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An artificial proof of a geometric inequality in a triangle

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Abstract
In this paper, the authors give an artificial proof of a geometric inequality relating to the medians and the exradius in a triangle by making use of certain analytical techniques for systems of nonlinear algebraic equations.

MSC: 51M16; 52A40

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1 Introduction and main results

For a given $\triangle ABC$, let a, b and c denote the side-lengths facing the angles A, B and C, respectively. Also, let m_a, m_b and m_c denote the corresponding medians, r_a, r_b and r_c the corresponding exradii, $s = \frac{1}{2}(a+b+c)$ the semi-perimeter, Δ the area. In addition, we let

$$m_1 = \frac{1}{2}\sqrt{(b+c)^2 - a^2} = \sqrt{s(s-a)},$$
$$m_2 = \frac{1}{2}\sqrt{2a^2 + \frac{1}{4}(b+c)^2},$$

and

$$r_1 = \frac{a\sqrt{s(s-a)}}{2(s-a)}.$$

Throughout this paper, we will customarily use the cyclic sum symbols as follows:

$$\sum f(a) = f(a) + f(b) + f(c)$$

and

$$\sum f(b,c) = f(a,b) + f(b,c) + f(c,a).$$

In 2003, Liu [1] found the following interesting geometric inequality relating to the medians and the exradius in a triangle with the computer software BOTTEMA invented by Yang [2–5], and Liu thought this inequality cannot be proved by a human.

Theorem 1.1 *In $\triangle ABC$, the best constant k for the following inequality*

$$\sum (r_b - r_c)^2 \geq k \cdot \sum (m_b - m_c)^2 \tag{1.1}$$

is the real root on the interval $(3,4)$ *of the following equation*

$$6{,}561k^4 - 14{,}256k^3 - 18{,}080k^2 - 25{,}344k + 20{,}736 = 0. \tag{1.2}$$

Furthermore, the constant k *has its numerical approximation given by* 3.2817755127.

In this paper, the authors give an artificial proof of Theorem 1.1.

2 Preliminary results

In order to prove Theorem 1.1, we require the following results.

Lemma 2.1 *In* $\triangle ABC$, *if* $a \le b \le c$, *then*

$$r_a^2 + r_b^2 + r_c^2 - \left(r_1^2 + 2m_1^2\right) \ge \frac{3s(s-a)(b-c)^2}{4(s-b)(s-c)}. \tag{2.1}$$

Proof From $a = (s-b) + (s-c)$ and the formulas of the exradius $r_a = \frac{\Delta}{s-a} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a}$, *etc.*, we get

$$\begin{aligned}
&r_a^2 + r_b^2 + r_c^2 - \left(r_1^2 + 2m_1^2\right) \\
&= \left[\frac{1}{(s-a)^2} + \frac{1}{(s-b)^2} + \frac{1}{(s-c)^2}\right] s(s-a)(s-b)(s-c) - \frac{a^2 s(s-a)}{4(s-a)^2} - 2s(s-a) \\
&= \frac{1}{4}s(s-a)\left[\frac{4(s-b)(s-c)}{(s-a)^2} + \frac{4(s-b)(s-c)}{(s-b)^2} + \frac{4(s-b)(s-c)}{(s-c)^2} - \frac{a^2}{(s-a)^2} - 8\right] \\
&= \frac{1}{4}s(s-a)\left[\frac{4(s-b)(s-c) - a^2}{(s-a)^2} + 4\left(\frac{s-c}{s-b} + \frac{s-b}{s-c} - 2\right)\right] \\
&= \frac{1}{4}s(s-a)\left[-\frac{(b-c)^2}{(s-a)^2} + \frac{4(b-c)^2}{(s-b)(s-c)}\right] \\
&= \frac{1}{4}s(s-a)(b-c)^2\left[\frac{4}{(s-b)(s-c)} - \frac{1}{(s-a)^2}\right].
\end{aligned} \tag{2.2}$$

For $a \le b \le c$, we have

$$s-a \ge s-b \ge s-c > 0,$$

then

$$0 < \frac{1}{s-a} \le \frac{1}{s-b} \le \frac{1}{s-c},$$

hence

$$\frac{1}{(s-b)(s-c)} \ge \frac{1}{(s-a)^2} > 0. \tag{2.3}$$

Inequality (2.1) follows from inequalities (2.2)-(2.3) immediately. □

Lemma 2.2 *In* $\triangle ABC$, *we have*

$$(m_b + m_2)(m_c + m_2) \geq 4s\sqrt{(s-b)(s-c)} \tag{2.4}$$

and

$$a(m_b + m_c)^2 - 8s(s-b)(s-c) \geq \frac{3s\sqrt{(s-b)(s-c)}(b-c)^2}{a}. \tag{2.5}$$

Proof of inequality (2.4) From

$$m_2^2 - \frac{1}{2}as = \frac{1}{4}\left(a - \frac{b+c}{2}\right)^2 \geq 0,$$

we immediately obtain

$$m_2 \geq \sqrt{\frac{1}{2}as}. \tag{2.6}$$

In view of the *AM-GM* inequality, we get

$$\frac{a}{2} = \frac{(s-b)+(s-c)}{2} \geq \sqrt{(s-b)(s-c)}. \tag{2.7}$$

By the power mean inequality, we have

$$\sqrt{\frac{a}{2}} = \sqrt{\frac{(s-b)+(s-c)}{2}} \geq \frac{\sqrt{s-b}+\sqrt{s-c}}{2}. \tag{2.8}$$

By the well-known inequalities $m_b \geq \sqrt{s(s-b)}$ and $m_c \geq \sqrt{s(s-c)}$, together with inequalities (2.6)-(2.8), we obtain

$$\begin{aligned}
&(m_b + m_2)(m_c + m_2) \\
&\quad \geq \left(\sqrt{s(s-b)} + \sqrt{\frac{1}{2}as}\right)\left(\sqrt{s(s-c)} + \sqrt{\frac{1}{2}as}\right) \\
&\quad = s\left(\sqrt{s-b} + \sqrt{\frac{1}{2}a}\right)\left(\sqrt{s-c} + \sqrt{\frac{1}{2}a}\right) \\
&\quad = s\left[\frac{1}{2}a + \sqrt{\frac{1}{2}a}(\sqrt{s-b} + \sqrt{s-c}) + \sqrt{(s-b)(s-c)}\right] \\
&\quad \geq s\left[\frac{1}{2}(\sqrt{s-b} + \sqrt{s-c})^2 + 2\sqrt{(s-b)(s-c)}\right] \\
&\quad = s\left[\frac{1}{2}a + 3\sqrt{(s-b)(s-c)}\right] \\
&\quad \geq 4s\sqrt{(s-b)(s-c)}.
\end{aligned}$$

The proof of inequality (2.4) is thus complete. □

Proof of inequality (2.5) According to the well-known inequalities $m_b \geq \sqrt{s(s-b)}$, $m_c \geq \sqrt{s(s-c)}$ and inequality (2.7), we have

$$\begin{aligned}
&a(m_b+m_c)^2 - 8s(s-b)(s-c) \\
&\quad= \bigl[a - 2\sqrt{(s-b)(s-c)}\bigr](m_b+m_c)^2 \\
&\qquad + 2\sqrt{(s-b)(s-c)}\bigl[(m_b+m_c)^2 - 4s\sqrt{(s-b)(s-c)}\bigr] \\
&\quad\geq \bigl[a - 2\sqrt{(s-b)(s-c)}\bigr]\cdot 4m_bm_c + 2\sqrt{(s-b)(s-c)}\bigl[\bigl(\sqrt{s(s-b)}+\sqrt{s(s-c)}\bigr)^2 \\
&\qquad - 4s\sqrt{(s-b)(s-c)}\bigr] \\
&\quad\geq 4s\bigl[a - 2\sqrt{(s-b)(s-c)}\bigr]\sqrt{(s-b)(s-c)} + 2\sqrt{(s-b)(s-c)}\bigl[a - 2\sqrt{(s-b)(s-c)}\bigr] \\
&\quad= 6s\sqrt{(s-b)(s-c)}\bigl[a - 2\sqrt{(s-b)(s-c)}\bigr] \\
&\quad= \frac{6s\sqrt{(s-b)(s-c)}(b-c)^2}{a + 2\sqrt{(s-b)(s-c)}} \\
&\quad\geq \frac{3s\sqrt{(s-b)(s-c)}(b-c)^2}{a}.
\end{aligned} \tag{2.9}$$

Hence, we complete the proof of inequality (2.5). □

Lemma 2.3 *In* $\triangle ABC$, *we have*

$$m_bm_c \leq m_2^2. \tag{2.10}$$

Proof From the formulas of the medians, we have

$$\begin{aligned}
m_bm_c - m_2^2 &= \frac{m_b^2m_c^2 - m_2^4}{m_bm_c + m_2^2} \\
&= \frac{\frac{1}{16}(2c^2+2a^2-b^2)(2a^2+2b^2-c^2) - \frac{1}{16}(2a^2+\frac{1}{4}(b+c)^2)}{m_bm_c + m_2^2} \\
&= \frac{\{16[a^2-(b+c)^2] - (17b^2+17c^2+38bc)\}(b-c)^2}{256(m_bm_c+m_2^2)} \leq 0.
\end{aligned}$$

Therefore, inequality (2.10) holds true. □

Lemma 2.4 *In* $\triangle ABC$, *if* $a \leq b \leq c$, *then*

$$\begin{aligned}
&\frac{m_b+m_c}{m_a+m_1} + \frac{1}{4}\left(\frac{m_1}{m_2+m_b} + \frac{m_1}{m_2+m_c}\right) - \frac{9(b+c)^2}{8(m_b+m_c)^2} \\
&\quad\geq \frac{m_2}{m_1} + \frac{m_1}{4m_2} - \frac{9a(b+c)^2}{64s(s-b)(s-c)}.
\end{aligned} \tag{2.11}$$

Proof It is obvious that $m_b > c - \frac{b}{2}$ and $m_c > b - \frac{c}{2}$, then we have $m_b + m_c > \frac{1}{2}(b+c)$, thus

$$(m_b-m_c)^2 = \frac{(m_b^2-m_c^2)^2}{(m_b+m_c)^2} = \frac{9(b+c)^2(b-c)^2}{16(m_b+m_c)^2} \leq \frac{9}{4}(b-c)^2. \tag{2.12}$$

For $a \le b \le c$, we have that

$$m_a \ge \begin{cases} m_1 \\ m_b \end{cases} \ge m_2 \ge m_c. \tag{2.13}$$

By Lemma 2.3 and inequalities (2.12)-(2.13), we have

$$\begin{aligned}
&\frac{m_b + m_c}{m_a + m_1} + \frac{1}{4}\left(\frac{m_1}{m_2 + m_b} + \frac{m_1}{m_2 + m_c}\right) - \frac{m_2}{m_1} - \frac{m_1}{4m_2} \\
&\quad = \frac{m_b + m_c - 2m_2}{m_a + m_1} + \frac{m_2(m_1 - m_a)}{m_1(m_a + m_1)} + \frac{m_1(m_2^2 - m_b m_c)}{4m_2(m_2 + m_b)(m_2 + m_c)} \\
&\quad \ge \frac{(m_b + m_c)^2 - 4m_2^2}{(m_a + m_1)(m_b + m_c + 2m_2)} + \frac{m_2(m_1^2 - m_a^2)}{m_1(m_a + m_1)^2} \\
&\quad = \frac{2(m_b^2 + m_c^2) - (m_b - m_c)^2 - 4m_2^2}{(m_a + m_1)(m_b + m_c + 2m_2)} + \frac{m_2(m_1^2 - m_a^2)}{m_1(m_a + m_1)^2} \\
&\quad = \frac{\frac{1}{4}(b - c)^2 - (m_b - m_c)^2}{(m_a + m_1)(m_b + m_c + 2m_2)} - \frac{m_2(b - c)^2}{4m_1(m_a + m_1)^2} \\
&\quad \ge \frac{\frac{1}{4}(b - c)^2 - \frac{9}{4}(b - c)^2}{(m_a + m_1)(m_b + m_c + 2m_2)} - \frac{(b - c)^2}{4(m_a + m_1)^2} \\
&\quad = \frac{-2(b - c)^2}{(m_a + m_1)(m_b + m_c + 2m_2)} - \frac{(b - c)^2}{4(m_a + m_1)^2} \\
&\quad \ge \frac{-2(b - c)^2}{(m_a + m_1)(m_b + m_c)} - \frac{(b - c)^2}{4(m_a + m_1)^2} \\
&\quad = \frac{-2(b - c)^2}{(m_b + m_c)^2} - \frac{(b - c)^2}{4(m_b + m_c)^2} \\
&\quad \ge \frac{-9(b - c)^2}{4(m_b + m_c)^2}.
\end{aligned} \tag{2.14}$$

By inequality (2.5), (2.7) and $a \le b \le c$, we obtain that

$$\begin{aligned}
&\frac{9a(b + c)^2}{64s(s - b)(s - c)} - \frac{9(b + c)^2}{8(m_b + m_c)^2} \\
&\quad = \frac{9(b + c)^2[a(m_b + m_c)^2 - 8s(s - b)(s - c)]}{64s(s - b)(s - c)(m_b + m_c)^2} \\
&\quad \ge \frac{9(b + c)^2}{64s(s - b)(s - c)(m_b + m_c)^2} \cdot \frac{3s\sqrt{(s - b)(s - c)}(b - c)^2}{a} \\
&\quad = \frac{27(b + c)^2(b - c)^2}{64a\sqrt{(s - b)(s - c)}(m_b + m_c)^2} \\
&\quad \ge \frac{27(b + c)^2(b - c)^2}{32a^2(m_b + m_c)^2} \\
&\quad \ge \frac{27(b - c)^2}{8(m_b + m_c)^2}.
\end{aligned} \tag{2.15}$$

By inequalities (2.14)-(2.15), we have

$$\begin{aligned}
&\left[\frac{m_b+m_c}{m_a+m_1}+\frac{1}{4}\left(\frac{m_1}{m_2+m_b}+\frac{m_1}{m_2+m_c}\right)-\frac{9(b+c)^2}{8(m_b+m_c)^2}\right] \\
&\quad -\left[\frac{m_2}{m_1}+\frac{m_1}{4m_2}-\frac{9a(b+c)^2}{64s(s-b)(s-c)}\right] \\
&=\left[\frac{m_b+m_c}{m_a+m_1}+\frac{1}{4}\left(\frac{m_1}{m_2+m_b}+\frac{m_1}{m_2+m_c}\right)-\frac{m_2}{m_1}-\frac{m_1}{4m_2}\right] \\
&\quad +\left[\frac{9a(b+c)^2}{64s(s-b)(s-c)}-\frac{9(b+c)^2}{8(m_b+m_c)^2}\right] \\
&\geq \frac{-9(b-c)^2}{4(m_b+m_c)^2}+\frac{27(b-c)^2}{8(m_b+m_c)^2} \\
&=\frac{9(b-c)^2}{8(m_b+m_c)^2}\geq 0. \qquad (2.16)
\end{aligned}$$

Inequality (2.11) follows from inequality (2.16) immediately. □

Lemma 2.5 *In $\triangle ABC$, if $a \leq b \leq c$, then*

$$\frac{m_2}{m_1}+\frac{m_1}{4m_2}+\frac{3(b+c)^2}{16a^2}\geq 2 \qquad (2.17)$$

and

$$\frac{m_1+\sqrt{3}a}{s}\leq\sqrt{3}. \qquad (2.18)$$

Proof Without loss of generality, we can take $b+c=2$ and $a=x$, for $a \leq b \leq c$, we have $0 < x \leq 1$.

(i) First, we prove inequality (2.17).

$$\begin{aligned}
\frac{m_2}{m_1}+\frac{m_1}{4m_2}+\frac{3(b+c)^2}{16a^2}-2 &= \sqrt{\frac{1+2x^2}{4-x^2}}+\frac{1}{4}\sqrt{\frac{4-x^2}{1+2x^2}}+\frac{3}{4x^2}-2 \\
&= \frac{8+7x^2}{4\sqrt{(4-x^2)(1+2x^2)}}+\frac{3(1-x^2)}{4x^2}-\frac{5}{4} \\
&\geq \frac{8+7x^2}{4\cdot\frac{(4-x^2)+(1+2x^2)}{2}}+\frac{3(1-x^2)}{4x^2}-\frac{5}{4} \\
&= \frac{8+7x^2}{2(5+x^2)}+\frac{3(1-x^2)}{4x^2}-\frac{5}{4} \\
&= \frac{9(x^2-1)}{4(5+x^2)}+\frac{3(1-x^2)}{4x^2} \\
&\geq \frac{3(x^2-1)}{8}+\frac{3(1-x^2)}{4} \\
&= \frac{3(1-x^2)}{8}\geq 0. \qquad (2.19)
\end{aligned}$$

Inequality (2.19) terminates the proof of inequality (2.17).

(ii) Second, we prove inequality (2.18).

$$\begin{aligned} &m_1 + \sqrt{3}a - \sqrt{3}s \\ &\quad = \frac{1}{2}\sqrt{4-x^2} - \frac{\sqrt{3}}{2}(2-x) \\ &\quad = \frac{1}{2}\sqrt{2-x}\bigl(\sqrt{2+x} - \sqrt{3(2-x)}\bigr) \\ &\quad = \frac{-2\sqrt{2-x}(1-x)}{\sqrt{2+x}+\sqrt{3(2-x)}} \le 0. \end{aligned} \tag{2.20}$$

Inequality (2.18) follows from inequality (2.20) immediately. □

Lemma 2.6 *In $\triangle ABC$, if $a \le b \le c$, then*

$$m_a m_b + m_b m_c + m_c m_a - 2m_1 m_2 - m_2^2 \ge \frac{3}{8}(b-c)^2 - \frac{3s(s-a)(b-c)^2}{16(s-b)(s-c)}. \tag{2.21}$$

Proof By the *AM-GM* inequality, the well-known inequalities $m_b \ge \sqrt{s(s-b)}$ and $m_c \ge \sqrt{s(s-c)}$, we get

$$(m_b + m_c)^2 \ge 4m_b m_c \ge 4s\sqrt{(s-b)(s-c)} \ge 6a\sqrt{(s-b)(s-c)} \ge 12(s-b)(s-c)$$

or

$$m_b + m_c \ge 2\sqrt{3}\sqrt{(s-b)(s-c)}. \tag{2.22}$$

By inequalities (2.4), (2.10), (2.11), (2.17), (2.22), we obtain that

$$\begin{aligned} &m_a m_b + m_b m_c + m_c m_a - 2m_1 m_2 - m_2^2 \\ &\quad = \frac{(m_b+m_c)(m_a^2-m_1^2)}{m_a+m_1} + \frac{m_1(m_b^2-m_2^2)}{m_b+m_2} + \frac{m_1(m_c^2-m_2^2)}{m_c+m_2} - \frac{(m_b^2-m_c^2)^2}{2(m_b+m_c)^2} + \frac{1}{16}(b-c)^2 \\ &\quad = \frac{(m_b+m_c)(b-c)^2}{4(m_a+m_1)} + \frac{m_1(5b+7c)(c-b)}{16(m_b+m_2)} + \frac{m_1(7b+5c)(b-c)}{16(m_c+m_2)} \\ &\qquad - \frac{9(b+c)^2(b-c)^2}{32(m_b+m_c)^2} + \frac{1}{16}(b-c)^2 \\ &\quad = \frac{(m_b+m_c)(b-c)^2}{4(m_a+m_1)} + \frac{m_1(b-c)^2}{16(m_b+m_2)} + \frac{m_1(b-c)^2}{16(m_c+m_2)} \\ &\qquad - \frac{9m_1(b+c)^2(b-c)^2}{32(m_b+m_2)(m_c+m_2)(m_b+m_c)} \\ &\qquad - \frac{9(b+c)^2(b-c)^2}{32(m_b+m_c)^2} + \frac{1}{16}(b-c)^2 \\ &\quad \ge \frac{1}{4}\left(\frac{m_2}{m_1} + \frac{m_1}{4m_2} - \frac{9a(b+c)^2}{64s(s-b)(s-c)} - \frac{m_1(b+c)^2}{64\sqrt{3}s(s-b)(s-c)} + \frac{1}{4}\right)(b-c)^2 \\ &\quad = \frac{1}{4}\left(\frac{m_2}{m_1} + \frac{m_1}{4m_2} - \frac{9(m_1+\sqrt{3}a)(b+c)^2}{64\sqrt{3}s(s-b)(s-c)} + \frac{1}{4}\right)(b-c)^2 \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{4}\left(\frac{m_2}{m_1}+\frac{m_1}{4m_2}-\frac{9(b+c)^2}{64(s-b)(s-c)}+\frac{1}{4}\right)(b-c)^2\\
&=\frac{1}{4}\left(\frac{m_2}{m_1}+\frac{m_1}{4m_2}-\frac{3[(b+c)^2-a^2]}{16(s-b)(s-c)}+\frac{3[(b+c)^2-4a^2]}{64(s-b)(s-c)}+\frac{1}{4}\right)(b-c)^2\\
&\geq \frac{1}{4}\left(\frac{m_2}{m_1}+\frac{m_1}{4m_2}-\frac{3s(s-a)}{4(s-b)(s-c)}+\frac{3[(b+c)^2-4a^2]}{16a^2}+\frac{1}{4}\right)(b-c)^2\\
&=\frac{1}{4}\left(\frac{m_2}{m_1}+\frac{m_1}{4m_2}+\frac{3(b+c)^2}{16a^2}-\frac{3s(s-a)}{4(s-b)(s-c)}-\frac{1}{2}\right)(b-c)^2\\
&\geq \frac{1}{4}\left(2-\frac{3s(s-a)}{4(s-b)(s-c)}-\frac{1}{2}\right)(b-c)^2\\
&=\frac{3}{8}(b-c)^2-\frac{3s(s-a)(b-c)^2}{16(s-b)(s-c)}.
\end{aligned}$$

The proof of Lemma 2.6 is thus completed. □

Lemma 2.7 *In $\triangle ABC$, if inequality* (1.1) *holds, then $k \leq 4$.*

Proof Let $b = c = 1$ and $a = x$. For $a \leq b \leq c$, we have $x \in (0,1]$, then inequality (1.1) is equivalent to

$$\begin{aligned}
&2\left(\frac{x\sqrt{4-x^2}}{2(2-x)}-\frac{\sqrt{4-x^2}}{2}\right)^2 \geq 2k\left(\frac{\sqrt{4-x^2}}{2}-\frac{\sqrt{2x^2+1}}{2}\right)^2\\
&\quad\Longleftrightarrow\quad \frac{2+x}{2-x}\geq k\cdot\frac{9(1+x)^2}{4(\sqrt{4-x^2}+\sqrt{2x^2+1})^2}\\
&\quad\Longleftrightarrow\quad k\leq\frac{4(2+x)(\sqrt{4-x^2}+\sqrt{2x^2+1})^2}{9(2-x)(1+x)^2}.
\end{aligned}\tag{2.23}$$

Taking $x = 1$ in inequality (2.23), we obtain that $k \leq 4$. □

Lemma 2.8 *In $\triangle ABC$, if $a \leq b \leq c$ and $0 < k \leq 4$, then we have*

$$\sum(r_b-r_c)^2-k\cdot\sum(m_b-m_c)^2\geq 2(r_1-m_1)^2-2k(m_1-m_2)^2.\tag{2.24}$$

Proof For

$$\sum(r_b-r_c)^2=2\sum r_a^2-2\sum r_br_c=2\sum r_a^2-2s^2$$

and

$$\sum(m_b-m_c)^2=2\sum m_a^2-2\sum m_bm_c=\frac{3}{2}\sum a^2-2\sum m_bm_c,$$

hence, by Lemmas 2.1 and 2.6, we have

$$\begin{aligned}
&\sum(r_b-r_c)^2-k\cdot\sum(m_b-m_c)^2-2(r_1-m_1)^2+2k(m_1-m_2)^2\\
&=2\left[\sum r_a^2-r_1^2-2m_1^2\right]+2k\left[\sum m_bm_c-2m_1m_2-m_2^2-\frac{3}{8}(b-c)^2\right]
\end{aligned}$$

$$\begin{aligned} &\geq \frac{3s(s-a)(b-c)^2}{2(s-b)(s-c)} - \frac{3ks(s-a)(b-c)^2}{8(s-b)(s-c)} \\ &= \frac{3(4-k)s(s-a)(b-c)^2}{8(s-b)(s-c)} \geq 0. \end{aligned}$$

The proof of Lemma 2.8 is complete. □

Lemma 2.9 (see [4, 6, 7]) *Define*

$$F(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

and

$$G(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m.$$

If $a_0 \neq 0$ or $b_0 \neq 0$, then the polynomials $F(x)$ and $G(x)$ have a common root if and only if

$$R(F,G) := \begin{vmatrix} a_0 & a_1 & \cdots & a_n & & & \\ & a_0 & a_1 & \cdots & a_n & & \\ & & \ddots & \ddots & & \ddots & \\ & & & a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_m & & & \\ & b_0 & b_1 & \cdots & b_m & & \\ & & \ddots & \ddots & & \ddots & \\ & & & b_0 & b_1 & \cdots & b_m \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n \end{matrix} = 0,$$

where $R(F,G)$ ($(m+n) \times (m+n)$ determinant) is Sylvester's resultant of $F(x)$ and $G(x)$.

Lemma 2.10 (see [7, 8]) *Given a polynomial $f(x)$ with real coefficients*

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

if the number of the sign changes in the revised sign list of its discriminant sequence

$$\{D_1(f), D_2(f), \ldots, D_n(f)\}$$

is v, then the number of the pairs of distinct conjugate imaginary roots of $f(x)$ equals v. Furthermore, if the number of non-vanishing members in the revised sign list is l, then the number of the distinct real roots of $f(x)$ equals $l - 2v$.

3 The proof of Theorem 1.1

Proof If $k \leq 0$,we can easily find that inequality (1.1) holds. Hence, we only need to consider the case $k > 0$, and by Lemma 2.7, we only need to consider the case $0 < k \leq 4$.

Now we determine the best constant k such that inequality (1.1) holds. Since inequality (1.1) is symmetrical with respect to the side-lengths a, b and c, there is no harm in supposing $a \leq b \leq c$. Thus, by Lemma 2.8, we only need to determine the best constant k such that

$$2(r_1 - m_1)^2 - 2k(m_1 - m_2)^2 \geq 0$$

or, equivalently, that

$$\left(\frac{a\sqrt{(b+c)^2 - a^2}}{2(b+c-a)} - \frac{\sqrt{(b+c)^2 - a^2}}{2}\right)^2 - k\left(\frac{\sqrt{(b+c)^2 - a^2}}{2} - \frac{1}{2}\sqrt{2a^2 + \frac{1}{4}(b+c)^2}\right)^2 \geq 0. \tag{3.1}$$

Without loss of generality, we can assume that

$$a = t \quad \text{and} \quad \frac{b+c}{2} = 1 \quad (0 < t \leq 1),$$

because inequality (3.1) is homogeneous with respect to a and $\frac{b+c}{2}$. Thus, clearly, inequality (3.1) is equivalent to the following inequality:

$$\left(\frac{t\sqrt{4-t^2}}{2(2-t)} - \frac{\sqrt{4-t^2}}{2}\right)^2 - k\left(\frac{\sqrt{4-t^2}}{2} - \frac{\sqrt{2t^2+1}}{2}\right)^2 \geq 0. \tag{3.2}$$

We consider the following two cases separately.

Case 1. When $t = 1$, inequality (3.2) holds true for any $k \in R := (-\infty, +\infty)$.

Case 2. When $0 < t < 1$, inequality (3.2) is equivalent to the following inequality:

$$k \leq \frac{4(2+t)(\sqrt{4-t^2} + \sqrt{2t^2+1})^2}{9(2-t)(1+t)^2}. \tag{3.3}$$

Define the function

$$g(t) := \frac{4(2+t)(\sqrt{4-t^2} + \sqrt{2t^2+1})^2}{9(2-t)(1+t)^2}, \quad x \in (0,1).$$

Calculating the derivative for $g(t)$, we get

$$g'(t) = \frac{8(\sqrt{4-t^2} + \sqrt{2t^2+1}) \cdot \sqrt{4-t^2} \cdot [(2t^3 + 5t^2 + 10t - 2) - (2-t)\sqrt{4-t^2} \cdot \sqrt{2t^2+1}]}{9(2-t)^2(1+t)^3\sqrt{2t^2+1} \cdot \sqrt{4-t^2}}.$$

By setting $g'(t) = 0$, we obtain

$$\sqrt{4-t^2} \cdot \left[\left(2t^3 + 5t^2 + 10t - 2\right) - (2-t)\sqrt{4-t^2} \cdot \sqrt{2t^2+1}\right] = 0. \tag{3.4}$$

It is easily observed that the equation $\sqrt{4-t^2} = 0$ has no real root on the interval $(0,1)$. Hence, the roots of equation (3.4) are also solutions of the following equation:

$$\left(2t^3 + 5t^2 + 10t - 2\right) - (2-t)\sqrt{4-t^2} \cdot \sqrt{2t^2+1} = 0,$$

that is,

$$(1+t)^2\varphi(t) = 0, \tag{3.5}$$

where

$$\varphi(t) = t^4 + 10t^2 - 2.$$

It is obvious that the equation

$$(1+t)^2 = 0 \tag{3.6}$$

has no real root on the interval $(0,1)$.

It is easy to find that the equation

$$\varphi(t) = 0 \tag{3.7}$$

has one positive real root. Moreover, it is not difficult to observe that $\varphi(0) = -2 < 0$ and $\varphi(1) = 9 > 0$. We can thus find that equation (3.7) has one distinct real root on the interval $(0,1)$. So that equation (3.4) has only one real root t_0 given by $t_0 = 0.442890982868958\ldots$ on the interval $(0,1)$, and

$$g(t)_{\max} = g(t_0) \approx 3.2817755127 \in (3,4). \tag{3.8}$$

Now we prove $g(t_0)$ is the root of equation (1.2). For this purpose, we consider the following nonlinear algebraic equation system:

$$\begin{cases} \varphi(t_0) = 0, \\ 2t_0^2 + 1 - u_0^2 = 0, \\ 4 - t_0^2 - v_0^2 = 0, \\ 4(2+t)(u_0+v_0)^2 - 9(2-t)(1+t)^2 k = 0. \end{cases} \tag{3.9}$$

It is easy to see that $g(t_0)$ is also the solution of nonlinear algebraic equation system (3.9). If we eliminate the v_0, u_0 and t_0 ordinal by the resultant (by using Lemma 2.9), then we get

$$29{,}648{,}323{,}021{,}629{,}456 \cdot \phi_1^2(k) \cdot \phi_2^2(k) = 0, \tag{3.10}$$

where

$$\phi_1(k) = 6{,}561k^4 - 14{,}256k^3 - 18{,}080k^2 - 25{,}344k + 20{,}736$$

and

$$\phi_2(k) = 729k^4 - 7{,}344k^3 + 8{,}800k^2 - 13{,}056k + 2{,}304.$$

The revised sign list of the discriminant sequence of $\phi_1(k)$ is given by

$$[1, 1, -1, -1]. \tag{3.11}$$

The revised sign list of the discriminant sequence of $\phi_2(k)$ is given by

$$[1, 1, -1, -1]. \tag{3.12}$$

So the number of sign changes in the revised sign list of (3.11) and (3.12) are both 2. Thus, by applying Lemma 2.10, we find that the equations

$$\phi_1(k) = 0 \tag{3.13}$$

and

$$\phi_2(k) = 0 \tag{3.14}$$

both have two distinct real roots. In addition, it is easy to find that

$$\begin{aligned} &\phi_1(0) = 20{,}736 > 0; \qquad \phi_2(0) = 2{,}304 > 0, \\ &\phi_1(1) = -30{,}383 < 0; \qquad \phi_2(1) = -8{,}567 < 0, \\ &\phi_1(3) = -71{,}487 < 0; \qquad \phi_2(8) = -313{,}088 < 0 \end{aligned}$$

and

$$\phi_1(4) = 397{,}312 > 0; \qquad \phi_2(9) = 26{,}793 > 0.$$

We can thus find that equation (3.13) has two distinct real roots on the intervals

$$(0, 1) \quad \text{and} \quad (3, 4).$$

And equation (3.14) has two distinct real roots on the intervals

$$(0, 1) \quad \text{and} \quad (8, 9).$$

Hence, by (3.8), we can conclude that $g(t_0)$ is the root of equation (1.2). The proof of Theorem 1.1 is thus completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and read and approved the final manuscript.

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