB^*\big)\big)\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\phi_i}(X,\mu)}.
\end{aligned}$$

Therefore, we get for $x\in B$ that

$$\big|\mathcal{T}\big(f_1^{\infty},\dots,f_m^{\infty}\big)(x)\big|\le C\phi\big(\mu\big(kB^*\big)\big)\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\phi_i}(X,\mu)}. \tag{3.2}$$

It is left to consider the case that there is $1\le l<m$ and $\{i_1,\dots,i_l\}\subset\{1,\dots,m\}$ such that $\tau_i=0$ if $i\in\{i_1,\dots,i_l\}$, and $\tau_i=\infty$ if $i\notin\{i_1,\dots,i_l\}$. For $x\in B$, we can write that

$$\begin{aligned}
&\big|\mathcal{T}\big(f_1^{\tau_1},\dots,f_m^{\tau_m}\big)(x)\big|\\
&\quad\le\int_{(B^*)^l}\prod_{i\in\{i_1,\dots,i_l\}}\big|f_i(y_i)\big|\,d\mu(y_i)\int_{(X\setminus B^*)^{m-l}}\frac{\prod_{i\notin\{i_1,\dots,i_l\}}|f_i(y_i)|\,d\mu(y_i)}{(\sum_{i\notin\{i_1,\dots,i_l\}}\rho(x,y_i))^m}\\
&\quad:=E_1(x)\cdot E_2(x).
\end{aligned}$$

With the same argument as $A_1(x)$ we have

$$E_1(x)\le C\mu\big(kB^*\big)^l\prod_{i\in\{i_1,\dots,i_l\}}\phi_i\big(\mu\big(kB^*\big)\big)\|f_i\|_{L^{p_i,\phi_i}(X,\mu)}.$$

Using a similar argument as that for the estimate of $\mathcal{T}(f_1^{\infty},\dots,f_m^{\infty})(x)$, we can deduce that

$$\begin{aligned}
E_2(x)&\le C\int_{\mu(kB^*)}^{\infty}\left(\prod_{i\notin\{i_1,\dots,i_l\}}\phi_i(t)\right)\frac{dt}{t^{l+1}}\cdot\prod_{i\notin\{i_1,\dots,i_l\}}\|f_i\|_{L^{p_i,\phi_i}(X,\mu)}\\
&\le C\mu\big(kB^*\big)^{-l}\left[\prod_{i\notin\{i_1,\dots,i_l\}}\left(\int_{\mu(kB^*)}^{\infty}\phi_i(t)^{\frac{p_i}{p}}\frac{dt}{t}\right)^{\frac{p}{p_i}}\right]\prod_{i\notin\{i_1,\dots,i_l\}}\|f_i\|_{L^{p_i,\phi_i}(X,\mu)}\\
&\le C\mu\big(kB^*\big)^{-l}\left[\prod_{i\notin\{i_1,\dots,i_l\}}\phi_i\big(\mu\big(kB^*\big)\big)\right]\prod_{i\notin\{i_1,\dots,i_l\}}\|f_i\|_{L^{p_i,\phi_i}(X,\mu)}.
\end{aligned}$$

Hence we obtain that

$$\big|\mathcal{T}\big(f_1^{\tau_1},\dots,f_m^{\tau_m}\big)(x)\big|\le E_1(x)\cdot E_2(x)\le C\phi\big(\mu\big(kB^*\big)\big)\prod_{i=1}^{m}\|f_i\|_{L^{p_i,\phi_i}(X,\mu)}. \tag{3.3}$$

Therefore, combining inequalities (3.1), (3.2) and (3.3), we have

$$\left(\frac{1}{\mu(kB^*)}\int_B \left|\mathcal{T}(f_1,\ldots,f_m)(x)\right|^p d\mu(x)\right)^{\frac{1}{p}} \le C\phi\bigl(\mu\bigl(kB^*\bigr)\bigr)\prod_{i=1}^m \|f_i\|_{L^{p_i,\phi_i}(X,\mu)},$$

which completes the proof of Theorem 1.7. □

**Remark 3.1** We have actually proved Theorem 1.7 in the case $f_i \in L^{p_i}(X,\mu)\cap L^{p_i,\phi_i}(X,\mu)$. Here we need give some remarks about the definition and boundedness of $\mathcal{T}(f_1,\ldots,f_m)$ with $f_i \in L^{p_i,\phi_i}(X,\mu)$ for $i = 1,2,\ldots,m$. Fix any $x_0 \in X$ and $R > 0$, and use the same notations $f_i^{\tau_i} = f_i\chi_{B(x_0,2a_1R)}$ if $\tau_i = 0$, and $f_i^{\tau_i} = f_i - f_i^0$ if $\tau_i = \infty$. Using a similar argument as (3.2) and (3.3), we have, if some $\tau_i = \infty$,

$$\begin{aligned}&\int_{X^m}\left|\mathcal{K}(x,y_1,\ldots,y_m)f_1^{\tau_1}(y_1)\cdots f_m^{\tau_m}(y_m)\right| d\mu(y_1)\cdots d\mu(y_m)\\&\quad\le C\phi\bigl(\mu\bigl(B\bigl(x_0,3a_1^2R\bigr)\bigr)\bigr)\prod_{i=1}^m \|f_i\|_{L^{p_i,\phi_i}(X,\mu)}\end{aligned}$$

with the constant $C$ independent of $R$, for all $f_i \in L^{p_i,\phi_i}(X,\mu)$, $i = 1,\ldots,m$, and $x \in B(x_0, R) \subset X$.

In view of this fact, and if $\lim_{R\to\infty}\phi(\mu(B(x_0,3a_1^2R))) = 0$, then we can extend the definition of $\mathcal{T}$ for $f_i \in L^{p_i,\phi_i}(X,\mu)$ by

$$\begin{aligned}&\mathcal{T}\bigl(f_1^{\tau_1},\ldots,f_m^{\tau_m}\bigr)(x)\\&\quad= \lim_{R\to\infty}\Bigl(\mathcal{T}\bigl(f_1^0,\ldots,f_m^0\bigr)(x)\\&\qquad+\sum_{\text{some }\tau_i=\infty}\int_{X^m}\left|\mathcal{K}(x,y_1,\ldots,y_m)f_1^{\tau_1}(y_1)\cdots f_m^{\tau_m}(y_m)\right| d\mu(y_1)\cdots d\mu(y_m)\Bigr).\end{aligned}$$

By the definition, it is easy to see that the following properties hold.

(1) If $\rho(x_0,x) \le R_0$, then the terms in the brackets on the right-hand side of the equation above do not depend on $R$ as long as $R > R_0$.
(2) Suppose that $1 < p_1,\ldots,p_m < \infty$, and if $f_i \in L^{p_i}(X,\mu)\cap L^{p_i,\phi_i}(X,\mu)$, then the definitions of $\mathcal{T}(f_1,\ldots,f_m)$ for $f_i \in L^{p_i}(X,\mu)$ and for $f_i \in L^{p_i,\phi_i}(X,\mu)$ coincide.
(3) Theorem 1.7 holds for any admissible $f_i \in L^{p_i,\phi_i}(X,\mu)$, $i = 1,\ldots,m$.

Finally, we consider the multi-sublinear maximal function $\mathcal{M}_\kappa(f_1,\ldots,f_m)(x)$, which is strictly smaller than the $m$-fold produce of the maximal function $M_\kappa(f_i)(x)$. Hence we have the following lemma.

**Lemma 3.2** *If $\kappa > a_1^2$ and $p, p_i > 1$, and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, then there exists a constant $C$ independent of $f_i$ such that*

$$\left\|\mathcal{M}_\kappa(f_1,\ldots,f_m)\right\|_{L^p(X,\mu)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i}(X,\mu)}.$$

*Proof of Theorem* 1.9 With the same notions, we decompose each $f_i \in L^{p_i,\phi_i}(X,\mu)$ according to the ball $B^* := B(x_0, a_1(1+\lambda)r)$ as $f_i = f_i^0 + f_i^\infty$, where $\lambda$ is a large positive constant that will be determined later. We have

$$\left|\mathcal{M}_\kappa(f_1,\dots,f_m)(x)\right| \le \left|\mathcal{M}_\kappa\left(f_1^0,\dots,f_m^0\right)(x)\right| + \sum{}' \left|\mathcal{M}_\kappa\left(f_1^{\tau_1},\dots,f_m^{\tau_m}\right)(x)\right|,$$

where each term in $\sum'$ contains at least one $\tau_i \neq 0$.

It follows from Lemma 3.2 that for any $k > a_1$ and $\kappa > a_1^2$,

$$\begin{aligned}&\left(\frac{1}{\mu(kB^*)}\int_{B(x_0,r)}\left|\mathcal{M}_\kappa\left(f_1^0,\dots,f_m^0\right)(x)\right|^p\,d\mu(x)\right)^{\frac{1}{p}}\\&\quad\le C\phi\left(\mu\left(kB^*\right)\right)\prod_{i=1}^m \|f_i\|_{L^{p_i,\phi_i}(X,\mu)}.\end{aligned}$$

It is left to study the case $\tau_{i_1} = \cdots = \tau_{i_l} = 0$ and $\tau_{i_{l+1}} = \cdots = \tau_{i_m} = \infty$ for some $1 \le l < m$. Hence for $x \in B(x_0,r)$ we have

$$\begin{aligned}\mathcal{M}_\kappa\left(f_1^{\tau_1},\dots,f_m^{\tau_m}\right)(x) &= \sup_{x\in D\in\mathscr{B}(\mu)}\prod_{i=1}^m \frac{1}{\mu(\kappa D)}\int_D \left|f_i^{\tau_i}(y_i)\right|\,d\mu(y_i)\\&\le \sup_{x\in D\in\mathscr{B}(\mu)}\prod_{i\in\{i_1,\dots,i_l\}}\frac{1}{\mu(\kappa D)}\int_D\left|f_i^0(y_i)\right|\,d\mu(y_i)\\&\quad\cdot\prod_{i\notin\{i_1,\dots,i_l\}}\frac{1}{\mu(\kappa D)}\int_D\left|f_i^\infty(y_i)\right|\,d\mu(y_i).\end{aligned}$$

Let $r_D$ be the radius of the ball $D$ and $c_D$ be the center of $D$. We note that on the right-hand side of the inequality above, the balls $D$ in the integrals must satisfy that $x \in D \cap B(x_0,r)$ and some $y_{i_m} \in D \cap (X \setminus B^*)$, which implies

$$\begin{aligned}&a_1\rho(x,y_{i_m}) \ge \rho(x_0,y_{i_m}) - a_1\rho(x_0,x) \ge a_1\lambda r,\\&\rho(x,y_{i_m}) \le a_1a_0\rho(c_D,x) + a_1\rho(c_D,y_{i_m}) \le a_1(1+a_0)r_D.\end{aligned}$$

Further, a simple calculus yields

$$B := B(x_0,r) \subset \left(a_1 + a_1^3(1+a_0)^2\lambda^{-1}\right)D \subset \frac{\kappa + a_1^2}{2a_1}D$$

as long as we take $\lambda$ big enough, because of $\kappa > a_1^2$. Thus,

$$\mathcal{M}_\kappa\left(f_1^{\tau_1},\dots,f_m^{\tau_m}\right)(x) \le \sup_{B\subset D\in\mathscr{B}(\mu)}\prod_{i=1}^m \frac{1}{\mu(\frac{2a_1\kappa}{\kappa+a_1^2}D)}\int_D\left|f_i^{\tau_i}(y_i)\right|\,d\mu(y_i).$$

If let $k = 2a_1\kappa/(\kappa + a_1^2)$, then $k > a_1$, and we can get from the Hölder inequality and condition (1.6) on $\phi_i$ that

$$\begin{aligned}
\mathcal{M}_\kappa\big(f_1^{\tau_1},\ldots,f_m^{\tau_m}\big)(x) &\le \sup_{B\subset D\in\mathscr{B}(\mu)} \prod_{i=1}^m \frac{1}{\mu(kD)} \int_D \big|f_i^{\tau_i}(y_i)\big|\,d\mu(y_i) \\
&\le C \sup_{B\subset D\in\mathscr{B}(\mu)} \prod_{i=1}^m \|f_i\|_{L^{p_i,\phi_i}(X,\mu)} \phi_i\big(\mu(kD)\big) \\
&\le C\phi\big(\mu(kB)\big) \prod_{i=1}^m \|f_i\|_{L^{p_i,\phi_i}(X,\mu)}.
\end{aligned}$$

The theorem is proved. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
XT finished the proof and the revised writing work. SH gave XT the first version of the manuscript. All authors read and approved the final manuscript.

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