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Asymptotic decay estimate of solutions to the generalized damped Bq equation

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Abstract

In this paper, we investigate the Cauchy problem for the generalized damped Bq equation. By applying fixed point theorem, we prove the global existence and asymptotic decay estimate of solutions for all space dimensions $n \geq 1$ provided that the initial value is suitably small.

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Keywords: generalized damped Boussinesq equation; fixed point theorem; decay estimate

1 Introduction

We study asymptotic decay estimate of solution to the Cauchy problem for the generalized damped Bq equation:

$$u_{tt} - \Delta u - 2b\Delta u_t + \alpha \Delta^2 u = \Delta f(u) \quad (1.1)$$

with the initial value

$$t = 0: \quad u = u_0(x), \quad u_t = u_1(x). \quad (1.2)$$

Here $u = u(x, t)$ is the unknown function of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $t > 0$, $b > 0$ and $\alpha > 0$ are constants. The nonlinear term $f(u)$ is a given smooth function of $u \in \mathbb{R}$ satisfying $f(u) = O(u^2)$ for $u \rightarrow 0$.

It is well known that the equation (called the classical Bq equation)

$$u_{tt} + \alpha \Delta^2 u - \Delta u = \beta \Delta(u^2) \quad (1.3)$$

was derived by Boussinesq [1] in 1872 to describe shallow water waves, where $u(x, t)$ is an elevation of the free surface of fluid and the constant coefficients α and β depend on the depth of fluid and the characteristic speed of long waves. It is interesting to note that this equation governs nonlinear string oscillations as well. For (1.3) and the generalized Bq equation, there are lots of important results (see [2–5]).

Equation (1.3) takes into account dispersion and nonlinearity, but in real processes viscosity also plays an important role. Varlamov considered the following damped Boussinesq equation (see [6–8]):

$$u_{tt} - \Delta u - 2b\Delta u_t + \alpha \Delta^2 u = \beta \Delta(u^2), \quad (1.4)$$

where $b > 0$ and $\alpha > 0$ are constants. A classical solution to the Cauchy problem for (1.4) with small initial data was constructed by means of the application of both spectral and perturbation theories. Large time asymptotics of this solution was also obtained (see [6]). Varlamov [7] investigated the Cauchy problem for (1.4). For the cases of one, two and three space dimensions local in time existence and uniqueness of a solution is proved. The authors also showed that for discontinuous initial perturbations this solution is infinitely differentiable with respect to time t and space co-ordinates for $t > 0$ on a bounded time interval. Varlamov [8] considered the Cauchy problem for (1.4) with small initial data in two space dimensions. Existence and uniqueness of the classical solution was proved and the solution was constructed in the form of a series. The major term of its long-time asymptotics is calculated explicitly and a uniform in space estimate of the residual term was given.

The main purpose of this paper is to establish the following asymptotic decay estimate of solutions to (1.1), (1.2) for $n \geq 1$:

$$\|\partial_x^k u(t)\|_{H^{s+2-k}} \leq CE_0(1+t)^{-\frac{n}{4}-\frac{k}{2}-\frac{1}{2}}$$

for $0 \leq k \leq s + 2$ and $s \geq \max\{0, \lfloor \frac{n}{2} \rfloor - 1\}$. Here $E_0 := \|u_0\|_{\dot{W}^{-1,1}} + \|u_1\|_{\dot{W}^{-2,1}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}$.

Decay estimate of solutions to hyperbolic-type equations has been investigated by many authors. We refer to [9, 10] for hyperbolic equations, to [11–15] for damped wave equation and to [16–19] for various aspects of dissipation of the plate equation.

The paper is organized as follows. In Section 2 we derive the solution formula of our semi-linear problem. We study the decay property of the solution operators appearing in the solution formula in Section 3. In Section 4, asymptotic decay estimate of solutions to the cauchy problem (1.1), (1.2) is established by applying fixed point theorem.

Notations We give some notations which are used in this paper. Let $\mathcal{F}[u]$ denote the Fourier transform of u defined by

$$\hat{u}(\xi) = \mathcal{F}[u] = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) dx,$$

and we denote its inverse transform by \mathcal{F}^{-1} .

For $1 \leq p \leq \infty$, $L^p = L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. The usual Sobolev space of s is defined by $W^{s,p} = (I - \Delta)^{-\frac{s}{2}} L^p$ with the norm $\|f\|_{W^{s,p}} = \|(I - \Delta)^{\frac{s}{2}} f\|_{L^p}$; the homogeneous Sobolev space of s is defined by $\dot{W}^{s,p} = (-\Delta)^{-\frac{s}{2}} L^p$ with the norm $\|f\|_{\dot{W}^{s,p}} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^p}$; especially $H^s = W^{s,2}$, $\dot{H}^s = \dot{W}^{s,2}$. Moreover, we know that $W^{s,p} = L^p \cap \dot{W}^{s,p}$ for $s \geq 0$.

Finally, in this paper, we denote every positive constant by the same symbol C or c without confusion. $[\cdot]$ is the Gauss symbol.

2 Solution formula

The aim of this section is to derive the solution formula for the problem (1.1), (1.2). We first investigate the linearized equation of (1.1).

$$u_{tt} - \Delta u - 2b\Delta u_t + \alpha \Delta^2 u = 0$$

with the initial data in (1.2). Taking the Fourier transform, we have

$$\hat{u}_{tt} + 2b|\xi|^2\hat{u}_t + (|\xi|^2 + \alpha|\xi|^4)\hat{u} = 0. \tag{2.1}$$

The corresponding initial values are given as

$$t = 0: \quad \hat{u} = \hat{u}_0(\xi), \quad \hat{u}_t = \hat{u}_1(\xi). \tag{2.2}$$

The characteristic equation of (2.1) is

$$\lambda^2 + 2b|\xi|^2\lambda + |\xi|^2 + \alpha|\xi|^4 = 0. \tag{2.3}$$

Let $\lambda = \lambda_{\pm}(\xi)$ be the corresponding eigenvalues of (2.3), we obtain

$$\lambda_{\pm}(\xi) = -b|\xi|^2 \pm |\xi|\sqrt{(b^2 - \alpha)|\xi|^2 - 1}. \tag{2.4}$$

The solution to the problem (2.1)-(2.2) is given in the form

$$\hat{u}(\xi, t) = \hat{G}(\xi, t)\hat{u}_1(\xi) + \hat{H}(\xi, t)\hat{u}_0(\xi), \tag{2.5}$$

where

$$\hat{G}(\xi, t) = \frac{1}{\lambda_+(\xi) - \lambda_-(\xi)} (e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}) \tag{2.6}$$

and

$$\hat{H}(\xi, t) = \frac{1}{\lambda_+(\xi) - \lambda_-(\xi)} (\lambda_+(\xi)e^{\lambda_-(\xi)t} - \lambda_-(\xi)e^{\lambda_+(\xi)t}). \tag{2.7}$$

We define $G(x, t)$ and $H(x, t)$ by

$$G(x, t) = \mathcal{F}^{-1}[\hat{G}(\xi, t)](x) \tag{2.8}$$

and

$$H(x, t) = \mathcal{F}^{-1}[\hat{H}(\xi, t)](x), \tag{2.9}$$

respectively, where \mathcal{F}^{-1} denotes the inverse Fourier transform. Then, applying \mathcal{F}^{-1} to (2.5), we obtain

$$u(t) = G(t) * u_1 + H(t) * u_0. \tag{2.10}$$

By the Duhamel principle, we obtain the solution formula to (1.1), (1.2)

$$u(t) = G(t) * u_1 + H(t) * u_0 + \int_0^t G(t - \tau) * \Delta f(u)(\tau) d\tau. \tag{2.11}$$

3 Decay property

The aim of this section is to establish decay estimates of the solution operators $G(t)$ and $H(t)$ appearing in the solution formula (2.10).

Lemma 3.1 *The solution of the problem (2.1), (2.2) satisfies*

$$(|\xi|^2 + |\xi|^4)|\hat{u}(\xi, t)|^2 + |\hat{u}_t(\xi, t)|^2 \leq Ce^{-c|\xi|^2 t} ((|\xi|^2 + |\xi|^4)|\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2) \tag{3.1}$$

for $\xi \in \mathbb{R}^n$ and $t \geq 0$.

Proof Multiplying (2.1) by $\bar{\hat{u}}_t$ and taking the real part yields

$$\frac{1}{2} \frac{d}{dt} \{ |\hat{u}_t|^2 + |\xi|^2 |\hat{u}|^2 + \alpha |\xi|^4 |\hat{u}|^2 \} + 2b|\xi|^2 |\hat{u}_t|^2 = 0. \tag{3.2}$$

Multiplying (2.1) by $\bar{\hat{u}}$ and taking the real part, we obtain

$$\frac{1}{2} \frac{d}{dt} \{ 2b|\xi|^2 |\hat{u}|^2 + 2 \operatorname{Re}(\hat{u}_t \bar{\hat{u}}) \} + |\xi|^2 |\hat{u}|^2 + \alpha |\xi|^4 |\hat{u}|^2 - |\hat{u}_t|^2 = 0. \tag{3.3}$$

Multiplying both sides of (3.3) by $b|\xi|^2$ and summing up the resulting equation and (3.2) yield

$$\frac{d}{dt} E + F = 0, \tag{3.4}$$

where

$$E = \frac{1}{2} |\hat{u}_t|^2 + \frac{1}{2} |\xi|^2 |\hat{u}|^2 + \frac{\alpha}{2} |\xi|^4 |\hat{u}|^2 + b^2 |\xi|^4 |\hat{u}|^2 + b|\xi|^2 \operatorname{Re}(\hat{u}_t \bar{\hat{u}})$$

and

$$F = b|\xi|^2 |\hat{u}_t|^2 + b|\xi|^4 |\hat{u}|^2 + b\alpha |\xi|^6 |\hat{u}|^2.$$

A simple computation implies that

$$CE_0 \leq E \leq CE_0, \tag{3.5}$$

where

$$E_0 = (|\xi|^2 + |\xi|^4)|\hat{u}|^2 + |\hat{u}_t|^2.$$

Note that

$$F \geq c|\xi|^2 E_0.$$

It follows from (3.5) that

$$F \geq c|\xi|^2 E. \tag{3.6}$$

Using (3.4) and (3.6), we get

$$\frac{d}{dt}E + c|\xi|^2 E \leq 0.$$

Thus

$$E(\xi, t) \leq e^{-c|\xi|^2 t} E(\xi, 0),$$

which together with (3.5) proves the desired estimates (3.1). Then we have completed the proof of the lemma. \square

Lemma 3.2 *Let $\hat{G}(\xi, t)$ and $\hat{H}(\xi, t)$ be the fundamental solutions of the linear equation of (1.1) in the Fourier space, which are given in (2.6) and (2.7), respectively. Then we have the estimates*

$$(|\xi|^2 + |\xi|^4) |\hat{G}(\xi, t)|^2 + |\hat{G}_t(\xi, t)|^2 \leq C e^{-c|\xi|^2 t} \tag{3.7}$$

and

$$(|\xi|^2 + |\xi|^4) |\hat{H}(\xi, t)|^2 + |\hat{H}_t(\xi, t)|^2 \leq C (|\xi|^2 + |\xi|^4) e^{-c|\xi|^2 t} \tag{3.8}$$

for $\xi \in \mathbb{R}^n$ and $t \geq 0$.

Proof If $\hat{u}_0(\xi) = 0$, from (2.5), we obtain

$$\hat{u}(\xi, t) = \hat{G}(\xi, t) \hat{u}_1(\xi), \quad \hat{u}_t(\xi, t) = \hat{G}_t(\xi, t) \hat{u}_1(\xi).$$

Substituting the equalities into (3.1) with $\hat{u}_0(\xi) = 0$, we get (3.7).

In what follows, we consider $\hat{u}_1(\xi) = 0$, it follows from (2.5) that

$$\hat{u}(\xi, t) = \hat{H}(\xi, t) \hat{u}_0(\xi), \quad \hat{u}_t(\xi, t) = \hat{H}_t(\xi, t) \hat{u}_0(\xi).$$

Substituting the equalities into (3.1) with $\hat{u}_1(\xi) = 0$, we get the desired estimate (3.8). The lemma is proved. \square

Lemma 3.3 *Let $G(x, t)$ and $H(x, t)$ be the fundamental solutions of the linear equation of (1.1), which are given in (2.8) and (2.9), respectively. Then, for $k \geq 0$ and $j \geq 0$, we have*

$$\|\partial_x^k G(t) * \phi\|_{L^2} \leq C(1+t)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{j}{2} - \frac{1}{2}\right)} \|\phi\|_{\dot{W}^{-j,1}} + C e^{-ct} \|\partial_x^{(k-2)_+} \phi\|_{L^2}, \tag{3.9}$$

$$\|\partial_x^k H(t) * \phi\|_{L^2} \leq C(1+t)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{j}{2}\right)} \|\phi\|_{\dot{W}^{-j,1}} + C e^{-ct} \|\partial_x^k \phi\|_{L^2}, \tag{3.10}$$

$$\|\partial_x^k G_t(t) * \phi\|_{L^2} \leq C(1+t)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{j}{2}\right)} \|\phi\|_{\dot{W}^{-j,1}} + C e^{-ct} \|\partial_x^k \phi\|_{L^2}, \tag{3.11}$$

$$\|\partial_x^k H_t(t) * \phi\|_{L^2} \leq C(1+t)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{j}{2} + \frac{1}{2}\right)} \|\phi\|_{\dot{W}^{-j,1}} + C e^{-ct} \|\partial_x^{(k+2)} \phi\|_{L^2} \tag{3.12}$$

and

$$\|\partial_x^k G(t) * \Delta g\|_{L^2} \leq C(1+t)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{1}{2}\right)} \|g\|_{L^1} + Ce^{-ct} \|\partial_x^k g\|_{L^2}, \tag{3.13}$$

$$\|\partial_x^k G_t(t) * \Delta g\|_{L^2} \leq C(1+t)^{-\left(\frac{n}{4} + \frac{k}{2} + 1\right)} \|g\|_{L^1} + Ce^{-ct} \|\partial_x^{(k+2)} g\|_{L^2}, \tag{3.14}$$

where $(k-2)_+ = \max\{0, k-2\}$ in (3.9).

Proof By the Plancherel theorem and (3.7), the Hausdorff-Young inequality, we obtain

$$\begin{aligned} \|\partial_x^k G(t) * \phi\|_{L^2}^2 &= \int_{|\xi| \leq R_0} |\xi|^{2k} |\hat{G}(\xi, t)|^2 |\hat{\phi}(\xi)|^2 d\xi \\ &\quad + \int_{|\xi| \geq R_0} |\xi|^{2k} |\hat{G}(\xi, t)|^2 |\hat{\phi}(\xi)|^2 d\xi \\ &\leq \int_{|\xi| \leq R_0} |\xi|^{2k-2} e^{-c|\xi|^2 t} |\hat{\phi}(\xi)|^2 d\xi \\ &\quad + Ce^{-ct} \int_{|\xi| \geq R_0} |\xi|^{2k} (|\xi|^2 + |\xi|^4)^{-1} |\hat{\phi}(\xi)|^2 d\xi \\ &\leq \int_{|\xi| \leq R_0} |\xi|^{2k-2+2j} e^{-c|\xi|^2 t} |\xi|^{-2j} |\hat{\phi}(\xi)|^2 d\xi + Ce^{-ct} \|\partial_x^{(k-2)_+} \phi\|_{L^2}^2 \\ &\leq C \|\xi^{-j} \hat{\phi}(\xi)\|_{L^\infty}^2 \int_{|\xi| \leq R_0} |\xi|^{2k-2+2j} e^{-c|\xi|^2 t} d\xi + Ce^{-ct} \|\partial_x^{(k-2)_+} \phi\|_{L^2}^2 \\ &\leq C(1+t)^{-\left(\frac{n}{2} + k + j - 1\right)} \|(-\Delta)^{-\frac{j}{2}} \phi\|_{L^1}^2 + Ce^{-ct} \|\partial_x^{(k-2)_+} \phi\|_{L^2}^2, \end{aligned}$$

where $(k-2)_+ = \max\{0, k-2\}$ and R_0 is a small positive constant. Thus (3.9) follows.

Similarly, using (3.7) and (3.8), respectively, we can prove (3.10)-(3.12).

In what follows, we prove (3.13). By the Plancherel theorem, (3.7) and the Hausdorff-Young inequality, we have

$$\begin{aligned} \|\partial_x^k G(t) * \Delta g\|_{L^2}^2 &= \int_{|\xi| \leq R_0} |\xi|^{2|k|} |\hat{G}(\xi, t)|^2 |\xi|^4 |\hat{g}(\xi)|^2 d\xi \\ &\quad + \int_{|\xi| \geq R_0} |\xi|^{2k} |\hat{G}(\xi, t)|^2 |\xi|^4 |\hat{g}(\xi)|^2 d\xi \\ &\leq \int_{|\xi| \leq R_0} |\xi|^{2k+2} e^{-c|\xi|^2 t} |\hat{g}(\xi)|^2 d\xi + Ce^{-ct} \int_{|\xi| \geq R_0} |\xi|^{2k} |\hat{g}(\xi)|^2 d\xi \\ &\leq C \|\hat{g}(\xi)\|_{L^\infty}^2 \int_{|\xi| \leq R_0} |\xi|^{2k+2} e^{-c|\xi|^2 t} d\xi + Ce^{-ct} \|\partial_x^k g\|_{L^2}^2 \\ &\leq C(1+t)^{-\left(\frac{n}{2} + k + 1\right)} \|g\|_{L^1}^2 + Ce^{-ct} \|\partial_x^k g\|_{L^2}^2, \end{aligned}$$

where R_0 is a small positive constant. Thus (3.13) follows. Similarly, we can prove (3.14). Thus we have completed the proof of the lemma. \square

4 Asymptotic decay of solutions to (1.1), (1.2)

The purpose of this section is to prove asymptotic decay of solutions to the Cauchy problem (1.1), (1.2). We need the following lemma, which comes from [20] (see also [21]).

Lemma 4.1 *Assume that $f = f(v)$ is a smooth function. Suppose that $f(v) = O(|v|^{1+\theta})$ ($\theta \geq 1$ is an integer) when $|v| \leq v_0$. Then, for integer $m \geq 0$, if $v, w \in W^{m,q}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\|v\|_{L^\infty} \leq v_0, \|w\|_{L^\infty} \leq v_0$, then $f(v) - f(w) \in W^{m,r}(\mathbb{R}^n)$. Furthermore, the following inequalities hold:*

$$\|\partial_x^m f(v)\|_{L^r} \leq C \|v\|_{L^p} \|\partial_x^m v\|_{L^q} \|v\|_{L^\infty}^{\theta-1}$$

and

$$\begin{aligned} \|\partial_x^m (f(v) - f(w))\|_{L^r} \leq C \{ & (\|\partial_x^m v\|_{L^q} + \|\partial_x^m w\|_{L^q}) \|v - w\|_{L^p} \\ & + (\|v\|_{L^p} + \|w\|_{L^p}) \|\partial_x^m (v - w)\|_{L^q} \} (\|v\|_{L^\infty} + \|w\|_{L^\infty})^{\theta-1}, \end{aligned}$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, 1 \leq p, q, r \leq +\infty$.

In the previous section, we showed the decay estimates for the solution operators. With this preparation, we can prove the global existence and asymptotic decay of solutions to the integral equation (2.11) and hence to the problem (1.1), (1.2). So we define the following solution space:

$$X = \{u \in C([0, \infty); H^{s+2}(\mathbb{R}^n)) \cap C^1([0, \infty); H^s(\mathbb{R}^n)) : \|u\|_X < \infty\},$$

where

$$\|u\|_X = \sup_{t \geq 0} \left\{ \sum_{k \leq s+2} (1+t)^{\frac{n}{4} + \frac{k}{2} + \frac{1}{2}} \|\partial_x^k u(t)\|_{L^2} + \sum_{l \leq s} (1+t)^{\frac{n}{4} + \frac{l}{2} + 1} \|\partial_x^l u_t(t)\|_{L^2} \right\}.$$

For $R > 0$, we define

$$X_R = \{u \in X : \|u\|_X \leq R\}.$$

Using Gagliardo-Nirenberg inequality, we obtain

$$\|u(t)\|_{L^\infty} \leq C \|\partial_x^{s_0} u(t)\|_{L^2}^\theta \|u(t)\|_{L^2}^{1-\theta} \leq CR(1+t)^{-\frac{n}{4} - \frac{1}{2}}, \tag{4.1}$$

where

$$s_0 = \left[\frac{n}{2} \right] + 1, \quad \theta = \frac{n}{2s_0}, \quad s_0 \leq s + 2 \left(\text{i.e., } s \geq \left[\frac{n}{2} \right] - 1 \right).$$

Theorem 4.1 *Assume that $u_0 \in H^{s+2}(\mathbb{R}^n) \cap \dot{W}^{-1,1}(\mathbb{R}^n), u_1 \in H^s(\mathbb{R}^n) \cap \dot{W}^{-2,1}(\mathbb{R}^n)$ ($s \geq \max\{0, [\frac{n}{2}] - 1\}$). Put*

$$E_0 := \|u_0\|_{\dot{W}^{-1,1}} + \|u_1\|_{\dot{W}^{-2,1}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}.$$

If E_0 is suitably small, the Cauchy problem (1.1), (1.2) has a unique global classical solution $u(x, t)$ satisfying

$$u \in C^0([0, \infty); H^{s+2}(\mathbb{R}^n)) C^1([0, \infty); H^s(\mathbb{R}^n)).$$

Moreover, the solution satisfies the decay estimate

$$\|\partial_x^k u(t)\|_{L^2} \leq CE_0(1+t)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{1}{2}\right)} \tag{4.2}$$

and

$$\|\partial_x^l u_t(t)\|_{L^2} \leq CE_0(1+t)^{-\left(\frac{n}{4} + \frac{l}{2} + 1\right)} \tag{4.3}$$

for $k \leq s + 2$ and $l \leq s$.

Proof Define the mapping

$$\mathbb{T}(u) = G(t) * u_1 + H(t) * u_0 + \int_0^t G(t-\tau) * \Delta f(u(\tau)) d\tau. \tag{4.4}$$

Using (3.9) and (3.10) with $j = 2, 1$, respectively, (3.13), Lemma 4.1 and (4.1), for $k \leq s + 2$, we obtain

$$\begin{aligned} \|\partial_x^k \mathbb{T}(u)\|_{L^2} &\leq C \|\partial_x^k G(t) * u_1\|_{L^2} + C \|\partial_x^k H(t) * u_0\|_{L^2} \\ &\quad + C \int_0^t \|\partial_x^k G(t-\tau) * \Delta f(u(\tau))\|_{L^2} d\tau \\ &\leq C(1+t)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{1}{2}\right)} (\|u_0\|_{\dot{W}^{-1,1}} + \|u_1\|_{\dot{W}^{-2,1}}) + Ce^{-ct} (\|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}) \\ &\quad + C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{1}{2}\right)} \|f(u)\|_{L^1} d\tau \\ &\quad + C \int_{\frac{t}{2}}^t (1+t-\tau)^{-\left(\frac{n}{4} + \frac{1}{2}\right)} \|\partial_x^k f(u)\|_{L^1} d\tau \\ &\quad + C \int_0^t e^{-c(t-\tau)} \|\partial_x^k f(u)\|_{L^2} d\tau \\ &\leq C(1+t)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{1}{2}\right)} (\|u_0\|_{\dot{W}^{-1,1}} + \|u_1\|_{\dot{W}^{-2,1}}) + Ce^{-ct} (\|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}) \\ &\quad + C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{1}{2}\right)} \|u\|_{L^2}^2 d\tau \\ &\quad + C \int_{\frac{t}{2}}^t (1+t-\tau)^{-\left(\frac{n}{4} + \frac{1}{2}\right)} \|u\|_{L^2} \|\partial_x^k u\|_{L^2} d\tau \\ &\quad + C \int_0^t e^{-c(t-\tau)} \|\partial_x^k u\|_{L^2} \|u\|_{L^\infty} d\tau \\ &\leq C(1+t)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{1}{2}\right)} (\|u_0\|_{\dot{W}^{-1,1}} + \|u_1\|_{\dot{W}^{-2,1}}) + Ce^{-ct} (\|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}) \\ &\quad + CR^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{1}{2}\right)} (1+\tau)^{-\left(\frac{n}{2} + 1\right)} d\tau \\ &\quad + CR^2 \int_{\frac{t}{2}}^t (1+t-\tau)^{-\left(\frac{n}{4} + \frac{1}{2}\right)} (1+\tau)^{-\left(\frac{n}{2} + \frac{k}{2} + 1\right)} d\tau \\ &\quad + CR^2 \int_0^t e^{-c(t-\tau)} (1+\tau)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{1}{2}\right)} (1+\tau)^{-\left(\frac{n}{2} + \frac{1}{2}\right)} d\tau \\ &\leq C(1+t)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{1}{2}\right)} \{ (\|u_0\|_{\dot{W}^{-1,1}} + \|u_1\|_{\dot{W}^{-2,1}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}) + R^2 \}. \end{aligned}$$

Thus

$$(1+t)^{\frac{n}{4}+\frac{k}{2}+\frac{1}{2}} \|\partial_x^k \mathbb{T}(u)\|_{L^2} \leq CE_0 + CR^2. \tag{4.5}$$

It follows from (4.4) that

$$\mathbb{T}(u)_t = G_t(t) * u_1 + H_t(t) * u_0 + \int_0^t G_t(t-\tau) * \Delta f(u(\tau)) \, d\tau. \tag{4.6}$$

By using (3.11) and (3.12) with $j = 2, 1$, respectively, (3.14) Lemma 4.1 and (4.1), for $l \leq s$, we have

$$\begin{aligned} \|\partial_x^l \mathbb{T}(u)_t\|_{L^2} &\leq C \|\partial_x^l G_t(t) * u_1\|_{L^2} + C \|\partial_x^l H_t(t) * u_0\|_{L^2} \\ &\quad + C \int_0^t \|\partial_x^l G_t(t-\tau) * \Delta f(u(\tau))\|_{L^2} \, d\tau \\ &\leq C(1+t)^{-\left(\frac{n}{4}+\frac{l}{2}+1\right)} (\|u_0\|_{\dot{W}^{-1,1}} + \|u_1\|_{\dot{W}^{-2,1}}) + Ce^{-ct} (\|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}) \\ &\quad + C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\left(\frac{n}{4}+\frac{l}{2}+1\right)} \|f(u)\|_{L^1} \, d\tau \\ &\quad + C \int_{\frac{t}{2}}^t (1+t-\tau)^{-\left(\frac{n}{4}+1\right)} \|\partial_x^l f(u)\|_{L^1} \, d\tau + C \int_0^t e^{-c(t-\tau)} \|\partial_x^{l+2} f(u)\|_{L^2} \, d\tau \\ &\leq C(1+t)^{-\left(\frac{n}{4}+\frac{l}{2}+1\right)} (\|u_0\|_{\dot{W}^{-1,1}} + \|u_1\|_{\dot{W}^{-2,1}}) + Ce^{-ct} (\|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}) \\ &\quad + C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\left(\frac{n}{4}+\frac{l}{2}+1\right)} \|u\|_{L^2}^2 \, d\tau \\ &\quad + C \int_{\frac{t}{2}}^t (1+t-\tau)^{-\left(\frac{n}{4}+1\right)} \|u\|_{L^2} \|\partial_x^l u\|_{L^2} \, d\tau \\ &\quad + C \int_0^t e^{-c(t-\tau)} \|\partial_x^{l+2} u\|_{L^2} \|u\|_{L^\infty} \, d\tau \\ &\leq C(1+t)^{-\left(\frac{n}{4}+\frac{l}{2}+1\right)} (\|u_0\|_{\dot{W}^{-1,1}} + \|u_1\|_{\dot{W}^{-2,1}}) + Ce^{-ct} (\|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}) \\ &\quad + CR^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-\left(\frac{n}{4}+\frac{l}{2}+1\right)} (1+\tau)^{-\left(\frac{n}{2}+1\right)} \, d\tau \\ &\quad + CR^2 \int_{\frac{t}{2}}^t (1+t-\tau)^{-\left(\frac{n}{4}+1\right)} (1+\tau)^{-\left(\frac{n}{2}+\frac{l}{2}+1\right)} \, d\tau \\ &\quad + CR^2 \int_0^t e^{-c(t-\tau)} (1+\tau)^{-\left(\frac{n}{4}+\frac{l+2}{2}+\frac{1}{2}\right)} (1+\tau)^{-\left(\frac{n}{2}+\frac{1}{2}\right)} \, d\tau \\ &\leq C(1+t)^{-\left(\frac{n}{4}+\frac{l}{2}+1\right)} \{ (\|u_0\|_{\dot{W}^{-1,1}} + \|u_1\|_{\dot{W}^{-2,1}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s}) + R^2 \}. \end{aligned}$$

Thus

$$(1+t)^{\frac{n}{4}+\frac{l}{2}+1} \|\partial_x^l \mathbb{T}(u)_t\|_{L^2} \leq CE_0 + CR^2. \tag{4.7}$$

Combining (4.5), (4.7) and taking E_0 and R suitably small yields

$$\|\mathbb{T}(u)\|_X \leq R. \tag{4.8}$$

For $\tilde{u}, \bar{u} \in X_R$, by using (4.4), we have

$$\mathbb{T}(\tilde{u}) - \mathbb{T}(\bar{u}) = \int_0^t G(t - \tau) * \Delta[f(\tilde{u}) - f(\bar{u})] d\tau. \tag{4.9}$$

Thanks to (4.9), (3.13) and Lemma 4.1, (4.1), for $k \leq s + 2$, we obtain

$$\begin{aligned} \|\partial_x^k(\mathbb{T}(\tilde{u}) - \mathbb{T}(\bar{u}))\|_{L^2} &\leq \int_0^t \|\partial_x^k G(t - \tau) * \Delta[f(\tilde{u}) - f(\bar{u})]\|_{L^2} d\tau \\ &\leq C \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{1}{2}\right)} \|f(\tilde{u}) - f(\bar{u})\|_{L^1} d\tau \\ &\quad + C \int_{\frac{t}{2}}^t (1 + t - \tau)^{-\left(\frac{n}{4} + \frac{1}{2}\right)} \|\partial_x^k(f(\tilde{u}) - f(\bar{u}))\|_{L^1} d\tau \\ &\quad + C \int_0^t e^{-c(t-\tau)} \|\partial_x^k(f(\tilde{u}) - f(\bar{u}))\|_{L^2} d\tau \\ &\leq C \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{1}{2}\right)} (\|\tilde{u}\|_{L^2} + \|\bar{u}\|_{L^2}) \|\tilde{u} - \bar{u}\|_{L^2} \\ &\quad + C \int_{\frac{t}{2}}^t (1 + t - \tau)^{-\left(\frac{n}{4} + \frac{1}{2}\right)} \{(\|\partial_x^k \tilde{u}\|_{L^2} + \|\partial_x^k \bar{u}\|_{L^2}) \|\tilde{u} - \bar{u}\|_{L^2} \\ &\quad + (\|\tilde{u}\|_{L^2} + \|\bar{u}\|_{L^2}) \|\partial_x^k(\tilde{u} - \bar{u})\|_{L^2}\} d\tau \\ &\quad + C \int_0^t e^{-c(t-\tau)} \{(\|\partial_x^k \tilde{u}\|_{L^2} + \|\partial_x^k \bar{u}\|_{L^2}) \|\tilde{u} - \bar{u}\|_{L^\infty} \\ &\quad + (\|\tilde{u}\|_{L^\infty} + \|\bar{u}\|_{L^\infty}) \|\partial_x^k(\tilde{u} - \bar{u})\|_{L^2}\} d\tau \\ &\leq CR \|\tilde{u} - \bar{u}\|_X \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{1}{2}\right)} (1 + \tau)^{-\left(\frac{n}{2} + 1\right)} d\tau \\ &\quad + CR \|\tilde{u} - \bar{u}\|_X \int_{\frac{t}{2}}^t (1 + t - \tau)^{-\left(\frac{n}{4} + \frac{1}{2}\right)} (1 + \tau)^{-\left(\frac{n}{2} + \frac{k}{2} + 1\right)} d\tau \\ &\quad + CCR \|\tilde{u} - \bar{u}\|_X \int_0^t e^{-c(t-\tau)} (1 + \tau)^{-\left(\frac{3n}{4} + \frac{k}{2} + 1\right)} d\tau \\ &\leq CR(1 + t)^{-\left(\frac{n}{4} + \frac{k}{2} + \frac{1}{2}\right)} \|\tilde{u} - \bar{u}\|_X, \end{aligned}$$

which implies

$$(1 + t)^{\frac{n}{4} + \frac{k}{2} + \frac{1}{2}} \|\partial_x^k(\mathbb{T}(\tilde{u}) - \mathbb{T}(\bar{u}))\|_{L^2} \leq CR \|\tilde{u} - \bar{u}\|_X. \tag{4.10}$$

Similarly, for $l \leq s$, from (4.6), (3.14) and (4.1), we have

$$\begin{aligned} \|\partial_x^l(\mathbb{T}(\tilde{u}) - \mathbb{T}(\bar{u}))_t\|_{L^2} &\leq \int_0^t \|\partial_x^l G_t(t - \tau) * \Delta[f(\tilde{u}) - f(\bar{u})]\|_{L^2} d\tau \\ &\leq C \int_0^{\frac{t}{2}} (1 + t - \tau)^{-\frac{n}{4} - \frac{l}{2} - 1} \|f(\tilde{u}) - f(\bar{u})\|_{L^1} d\tau \\ &\quad + C \int_{\frac{t}{2}}^t (1 + t - \tau)^{-\frac{n}{4} - 1} \|\partial_x^l(f(\tilde{u}) - f(\bar{u}))\|_{L^1} d\tau \end{aligned}$$

$$\begin{aligned} &+ C \int_0^t e^{-c(t-\tau)} \|\partial_x^{l+2}(f(\tilde{u}) - f(\bar{u}))\|_{L^2} d\tau \\ &\leq CR(1+t)^{-\left(\frac{n}{4} + \frac{l}{2} + 1\right)} \|\tilde{u} - \bar{u}\|_X, \end{aligned}$$

which implies

$$(1+t)^{\frac{n}{4} + \frac{l}{2} + 1} \|\partial_x^l(\mathbb{T}(\tilde{u}) - \mathbb{T}(\bar{u}))\|_{L^2} \leq CR \|\tilde{u} - \bar{u}\|_X. \quad (4.11)$$

Combining (4.10), (4.11) and taking R suitably small yields

$$\|\mathbb{T}(\tilde{u}) - \mathbb{T}(\bar{u})\|_X \leq \frac{1}{2} \|\tilde{u} - \bar{u}\|_X. \quad (4.12)$$

From (4.8) and (4.12), we know that \mathbb{T} is a strictly contracting mapping. Consequently, we conclude that there exists a fixed point $u \in X_R$ of the mapping \mathbb{T} , which is a classical solution to (1.1), (1.2). We have completed the proof of the theorem. \square

Competing interests

The author declares that she has no competing interests.

Author's contributions

The author completed the paper by herself. The author read and approved the final manuscript.

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