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Some results on the generalized Mittag-Leffler function operator

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Abstract

This paper is devoted to the study of a generalized Mittag-Leffler function operator introduced by Shukla and Prajapati (J. Math. Anal. Appl. 336:797-811, 2007). Laplace and Mellin transforms of this operator are investigated in this paper. Some special cases of the established results are also deduced as corollaries. The results obtained are useful where the Mittag-Leffler function occurs naturally.

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1 Introduction

In 1903, the Swedish mathematician Mittag-Leffler [1, 2] introduced the function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad \bigl(\alpha \in \mathbb{C}; \operatorname{Re}(\alpha) > 0\bigr). \tag{1.1}$$

A generalization of (1.1) was given by Wiman [3] in 1905 in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad \bigl(\alpha, \beta \in \mathbb{C}; \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0\bigr). \tag{1.2}$$

In 1971, in connection with the solution of certain singular integral equations, a further interesting and useful generalization of (1.2) was introduced by Prabhakar [4] in the form

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta) n!} \tag{1.3}$$

$$\bigl(\alpha, \beta, \gamma \in \mathbb{C}; \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0, \operatorname{Re}(\gamma) > 0\bigr),$$

where $(\gamma)_n$ is the Pochhammer symbol defined by

$$(\gamma)_0 = 1; \qquad (\gamma)_n = \gamma(\gamma + 1)\cdots(\gamma + n - 1) = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)}; \qquad \gamma \neq 0. \tag{1.4}$$

A generalization of (1.3) is given by Shukla and Prajapati [5] in the following form:

$$E_{\alpha,\beta}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)n!},\tag{1.5}$$

where ($\alpha,\beta,\gamma\in\mathbb{C}$; $\operatorname{Re}(\alpha)>0$, $\operatorname{Re}(\beta)>0$, $\operatorname{Re}(\gamma)>0$, $q\in(0,1)\cup\mathbb{N}$).

The above generalization studied by Shukla and Prajapati is shown in a series of papers [5–8]. A generalization of Mittag-Leffler functions defined by (1.3) and (1.5) is introduced and studied by Srivastava and Tomovski [9] in the form

$$E_{\alpha,\beta}^{\gamma,\kappa}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\kappa}z^n}{\Gamma(\alpha n+\beta)n!},\tag{1.6}$$

where $\alpha,\beta,\gamma,\kappa\in\mathbb{C}$; $\operatorname{Re}(\alpha)=\operatorname{Re}(\kappa)-1>0$, $\operatorname{Re}(\beta)>0$, $\operatorname{Re}(\gamma)>0$, $\operatorname{Re}(\kappa)>0$ and the Pochhammer symbol for $\lambda,\mu\in\mathbb{C}$ is defined by

$$(\lambda)_\mu=\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)}=\begin{cases}1 & (\mu=0,\lambda\in\mathbb{C}\backslash\{0\});\\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \mu=n\in\mathbb{N};\lambda\in\mathbb{C}.\end{cases}\tag{1.7}$$

The object of this paper is to derive the Laplace and Mellin transforms of the following integral operator associated with the generalized Mittag-Leffler function, defined by Shukla and Prajapati [6] as well as Srivastava and Tomovski [9], in the next sections:

$$\left(E_{\alpha,\beta,\omega;0_+}^{\gamma,q}f\right)(x)=\int_0^x(x-t)^{\beta-1}E_{\alpha,\beta}^{\gamma,q}\left[w(x-t)^\alpha\right]f(t)\,dt,\tag{1.8}$$

where $\alpha,\beta,\gamma\in\mathbb{C}$; $\operatorname{Re}(\alpha)>0$, $\operatorname{Re}(\beta)>0$, $\operatorname{Re}(\gamma)>0$; $q\in(0,1)\cup\mathbb{N}$.

Shukla and Prajapati [6] have shown that the Mellin-Barnes integral for the function defined by (1.5) is given by

$$E_{\alpha,\beta}^{\gamma,q}(z)=\frac{1}{2\pi i\Gamma(\gamma)}\int_{-i\infty}^{i\infty}\frac{\Gamma(-\xi)\Gamma(\gamma+q\xi)}{\Gamma(\alpha\xi+\beta)}(-z)^\xi\,d\xi.\tag{1.9}$$

As $\gamma\longrightarrow0$, then by virtue of the limit formula,

$$\lim_{\gamma\longrightarrow0}E_{\alpha,\beta}^{\gamma}(z)=\frac{1}{\Gamma(\beta)}\tag{1.10}$$

reduces to the familiar Reimann-Liouville fractional integral

$$\left(I_{0_+}^\alpha f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)\,dt\quad\left(\alpha\in\mathbb{C},\operatorname{Re}(\alpha)>0\right).\tag{1.11}$$

In the following, we will use the representation of a high transcendental function in terms of the so-called H-function defined as [10]

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right]=\frac{1}{2\pi i}\int_L\theta(\xi)z^{-\xi}\,d\xi,\tag{1.12}$$

where

$$\theta(\xi) = \frac{[\prod_{j=1}^{m}\Gamma(b_j + B_j\xi)][\prod_{j=1}^{n}\Gamma(1 - a_j - A_j\xi)]}{[\prod_{j=m+1}^{q}\Gamma(1 - b_j - B_j\xi)][\prod_{j=n+1}^{p}\Gamma(a_j + A_j\xi)]}, \tag{1.13}$$

and an empty product is always interpreted as unity; $m, n, p, q \in \mathbb{N}_0$ with $0 \le n \le p$, $1 \le m \le q$, $A_i, B_j \in R^+$, $a_i, b_j \in \mathbb{R}$ or $\mathbb{C}$ $(i = 1, \dots, p; j = 1, \dots, q)$ such that

$$A_i(b_j + k) \ne B_j(a_i - l - 1) \quad (k, l \in N_0; i = 1, \dots, n; j = 1, \dots, m). \tag{1.14}$$

The contour L is the path of integration in the complex ξ-plane running from $\gamma - i\infty$ to $\gamma + i\infty$ for some real number γ.

2 Mellin transform of the operator (1.8)

Theorem 2.1 *It is shown here that*

$$M\{(E^{\gamma,q}_{\alpha,\beta,\omega;0_+}f)(x); s\} = \frac{1}{\Gamma(\gamma)\Gamma(1-s)} \cdot H^{2,1}_{1,2}\left[-wt^{\alpha} \,\middle|\, \begin{matrix}(1-\gamma, q)\\ (0,1), (1-s-\beta, \alpha)\end{matrix}\right] M\{t^{\beta}f(t); s\}, \tag{2.1}$$

where $(\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0)$; $q \in (0,1) \cup \mathbb{N}$, $\mathrm{Re}(1 - s - \beta) > 0$ *and* $H^{2,1}_{1,2}(\cdot)$ *is the H-function defined by* (1.12).

Proof Mellin transform is defined as

$$M\{f(x); s\} = \int_0^{\infty} x^{s-1} f(x)\, dx.$$

Therefore, we have

$$M\{(E^{\gamma,q}_{\alpha,\beta,\omega;0_+}f)(x); s\} = \int_0^{\infty} x^{s-1} \int_0^{x} (x-t)^{\beta-1} E^{\gamma,q}_{\alpha,\beta}[\omega(x-t)^{\alpha}] f(t)\, dt\, dx.$$

Interchanging the order of integration, which is permissible under the conditions given in Theorem 2.1, we find that

$$M\{(E^{\gamma,q}_{\alpha,\beta,\omega;0_+}f)(x); s\} = \int_0^{\infty} f(t)\, dt \int_t^{\infty} x^{s-1}(x-t)^{\beta-1} E^{\gamma,q}_{\alpha,\beta}[\omega(x-t)^{\alpha}]\, dx. \tag{2.2}$$

If we consider $x = t + u$ in the r.h.s. of (2.2), we get

$$M\{(E^{\gamma,q}_{\alpha,\beta,\omega;0_+}f)(x); s\} = \int_0^{\infty} f(t)\, dt \int_t^{\infty} (t+u)^{s-1} u^{\beta-1} E^{\gamma,q}_{\alpha,\beta}[\omega u^{\alpha}]\, du. \tag{2.3}$$

To evaluate the u-integral, we express the Mittag-Leffler function in terms of its Mellin-Barnes contour integral by means of the formula (1.9), then the above expression transforms into the form

$$M\{(E^{\gamma,q}_{\alpha,\beta,\omega;0_+}f)(x);s\} = \int_0^\infty f(t)\,dt \frac{1}{2\pi i\Gamma(\gamma)} \int_{-i\infty}^{i\infty} \frac{\Gamma(-\xi)\Gamma(\gamma+q\xi)}{\Gamma(\alpha\xi+\beta)}(-\omega)^\xi \cdot \int_0^\infty (t+u)^{s-1}u^{\beta+\alpha\xi-1}\,du\,d\xi. \tag{2.4}$$

If the u-integral is evaluated with the help of the formula

$$\int_0^\infty x^{\nu-1}(x+a)^{-\rho}\,dx = \frac{\Gamma(\nu)\Gamma(\rho-\nu)}{\Gamma(\rho)}; \quad \mathrm{Re}(\rho) > \mathrm{Re}(\nu) > 0, \tag{2.5}$$

then after some simplification, it is seen that the right-hand side of above equation (2.4) simplifies to

$$\frac{1}{2\pi i\Gamma(\gamma)\Gamma(1-s)} \int_{-i\infty}^{i\infty} \Gamma(-\xi)\Gamma(\gamma+q\xi)\Gamma(1-s-\beta-\alpha\xi)(-\omega t^\alpha)^\xi\,d\xi \cdot \int_0^\infty t^{\beta+s-1}f(t)\,dt \tag{2.6}$$

which, on being interpreted by the definition of H-function (1.12), yields the desired result. □

For $q = 1$, Theorem 2.1 reduces to the following corollary.

Corollary 2.1 *The following result holds*:

$$M\{(E^{\gamma}_{\alpha,\beta,\omega;0_+}f)(x);s\} = \frac{1}{\Gamma(\gamma)\Gamma(1-s)} \cdot H^{2,1}_{1,2}\left[-wt^\alpha \,\middle|\, \begin{matrix} (1-\gamma,1) \\ (0,1),(1-s-\beta,\alpha) \end{matrix}\right] M\{t^\beta f(t);s\}, \tag{2.7}$$

where $(\mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0,\ \mathrm{Re}(\gamma) > 0)$; $\mathrm{Re}(1-s-\beta) > 0$ *and* $H^{2,1}_{1,2}(\cdot)$ *is the H-function defined by* (1.12).

Theorem 2.2 *It is shown here that*

$$M\{(E^{\gamma,\kappa}_{\alpha,\beta,\omega;0_+}f)(x);s\} = \frac{1}{\Gamma(\gamma)\Gamma(1-s)} \cdot H^{2,1}_{1,2}\left[-wt^\alpha \,\middle|\, \begin{matrix} (1-\gamma,\kappa) \\ (0,1),(1-s-\beta,\alpha) \end{matrix}\right] M\{t^\beta f(t);s\}, \tag{2.8}$$

where $\mathrm{Re}(\alpha) = \mathrm{Re}(\kappa) - 1 > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\kappa) > 0$, $\mathrm{Re}(1-s-\beta) > 0$ *and* $H^{2,1}_{1,2}(\cdot)$ *is the H-function defined by* (1.12).

3 Laplace transform of the operator (1.8)

Theorem 3.1 *The following result holds*:

$$L\{(E^{\gamma,q}_{\alpha,\beta,\omega;0_+}f)(x);p\} = \frac{1}{\Gamma(\gamma)} p^{-\beta}\, {}_1\psi_0\left[\begin{matrix} (\gamma,q); & \omega/p^\alpha \\ -; & \end{matrix}\right] F(p), \tag{3.1}$$

where $(\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma) > 0)$; $\mathrm{Re}(p) > |\omega|^{1/\mathrm{Re}(\alpha)}$ *and* $F(p)$ *is the Laplace transform of* $f(t)$ *defined by*

$$L\{f(t);p\} = F(p) = \int_0^\infty e^{-pt} f(t)\,dt, \tag{3.2}$$

where $\mathrm{Re}(p) > 0$ *and the integral is convergent.*

Proof By virtue of the definitions (1.8) and (3.2), it follows that

$$L\{(E^{\gamma,q}_{\alpha,\beta,\omega;0_+}f)(x);p\} = \int_0^\infty e^{-px}\int_0^x (x-t)^{\beta-1}E^{\gamma,q}_{\alpha,\beta}[\omega(x-t)^\alpha]f(t)\,dt\,dx.$$

Interchanging the order of integration, which is permissible under the conditions given in Theorem 3.1, we find that

$$L\{(E^{\gamma,q}_{\alpha,\beta,\omega;0_+}f)(x);p\} = \int_0^\infty f(t)\,dt\int_t^\infty e^{-px}(x-t)^{\beta-1}E^{\gamma,q}_{\alpha,\beta}[\omega(x-t)^\alpha]\,dx.$$

If we consider $x = t + u$, we obtain

$$L\{(E^{\gamma,q}_{\alpha,\beta,\omega;0_+}f)(x);p\} = \int_0^\infty e^{-pt}f(t)\,dt\int_0^\infty e^{-pu}u^{\beta-1}E^{\gamma,q}_{\alpha,\beta}[\omega u^\alpha]\,du.$$

On making use of the series definition (1.5), the above expression becomes

$$\begin{aligned}
&L\{(E^{\gamma,q}_{\alpha,\beta,\omega;0_+}f)(x);p\}\\
&\quad= \sum_{r=0}^\infty \frac{(\gamma)_{qr}\omega^r}{\Gamma(\alpha r+\beta)(r!)}\int_0^\infty e^{-pt}f(t)\,dt\int_0^\infty e^{-pu}u^{\beta+\alpha r-1}\,du\\
&\quad= \sum_{r=0}^\infty \frac{(\gamma)_{qr}\omega^r}{p^{\beta+\alpha r}(r!)}\int_0^\infty e^{-pt}f(t)\,dt\\
&\quad= \frac{1}{\Gamma(\gamma)}p^{-\beta}\,{}_1\psi_0\left[\begin{matrix}(\gamma,q); & \omega/p^\alpha\\ -; & \end{matrix}\right]F(p),
\end{aligned}$$

and $F(p)$ is the Laplace transform of $f(t)$. □

For $q = 1$, Theorem 3.1 reduces to the following corollary.

Corollary 3.1 *The following result holds*:

$$Ł\{(E^{\gamma}_{\alpha,\beta,\omega;0_+}f)(x);p\} = p^{-\beta}(1-ap^\alpha)^\gamma F(p), \tag{3.3}$$

where $(\mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0,\ \mathrm{Re}(\gamma) > 0)$; $\mathrm{Re}(p) > 0$ *and the integral operator is the one discussed by Prabhakar* [4] *defined by* (1.10).

Theorem 3.2 *The following result holds*:

$$L\{(E^{\gamma,\kappa}_{\alpha,\beta,\omega;0_+}f)(x);p\} = \frac{1}{\Gamma(\gamma)}p^{-\beta}\,{}_1\psi_0\left[\begin{matrix}(\gamma,\kappa); & \omega/p^\alpha\\ -; & \end{matrix}\right]F(p), \tag{3.4}$$

where $\mathrm{Re}(\alpha) = \mathrm{Re}(\kappa) - 1 > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\kappa) > 0$; $\mathrm{Re}(p) > |\omega|^{1/\mathrm{Re}(\alpha)}$ *and* $F(p)$ *is the Laplace transform of* $f(t)$ *defined by* (3.2).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this article. The authors read and approved the final manuscript.

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