Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Non-squareness properties of Orlicz-Lorentz function spaces

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Abstract

In this paper, criteria for non-squareness and uniform non-squareness of Orlicz-Lorentz function spaces $\Lambda_{\varphi,\omega}$ are given. Since degenerated Orlicz functions φ and degenerated weight functions ω are also admitted, this investigation concerns the most possible wide class of Orlicz-Lorentz function spaces.

It is worth recalling that uniform non-squareness is an important property, because it implies super-reflexivity as well as the fixed point property (see James in Ann. Math. 80:542-550, 1964; Pacific J. Math. 41:409-419, 1972 and García-Falset *et al.* in J. Funct. Anal. 233:494-514, 2006).

MSC: 46B20; 46B42; 46A80; 46E30

Keywords: uniform non-squareness; non-squareness; Orlicz-Lorentz space; Lorentz space; Orlicz function; Luxemburg norm; strict monotonicity; uniform monotonicity; reflexivity; super-reflexivity; fixed point property

Uniform non-squareness of Banach spaces has been defined by James as the geometric property which implies super-reflexivity (see [1, 2]). So, after proving this property for a Banach space, we know, without any characterization of the dual space, that it is super-reflexive, so reflexive as well. Recently, García-Falset, Llorens-Fuster and Mazcuñan-Navarro have shown that uniformly non-square Banach spaces have the fixed point property (see [3]).

Therefore, it was natural and interesting to look for criteria of non-squareness properties in various well-known classes of Banach spaces. Among a great number of papers concerning this topic, we list here [4–12].

The problem of uniform non-squareness of Calderón-Lozanovskiĭ spaces was initiated by Cerdà, Hudzik and Mastyło in [13]. Since the class of Orlicz-Lorentz spaces is a subclass of Calderón-Lozanovskiĭ spaces, we can say that also the problem of uniform non-squareness of Orlicz-Lorentz spaces was initiated in [13]. However, the results of our paper show that those results were only some sufficient conditions for uniform non-squareness which were very far from being necessary and sufficient. Analogous results for Orlicz-Lorentz sequence spaces were presented in [14], but the techniques of the proofs in the function case are different (in some parts completely different) than in the sequence case.

1 Preliminaries

We say that a Banach space $(X, \|\cdot\|)$ is non-square if $\min(\|\frac{x-y}{2}\|, \|\frac{x+y}{2}\|) < 1$ for any x and y from $S(X)$ (the unit sphere of X). A Banach space X is said to be uniformly non-square if there exists $\delta \in (0,1)$ such that $\min(\|\frac{x-y}{2}\|, \|\frac{x+y}{2}\|) \leq 1-\delta$ for any $x, y \in B(X)$ (the unit ball of X). In the last definition, the unit ball $B(X)$ can be replaced, equivalently, by the unit sphere $S(X)$.

Let $L^0 = L^0([0,\gamma))$ be the space of all (equivalence classes of) Lebesgue measurable real-valued functions defined on the interval $[0,\gamma)$, where $\gamma \leq \infty$. For any $x, y \in L^0$, we write $x \leq y$ if $x(t) \leq y(t)$ almost everywhere with respect to the Lebesgue measure m on the interval $[0,\gamma)$.

Given any $x \in L^0$, we define its distribution function $\mu_x : [0,+\infty) \to [0,\gamma]$ by

$$\mu_x(\lambda) = m(\{t \in [0,\gamma) : |x(t)| > \lambda\})$$

(see [15, 16] and [17]) and the non-increasing rearrangement $x^* : [0,\gamma) \to [0,\infty]$ of x as

$$x^*(t) = \inf\{\lambda \geq 0 : \mu_x(\lambda) \leq t\}$$

(under the convention $\inf \emptyset = \infty$). We say that two functions $x, y \in L^0$ are equimeasurable if $\mu_x(\lambda) = \mu_y(\lambda)$ for all $\lambda \geq 0$. Then we obviously have $x^* = y^*$.

Let (R_1, Σ_1, μ_1) and (R_2, Σ_2, μ_2) be totally σ-finite measure spaces. A map σ from R_1 into R_2 is called a measure preserving transformation if for each Σ_2-measurable subset A from R_2, the set $\sigma^{-1}(A) = \{t \in R_1 : \sigma(t) \in A\}$ is a Σ_1-measurable subset of R_1 and $\mu_1(\sigma^{-1}(A)) = \mu_2(A)$ (see [15]). It is well known that a measure preserving transformation induces equimeasurability, that is, if σ is a measure preserving transformation, then x and $x \circ \sigma$ are equimeasurable functions. The converse is false (see [15]).

A Banach space $E = (E, \leq, \|\cdot\|)$, where $E \subset L^0$, is said to be a Köthe space if the following conditions are satisfied:

(i) if $x \in E$, $y \in L^0$ and $|y| \leq |x|$, then $y \in E$ and $\|y\| \leq \|x\|$,

(ii) there exists a function x in E that is strictly positive on the whole $[0,\gamma)$.

Recall that a Köthe space E is called a symmetric space if E is rearrangement invariant which means that if $x \in E$, $y \in L^0$ and $x^* = y^*$, then $y \in E$ and $\|x\| = \|y\|$ (see [18]). For basic properties of symmetric spaces, we refer to [15, 16] and [17].

In the whole paper, φ denotes an Orlicz function (see [19–21]), that is, $\varphi : [-\infty,\infty] \to [0,\infty]$ (our definition is extended from R into R^e by assuming $\varphi(-\infty) = \varphi(\infty) = \infty$) and φ is convex, even, vanishing and continuous at zero, left continuous on $(0,\infty)$ and not identically equal to zero on $(-\infty,\infty)$. Let us denote

$$a_\varphi = \sup\{u \geq 0 : \varphi(u) = 0\},$$
$$b_\varphi = \sup\{u \geq 0 : \varphi(u) < \infty\}$$

and

$$\delta = \sup\left\{u \geq 0 : \varphi\left(\frac{u}{2}\right) = \frac{1}{2}\varphi(u)\right\}.$$

Let us note that if $a_\varphi > 0$, then $\delta = a_\varphi$, while left continuity of φ on $(0,\infty)$ is equivalent to the fact that $\lim_{u \to (b_\varphi)^-} \varphi(u) = \varphi(b_\varphi)$.

Recall that an Orlicz function φ satisfies the condition Δ_2 for all $u \in \mathbb{R}$ ($\varphi \in \Delta_2(\mathbb{R})$ for short) if there exists a constant $K > 0$ such that the inequality

$$\varphi(2u) \leq K\varphi(u) \tag{1}$$

holds for any $u \in \mathbb{R}$ (then we have $a_\varphi = 0$ and $b_\varphi = \infty$). Analogously, we say that an Orlicz function φ satisfies the condition Δ_2 at infinity ($\varphi \in \Delta_2(\infty)$ for short) if there exist a constant $K > 0$ and a constant $u_0 \geq 0$ such that $\varphi(u_0) < \infty$ and inequality (1) holds for any $u \geq u_0$ (then we obtain $b_\varphi = \infty$).

For any Orlicz function φ, we define its complementary function in the sense of Young by the formula

$$\psi(u) = \sup_{v>0}\{|u|v - \varphi(v)\}$$

for all $u \in \mathbb{R}$. It is easy to show that ψ is also an Orlicz function.

Let $\omega : [0, \gamma) \to \mathbb{R}_+$ be a non-increasing and locally integrable function called a weight function. Let us define

$$\gamma_0 = \sup\{t \geq 0 : \omega \text{ is constant on } (0, t)\},$$
$$\alpha = \sup\{t \geq 0 : \omega(t) > 0\}.$$

We say that a weight function ω is regular if there exists $\eta > 0$ such that

$$\int_0^{2t} \omega(t)\,dt \geq (1 + \eta) \int_0^t \omega(t)\,dt$$

for any $t \in [0, \gamma/2)$ (see [22–25]). Note that if the weight function ω is regular, then $\int_0^\infty \omega(t)\,dt = \infty$ in the case when $\gamma = \infty$ and $\alpha > \gamma/2$ whenever $\gamma < \infty$.

Now we recall the definition of Orlicz-Lorentz spaces. These spaces were introduced by Kamińska (see [26, 27] and [24]) at the beginning of 1990s. Her investigations gave an impulse to further investigations of the spaces, results of which have been published, among others, in the papers [14, 28–42].

Given any Orlicz function φ and a weight function ω, we define on L^0 the convex modular

$$I_{\varphi,\omega}(x) = \int_0^\gamma \varphi(x^*(t))\omega(t)\,dt$$

(see [26] and [28]) and the Orlicz-Lorentz function space

$$\Lambda_{\varphi,\omega} = \Lambda_{\varphi,\omega}([0, \gamma)) = \{x \in L^0 : I_{\varphi,\omega}(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

(see [26] and [28]), which becomes a Banach symmetric space under the Luxemburg norm

$$\|x\| = \inf\{\lambda > 0 : I_{\varphi,\omega}(x/\lambda) \leq 1\}.$$

In our investigations, we apply the results concerning the monotonicity properties of Lorentz function spaces that were presented in [25, 43, 44]. Let us recall that the Lorentz function spaces Λ_ω (see [10, 18, 22, 23, 45–50]) are defined by the formula

$$\Lambda_\omega = \Lambda_\omega\big([0,\gamma)\big) = \left\{x \in L^0 : \|x\|_\omega = \int_0^\gamma x^*(t)\omega(t)\,dt < \infty\right\}.$$

A Banach lattice $E = (E, \leq, \|\cdot\|)$ is said to be strictly monotone if $x, y \in E$, $0 \leq y \leq x$ and $y \neq x$ imply that $\|y\| < \|x\|$. We say that E is uniformly monotone if for any $\varepsilon \in (0,1)$, there is $\delta(\varepsilon) \in (0,1)$ such that $\|x - y\| \leq 1 - \delta(\varepsilon)$ whenever $x, y \in E$, $0 \leq y \leq x$, $\|x\| = 1$ and $\|y\| \geq \varepsilon$ (see [51]). Recall (see [52]) that in Banach lattices E, strict monotonicity and uniform monotonicity are restrictions of rotundity and uniform rotundity (respectively) to couples of comparable elements in the positive cone E_+ only.

Theorem 1.1 ([25], Theorem 2 and [43], Lemma 3.1) *The Lorentz function space Λ_ω is strictly monotone if and only if ω is positive on $[0,\gamma)$ and $\int_0^\gamma \omega(t)\,dt = \infty$ whenever $\gamma = \infty$.*

The following theorem has been proved in [25, Theorem 1] for $\gamma = \infty$. Moreover, applying some ideas from the proof of Theorem 3.7 (see Case 2 on p.2722) in [53], this theorem can be also shown for $\gamma < \infty$.

Theorem 1.2 *The Lorentz function space Λ_ω is uniformly monotone if and only if the weight function ω is regular and ω is positive on $[0,\gamma)$ whenever $\gamma < \infty$.*

In our further investigations, we will also apply Lemma 1.1 and Remark 1.1. By convexity of the modular $I_{\varphi,\omega}$, Lemma 1.1 can be proved analogously as in the case of Orlicz spaces (*cf.* also [43] for considering a more general case).

Lemma 1.1 *Suppose that the Orlicz function φ satisfies a suitable condition Δ_2, that is, $\varphi \in \Delta(\mathbb{R})$ if $\gamma = \infty$ and $\int_0^\infty \omega(t)\,dt = \infty$, and $\varphi \in \Delta(\infty)$ otherwise. Then, for any $\varepsilon \in (0,1)$, there exists $\delta = \delta(\varepsilon) \in (0,1)$ such that $\|x\| \leq 1 - \delta$ for any $x \in \Lambda_{\varphi,\omega}$ whenever $I_{\varphi,\omega}(x) \leq 1 - \varepsilon$.*

In particular, for any $x \in \Lambda_{\varphi,\omega}$, we then have that $\|x\| = 1$ if and only if $I_{\varphi,\omega}(x) = 1$.

Remark 1.1 Let $x, y \in \Lambda_{\varphi,\omega}$ and $t \in (0,\gamma)$ be such that $(\frac{x+y}{2})^*(t) > \lim_{s\to\infty}(\frac{x+y}{2})^*(s) = (\frac{x+y}{2})^*(\infty)$. By [16, Property 7°, p.64], there exists a set $e_t = e_t(\frac{x+y}{2})$ such that $m(e_t) = t$ and

$$\int_0^t \left(\frac{x+y}{2}\right)^*(s)\,ds = \int_{e_t}\left|\frac{x+y}{2}\right|(s)\,ds.$$

Defining $t(x) = m(\operatorname{supp} x \cap e_t)$ and $t(y) = m(\operatorname{supp} y \cap e_t)$, by convexity of the modular $I_{\varphi,\omega}$, we have

$$\begin{aligned}\int_0^t \varphi\left(\left(\frac{x+y}{2}\right)^*(s)\right)\omega(s)\,ds &= I_{\varphi,\omega}\left(\left(\frac{x+y}{2}\right)\chi_{e_t}\right) \leq \frac{1}{2}I_{\varphi,\omega}(x\chi_{e_t}) + \frac{1}{2}I_{\varphi,\omega}(y\chi_{e_t}) \\ &= \frac{1}{2}\int_0^{t(x)} \varphi\big((x\chi_{e_t})^*(s)\big)\omega(s)\,ds \\ &\quad + \frac{1}{2}\int_0^{t(y)} \varphi\big((y\chi_{e_t})^*(s)\big)\omega(s)\,ds.\end{aligned}$$

Denoting $A_t = [0,\gamma)\backslash e_t$, $a(x) = m(\operatorname{supp} x \cap A_t)$, $a(y) = m(\operatorname{supp} y \cap A_t)$ and applying convexity of the modular $I_{\varphi,t}$, defined by the formula

$$I_{\varphi,t}(x) = \int_0^\gamma \varphi\big(x^*(s)\big)\omega(t+s)\,ds$$

(if $\gamma < \infty$, we assume that $\omega(t+s) = 0$ for $s \geq \gamma - t$), we get

$$\begin{aligned}
\int_t^\gamma \varphi\left(\left(\frac{x+y}{2}\right)^*(s)\right)\omega(s)\,ds &= \int_0^\gamma \varphi\left(\left(\left(\frac{x+y}{2}\right)\chi_{A_t}\right)^*(s)\right)\omega(t+s)\,ds \\
&= I_{\varphi,t}\left(\left(\frac{x+y}{2}\right)\chi_{A_t}\right) \leq \frac{1}{2}I_{\varphi,t}(x\chi_{A_t}) + \frac{1}{2}I_{\varphi,t}(y\chi_{A_t}) \\
&= \frac{1}{2}\int_0^{a(x)} \varphi\big((x\chi_{A_t})^*(s)\big)\omega(t+s)\,ds \\
&\quad + \frac{1}{2}\int_0^{a(y)} \varphi\big((y\chi_{A_t})^*(s)\big)\omega(t+s)\,ds \\
&= \frac{1}{2}\int_t^{t+a(x)} \varphi\big((x\chi_{A_t})^*(s-t)\big)\omega(s)\,ds \\
&\quad + \frac{1}{2}\int_t^{t+a(y)} \varphi\big((y\chi_{A_t})^*(s-t)\big)\omega(s)\,ds.
\end{aligned} \tag{2}$$

2 Results

We start with the following

Theorem 2.1 *Let $\gamma = \infty$. Then the Orlicz-Lorentz function space $\Lambda_{\varphi,\omega}$ is non-square if and only if $\int_0^\infty \omega(t)\,dt = \infty$, $\varphi \in \Delta_2(\mathbb{R})$ and $\int_0^{\gamma_0/2} \varphi(\delta)\omega(t)\,dt < 1$.*

Proof Necessity. If $\int_0^\infty \omega(t)\,dt < \infty$ or $\varphi \notin \Delta_2(\mathbb{R})$, then $\Lambda_{\varphi,\omega}$ contains an order isometric copy of l^∞ (see [26, Theorem 2.4]). Finally, suppose that $\int_0^{\gamma_0/2} \varphi(\delta)\omega(t)\,dt \geq 1$. Taking

$$x = a\chi_{[0,\gamma_0/2)} \quad \text{and} \quad y = a\chi_{[\gamma_0/2,\gamma_0)},$$

where $a \leq \delta$ is such that $\int_0^{\gamma_0/2} \varphi(a)\omega(t)\,dt = 1$, we get $I_{\varphi,\omega}(x) = I_{\varphi,\omega}(y) = I_{\varphi,\omega}(\frac{x+y}{2}) = I_{\varphi,\omega}(\frac{x-y}{2}) = 1$ and, consequently, $\|x\| = \|y\| = \|\frac{x+y}{2}\| = \|\frac{x-y}{2}\| = 1$. Thus, $\Lambda_{\varphi,\omega}$ is not non-square.

Sufficiency. Let $x, y \in S(\Lambda_{\varphi,\omega})$. Since φ satisfies the condition $\Delta_2(\mathbb{R})$, by Lemma 1.1, it is enough to show that $\min(I_{\varphi,\omega}(\frac{x-y}{2}), I_{\varphi,\omega}(\frac{x+y}{2})) < 1$. Let us denote

$$\begin{aligned}
A_1 &= \big\{t \in (0,\infty) : x(t)y(t) > 0\big\}, \\
A_2 &= \big\{t \in (0,\infty) : x(t)y(t) < 0\big\}, \\
A_3 &= \big\{t \in (0,\infty) : x(t)y(t) = 0 \text{ and } \max\big(|x(t)|, |y(t)|\big) > \delta\big\}, \\
A_4 &= \big\{t \in (0,\infty) : x(t)y(t) = 0 \text{ and } 0 < \max\big(|x(t)|, |y(t)|\big) \leq \delta\big\}.
\end{aligned} \tag{3}$$

By $\varphi \in \Delta_2(\mathbb{R})$, we have $a_\varphi = 0$ and $b_\varphi = \infty$. Therefore,

$$\varphi\left(\frac{u-v}{2}\right) < \varphi\left(\frac{\max(|u|,|v|)}{2}\right) < \frac{1}{2}\big\{\varphi(u) + \varphi(v)\big\}$$

if $uv > 0$ and

$$\varphi\left(\frac{u+v}{2}\right) < \varphi\left(\frac{\max(|u|,|v|)}{2}\right) < \frac{1}{2}\{\varphi(u)+\varphi(v)\}$$

whenever $uv < 0$. Moreover, if $u > \delta$, then $\varphi(\frac{u}{2}) < \frac{1}{2}\varphi(u)$. Consequently,

$$\varphi\circ\left(\frac{x-y}{2}\right) \lneqq \frac{1}{2}\{\varphi\circ(x)+\varphi\circ(y)\} \quad \text{if } m(A_1) > 0,$$
$$\varphi\circ\left(\frac{x+y}{2}\right) \lneqq \frac{1}{2}\{\varphi\circ(x)+\varphi\circ(y)\} \quad \text{if } m(A_2\cup A_3) > 0.$$

Hence, by strict monotonicity of the Lorentz space Λ_ω (see Theorem 1.1), we get

$$I_{\varphi,\omega}\left(\frac{x-y}{2}\right) = \left\|\varphi\circ\left(\frac{x-y}{2}\right)\right\|_\omega < \left\|\frac{1}{2}\varphi\circ x + \frac{1}{2}\varphi\circ y\right\|_\omega \le 1 \quad \text{if } m(A_1) > 0,$$
$$I_{\varphi,\omega}\left(\frac{x+y}{2}\right) = \left\|\varphi\circ\left(\frac{x+y}{2}\right)\right\|_\omega < \left\|\frac{1}{2}\varphi\circ x + \frac{1}{2}\varphi\circ y\right\|_\omega \le 1 \quad \text{if } m(A_2\cup A_3) > 0.$$

Therefore, if $m(A_1\cup A_2\cup A_3) > 0$, we have $\min(I_{\varphi,\omega}(\frac{x-y}{2}), I_{\varphi,\omega}(\frac{x+y}{2})) < 1$.

Finally, suppose that $m(A_1\cup A_2\cup A_3) = 0$. Then $\delta > 0$ and $I_{\varphi,\omega}(\frac{x-y}{2}) = I_{\varphi,\omega}(\frac{x+y}{2})$. We will prove that

$$\begin{aligned} I_{\varphi,\omega}\left(\frac{x\pm y}{2}\right) &= \int_0^\infty \varphi\left(\left(\frac{x\pm y}{2}\right)^*(t)\right)\omega(t)\,dt \\ &< \frac{1}{2}\int_0^\infty \varphi\big(x^*(t)\big)\omega(t)\,dt + \frac{1}{2}\int_0^\infty \varphi\big(y^*(t)\big)\omega(t)\,dt = 1. \end{aligned} \tag{4}$$

In order to do this, we will consider two cases.

Case 1. Suppose that $\gamma_0 > 0$. Since $I_{\varphi,\omega}(x) = I_{\varphi,\omega}(y) = 1$, by the condition $\int_0^{\gamma_0/2}\varphi(\delta)\omega(t)\,dt < 1$, we have $m(\operatorname{supp} x) > \gamma_0/2$ and $m(\operatorname{supp} y) > \gamma_0/2$. Hence, by $m(\operatorname{supp} x\cap\operatorname{supp} y) = 0$, we obtain $m(\operatorname{supp} x\cup\operatorname{supp} y) > \gamma_0$. By the condition $\int_0^\infty \omega(t)\,dt = \infty$, we have $\lim_{t\to\infty}(\frac{x+y}{2})^*(t) = (\frac{x+y}{2})^*(\infty) = 0$, whence we get $(\frac{x+y}{2})^*(\gamma_0) > (\frac{x+y}{2})^*(\infty)$. Then there exists a set $e_{\gamma_0} = e_{\gamma_0}(\frac{x+y}{2})$ with $m(e_{\gamma_0}) = \gamma_0$ and

$$\int_0^{\gamma_0}\left(\frac{x+y}{2}\right)^*(t)\,dt = \int_{e_{\gamma_0}}\left|\frac{x+y}{2}\right|(t)\,dt$$

(see [16, Property 7°, p.64]). Defining

$$\gamma_0(x) = m(e_{\gamma_0}\cap\operatorname{supp} x) \quad\text{and}\quad \gamma_0(y) = m(e_{\gamma_0}\cap\operatorname{supp} y),$$

we have $\gamma_0(x)+\gamma_0(y) = \gamma_0$ and, by convexity of the modular $I_{\varphi,\omega}$,

$$\begin{aligned} \int_0^{\gamma_0}\varphi\left(\left(\frac{x+y}{2}\right)^*(t)\right)\omega(t)\,dt &= I_{\varphi,\omega}\left(\left(\frac{x+y}{2}\right)\chi_{e_{\gamma_0}}\right) \le \frac{1}{2}I_{\varphi,\omega}(x\chi_{e_{\gamma_0}}) + \frac{1}{2}I_{\varphi,\omega}(y\chi_{e_{\gamma_0}}) \\ &= \frac{1}{2}\int_0^{\gamma_0(x)}\varphi\big(x^*(t)\big)\omega(t)\,dt + \frac{1}{2}\int_0^{\gamma_0(y)}\varphi\big(y^*(t)\big)\omega(t)\,dt. \end{aligned} \tag{5}$$

Setting $A_{\gamma_0} = [0,\gamma)\backslash e_{\gamma_0}$, by inequality (2) from Remark 1.1, we get

$$\begin{aligned}
&\int_{\gamma_0}^{\infty} \varphi\left(\left(\frac{x+y}{2}\right)^*(t)\right)\omega(t)\,dt \\
&\quad \le \frac{1}{2}\int_{\gamma_0}^{\infty} \varphi\big((x\chi_{A_{\gamma_0}})^*(t-\gamma_0)\big)\omega(t)\,dt + \frac{1}{2}\int_{\gamma_0}^{\infty} \varphi\big((y\chi_{A_{\gamma_0}})^*(t-\gamma_0)\big)\omega(t)\,dt. \qquad (6)
\end{aligned}$$

Since $\varphi((\frac{x+y}{2})^*(\gamma_0)) > 0$, we may assume without loss of generality that

$$\int_{\gamma_0}^{\infty} \varphi\big((x\chi_{A_{\gamma_0}})^*(t-\gamma_0)\big)\omega(t)\,dt > 0.$$

Denote $\omega(t) = \omega$ for $t \in (0,\gamma_0)$. If $\gamma_0(x) < \gamma_0$, applying the inequality $\omega(t) < \omega$ for $t > \gamma_0$, we get

$$\begin{aligned}
&\frac{1}{2}\int_0^{\gamma_0(x)} \varphi\big(x^*(t)\big)\omega(t)\,dt + \frac{1}{2}\int_{\gamma_0}^{\infty} \varphi\big((x\chi_{A_{\gamma_0}})^*(t-\gamma_0)\big)\omega(t)\,dt \\
&\quad < \frac{1}{2}\int_0^{\gamma_0(x)} \varphi\big(x^*(t)\big)\omega(t)\,dt + \frac{1}{2}\int_{\gamma_0(x)}^{\infty} \varphi\big((x\chi_{A_{\gamma_0}})^*\big(t-\gamma_0(x)\big)\big)\omega(t)\,dt \\
&\quad = \frac{1}{2}\int_0^{\infty} \varphi\big(x^*(t)\big)\omega(t)\,dt. \qquad (7)
\end{aligned}$$

Suppose now that $\gamma_0(x) = \gamma_0$. Then $\gamma_0(y) = 0$, whence $\operatorname{supp} y \subset A_{\gamma_0}$ and consequently,

$$0 < \frac{1}{2}\int_{\gamma_0}^{\infty} \varphi\big((y\chi_{A_{\gamma_0}})^*(t-\gamma_0)\big)\omega(t)\,dt < \frac{1}{2}\int_0^{\infty} \varphi\big(y^*(t)\big)\omega(t)\,dt. \qquad (8)$$

Applying inequalities (5), (6), (7) and (8), we obtain (4).

Case 2. Let now $\gamma_0 = 0$. Then there exists ν such that $(\frac{x+y}{2})^*(\nu) > 0$ and $\omega(t) > \omega(s)$ for any t and s satisfying $t < \nu < s$. Proceeding similarly as in the above Case 1, but with ν instead of γ_0, we get again inequality (4). □

Theorem 2.2 *If $\gamma < \infty$, then the Orlicz-Lorentz function space $\Lambda_{\varphi,\omega}$ is non-square if and only if $\frac{\gamma}{2} < \alpha \le \gamma$, $\varphi \in \Delta_2(\infty)$ and $\int_0^{\gamma_0/2} \varphi(\delta)\omega(t)\,dt < 1$.*

Proof Necessity. The necessity of conditions $\varphi \in \Delta_2(\infty)$ and $\int_0^{\gamma_0/2} \varphi(\delta)\omega(t)\,dt < 1$ can be shown similarly as in Theorem 2.1. Suppose that $\alpha \le \frac{\gamma}{2}$. Since $\varphi \in \Delta_2(\infty)$, so $b_\varphi = \infty$, whence we can find $a > 0$ such that $\int_0^{\alpha} \varphi(a)\omega(t)\,dt = 1$. Putting

$$\begin{aligned}
x &= a\chi_{[0,2\alpha)}, \\
y &= a\chi_{[0,\alpha)} - a\chi_{[\alpha,2\alpha)},
\end{aligned}$$

we have

$$I_{\varphi,\omega}(x) = I_{\varphi,\omega}(y) = I_{\varphi,\omega}\left(\frac{x+y}{2}\right) = I_{\varphi,\omega}\left(\frac{x-y}{2}\right) = 1,$$

which means that $\Lambda_{\varphi,\omega}$ is not non-square.

Sufficiency. Let $x, y \in S(\Lambda_{\varphi,\omega})$. Analogously as in the proof of Theorem 2.1, it is enough to show that $\min(I_{\varphi,\omega}(\frac{x-y}{2}), I_{\varphi,\omega}(\frac{x+y}{2})) < 1$. We divide the proof into several parts.

Case 1. Assume that $\alpha = \gamma$. Let us define the sets A_i, $i = 1, \ldots, 4$ as in (3) and

$$A_1' = \{t \in A_1 : \max(|x(t)|, |y(t)|) > a_\varphi\},$$
$$A_2' = \{t \in A_2 : \max(|x(t)|, |y(t)|) > a_\varphi\}.$$

If $m(A_1') > 0$, then

$$\begin{aligned} 0 &= \varphi\left(\frac{x(t)-y(t)}{2}\right) = \varphi\left(\frac{\max(|x(t)|,|y(t)|)}{2}\right) < \frac{1}{2}\varphi(\max(|x(t)|,|y(t)|)) \\ &\le \frac{1}{2}\{\varphi(x(t)) + \varphi(y(t))\} \end{aligned}$$

for $t \in A_1'$ whenever $\max(|x(t)|,|y(t)|)/2 \le a_\varphi$ and

$$\varphi\left(\frac{x(t)-y(t)}{2}\right) < \varphi\left(\frac{\max(|x(t)|,|y(t)|)}{2}\right) \le \frac{1}{2}\{\varphi(x(t)) + \varphi(y(t))\}$$

for $t \in A_1'$ whenever $\max(|x(t)|,|y(t)|)/2 > a_\varphi$. Analogously as in Theorem 2.1, by strict monotonicity of the Lorentz space Λ_ω (see Theorem 1.1), we have $I_{\varphi,\omega}(\frac{x-y}{2}) < 1$. Similarly, $I_{\varphi,\omega}(\frac{x+y}{2}) < 1$ provided $m(A_2') > 0$. Notice that if $0 = m(A_1' \cup A_2') < m(A_1 \cup A_2)$, then $\delta = a_\varphi > 0$, whence $m(A_3) > 0$ (because $I_{\varphi,\omega}(x) = I_{\varphi,\omega}(y) = 1$). Now we will consider the case $m(A_3) > 0$. Then

$$\begin{aligned} \varphi\left(\frac{x(t) \pm y(t)}{2}\right) &= \varphi\left(\frac{\max(|x(t)|,|y(t)|)}{2}\right) < \frac{1}{2}\varphi(\max(|x(t)|,|y(t)|)) \\ &= \frac{1}{2}\{\varphi(x(t)) + \varphi(y(t))\} \end{aligned}$$

for $t \in A_3$, whence by strict monotonicity of the Lorentz space Λ_ω, we have again $I_{\varphi,\omega}(\frac{x\pm y}{2}) < 1$. Finally, suppose that $m(A_1 \cup A_2 \cup A_3) = 0$. Then $0 = a_\varphi < \delta$ and $I_{\varphi,\omega}(x\chi_{A_4}) = I_{\varphi,\omega}(y\chi_{A_4}) = 1$. Analogously as in the proof of Theorem 2.1, we can show

$$I_{\varphi,\omega}\left(\frac{x \pm y}{2}\right) < \frac{1}{2}\int_0^\gamma \varphi(x^*(t))\omega(t)\,dt + \frac{1}{2}\int_0^\gamma \varphi(y^*(t))\omega(t)\,dt = 1. \tag{9}$$

Case 2. Now suppose that $\frac{\gamma}{2} < \alpha < \gamma$ and denote

$$A_{x,y} = \{t \in [0,\gamma) : \max(|x(t)|, |y(t)|) > a_\varphi\}.$$

Case 2.1. If $m(A_{x,y}) \le \alpha$, then we define

$$\widetilde{x} = x\chi_{A_{x,y}} \circ \sigma \quad \text{and} \quad \widetilde{y} = y\chi_{A_{x,y}} \circ \sigma,$$

where $\sigma : [0, m(A_{x,y})) \to A_{x,y}$ is a measure preserving transformation (see [54, Theorem 17, p.410]). Obviously, $\varphi \circ \widetilde{x}$, $\varphi \circ \widetilde{y}$, $\varphi \circ \frac{\widetilde{x}+\widetilde{y}}{2}$ and $\varphi \circ \frac{\widetilde{x}-\widetilde{y}}{2}$ are equimeasurable with $\varphi \circ x\chi_{A_{x,y}}$,

$\varphi \circ y\chi_{A_{x,y}}$, $\varphi \circ \frac{x+y}{2}\chi_{A_{x,y}}$ and $\varphi \circ \frac{x-y}{2}\chi_{A_{x,y}}$, respectively. Since $\Lambda_\omega([0,\alpha))$ is strictly monotone, repeating the proof from Case 1, we get

$$\begin{aligned}\min\left(I_{\varphi,\omega}\left(\frac{x-y}{2}\right), I_{\varphi,\omega}\left(\frac{x+y}{2}\right)\right) &= \min\left(I_{\varphi,\omega}\left(\left(\frac{x-y}{2}\right)\chi_{A_{x,y}}\right), I_{\varphi,\omega}\left(\left(\frac{x+y}{2}\right)\chi_{A_{x,y}}\right)\right) \\ &= \min\left(I_{\varphi,\omega}\left(\frac{\widetilde{x}-\widetilde{y}}{2}\right), I_{\varphi,\omega}\left(\frac{\widetilde{x}+\widetilde{y}}{2}\right)\right) < 1.\end{aligned}$$

Case 2.2. Assume now that $m(A_{x,y}) > \alpha$, that is,

$$\left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(\alpha) > 0. \tag{10}$$

By convexity of φ and appropriate properties of the rearrangement (see [15, Proposition 1.7, p.41]), we obtain

$$\varphi\left(\left(\frac{x \pm y}{2}\right)^*(t)\right) = \left(\varphi \circ \left(\frac{x \pm y}{2}\right)\right)^*(t) \leq \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(t) \tag{11}$$

for any $t \in [0,\gamma)$. If there exists $t \in [0,\alpha)$ such that inequality (11) is sharp for the sum or for the difference, then by the right continuity of the rearrangement, we get

$$\min\left(I_{\varphi,\omega}\left(\frac{x-y}{2}\right), I_{\varphi,\omega}\left(\frac{x+y}{2}\right)\right) < 1.$$

Consequently, in the remaining part of the proof, we will assume that for any $t \in [0,\alpha)$ in formula (11), we have equality for both the sum and the difference.

Case 2.2.1. Let $(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y)^*(0) > (\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y)^*(t)$ for all $t > \alpha$ and let us set in this case

$$t_0 = \sup\left\{s : \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(s) > \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(t) \text{ for each } t > \alpha\right\}.$$

By the right continuity of the rearrangement, we have $0 < t_0 \leq \alpha$ and

$$\left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(t_0) = \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(\alpha) > 0. \tag{12}$$

Moreover, if $t_0 = \alpha$, then $(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y)^*(s) > (\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y)^*(\alpha)$ for any $s < \alpha$ or $(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y)^*(\alpha) > (\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y)^*(t)$ for all $t > \alpha$. In the case when $t_0 < \alpha$, there exists $t > \alpha$ such that $(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y)^*(s) > (\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y)^*(t_0) = (\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y)^*(t)$ for any $s < t_0$. Let $e_{t_0} = e_{t_0}(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y)$ be the set such that $m(e_{t_0}) = t_0$ and

$$\int_0^{t_0} \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(t)\,dt = \int_{e_{t_0}} \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)(t)\,dt \tag{13}$$

(see [16, Property 7°, p.64]). By the proof of Property 7° from [16], we conclude that

$$\left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)(s) \geq \lim_{t \to t_0-} \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(t)$$

for m-a.e. $s \in e_{t_0}$. Hence, by the definition of t_0, we obtain

$$\left(\frac{1}{2}\varphi\circ x+\frac{1}{2}\varphi\circ y\right)(s)>\left(\frac{1}{2}\varphi\circ x+\frac{1}{2}\varphi\circ y\right)^*(t) \tag{14}$$

for m-a.e. $s \in e_{t_0}$ and each $t > t_0$. Moreover, using again the definition of t_0, we get that for m-a.e. $s \in [0,\gamma)\backslash e_{t_0}$, there exists $t(s) > t_0$ such that

$$\left(\frac{1}{2}\varphi\circ x+\frac{1}{2}\varphi\circ y\right)(s)\le\left(\frac{1}{2}\varphi\circ x+\frac{1}{2}\varphi\circ y\right)^*\big(t(s)\big). \tag{15}$$

Since for any $t \in [0,\alpha)$ we have equality in formula (11) for both the sum and the difference, we can find sets $e_{t_0}(+) = e_{t_0}(\varphi\circ(\frac{x+y}{2}))$ and $e_{t_0}(-) = e_{t_0}(\varphi\circ(\frac{x-y}{2}))$ such that $m(e_{t_0}(+)) = m(e_{t_0}(-)) = t_0$ and

$$\int_0^{t_0}\left(\frac{1}{2}\varphi\circ x+\frac{1}{2}\varphi\circ y\right)^*(t)\,dt=\int_{e_{t_0}(+)}\varphi\circ\left(\frac{x+y}{2}\right)(t)\,dt=\int_{e_{t_0}(-)}\varphi\circ\left(\frac{x-y}{2}\right)(t)\,dt. \tag{16}$$

Similarly as in the case of the set e_{t_0}, for m-a.e. $s \in e_{t_0}(+)$ and for each $t > t_0$, we get

$$\varphi\circ\left(\frac{x+y}{2}\right)(s)>\left(\frac{1}{2}\varphi\circ x+\frac{1}{2}\varphi\circ y\right)^*(t).$$

Hence, by convexity of the function φ and inequalities (14) and (15), we obtain $e_{t_0}(+) \subset e_{t_0}$. Since $m(e_{t_0}) = t_0 = m(e_{t_0}(+))$, so $e_{t_0}(+) = e_{t_0}$. Analogously, we derive the equality $e_{t_0}(-) = e_{t_0}$. Note also that convexity of the function φ and equations (13) and (16) imply the equalities

$$\varphi\circ\left(\frac{x+y}{2}\right)\chi_{e_{t_0}}=\varphi\circ\left(\frac{x-y}{2}\right)\chi_{e_{t_0}}=\left(\frac{1}{2}\varphi\circ x+\frac{1}{2}\varphi\circ y\right)\chi_{e_{t_0}},$$

whence, by inequality (10), we get $m(\operatorname{supp}(x\chi_{e_{t_0}}) \cap \operatorname{supp}(y\chi_{e_{t_0}})) = 0$ and

$$0=a_\varphi<\left(\frac{1}{2}\varphi\circ x+\frac{1}{2}\varphi\circ y\right)^*(0)\le\delta. \tag{17}$$

Denoting $t_0(x) = m(e_{t_0} \cap \operatorname{supp} x)$ and $t_0(y) = m(e_{t_0} \cap \operatorname{supp} y)$, we have

$$t_0(x)+t_0(y)=t_0. \tag{18}$$

Case 2.2.1.1. Suppose $t_0 = \alpha$. By convexity of the modular $I_{\varphi,\omega}$, we get

$$\begin{aligned}\int_0^{t_0}\varphi\left(\left(\frac{x+y}{2}\right)^*(t)\right)\omega(t)\,dt&=I_{\varphi,\omega}\left(\left(\frac{x+y}{2}\right)\chi_{e_{t_0}}\right)\le\frac{1}{2}I_{\varphi,\omega}(x\chi_{e_{t_0}})+\frac{1}{2}I_{\varphi,\omega}(y\chi_{e_{t_0}})\\&=\frac{1}{2}\int_0^{t_0(x)}\varphi\big(x^*(t)\big)\omega(t)\,dt+\frac{1}{2}\int_0^{t_0(y)}\varphi\big(y^*(t)\big)\omega(t)\,dt.\end{aligned}$$

If $t_0(y) = 0$ ($t_0(x) = 0$), then $I_{\varphi,\omega}(\frac{x+y}{2}) \le \frac{1}{2}I_{\varphi,\omega}(x) = \frac{1}{2}$ ($I_{\varphi,\omega}(\frac{x+y}{2}) \le \frac{1}{2}I_{\varphi,\omega}(y) = \frac{1}{2}$). So, $0 < t_0(x) < t_0$ and $0 < t_0(y) < t_0$. Furthermore, by equation (10), we may assume without loss of generality that $\beta(x) := m((A_{x,y} \backslash e_{t_0}) \cap \operatorname{supp} x) > 0$. Thus

$$\begin{aligned}\int_0^{t_0(x)} \varphi\big(x^*(t)\big)\omega(t)\,dt &< \int_0^{t_0(x)} \varphi\big(x^*(t)\big)\omega(t)\,dt \\ &\quad + \int_{t_0(x)}^{t_0(x)+\beta(x)} \varphi\big((x\chi_{A_{x,y}\backslash e_{t_0}})^*\big(t - t_0(x)\big)\big)\omega(t)\,dt \\ &= \int_0^{\alpha} \varphi\big(x^*(t)\big)\omega(t)\,dt = 1,\end{aligned}$$

whence we get $I_{\varphi,\omega}(\frac{x+y}{2}) < 1$.

Case 2.2.1.2. Let now $t_0 < \alpha$. Then, by the definition of t_0, there exists $t > \alpha$ satisfying

$$\left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(t) = \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(t_0).$$

Define

$$\begin{aligned}t_1 &= \sup\left\{t > \alpha : \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(t) = \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(t_0)\right\}, \\ A_{t_0} &= \left\{t \in [0,\gamma) : \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)(t) = \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(t_0)\right\}\end{aligned}$$

and

$$A_{t_0,x,y} = \big\{t \in A_{t_0} : \min\big(|x(t)|, |y(t)|\big) = 0\big\}.$$

Since for any $t \in [0,\alpha)$ we have equality in formula (11) for both the sum and the difference, we can find a set $e_\alpha = e_\alpha(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y)$ such that $m(e_\alpha) = \alpha$ and

$$\int_0^{\alpha} \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(t)\,dt = \int_{e_\alpha} \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)(t)\,dt = \int_{e_\alpha} \varphi \circ \left(\frac{x+y}{2}\right)(t)\,dt. \quad (19)$$

If $m(A_{t_0,x,y}) \geq \alpha - t_0$, then we can assume without loss of generality that $e_{t_0} \subset e_\alpha \subset e_{t_0} \cup A_{t_0,x,y}$, whence we get the equality $m(\operatorname{supp} x\chi_{e_\alpha} \cap \operatorname{supp} y\chi_{e_\alpha}) = 0$. Proceeding analogously as in Case 2.2.1.1, we obtain $I_{\varphi,\omega}(\frac{x+y}{2}) < 1$.

Let now $m(A_{t_0,x,y}) < \alpha - t_0$. Then we will suppose that $e_{t_0} \cup A_{t_0,x,y} \subset e_\alpha \subset e_{t_0} \cup A_{t_0}$ and consequently

$$m\big((A_{t_0} \backslash e_\alpha) \cap \operatorname{supp} x\big) = m\big((A_{t_0} \backslash e_\alpha) \cap \operatorname{supp} y\big) = m(A_{t_0} \backslash e_\alpha) = t_1 - \alpha =: d > 0.$$

Putting $\alpha(x) = m(e_\alpha \cap \operatorname{supp} x)$, $\alpha(y) = m(e_\alpha \cap \operatorname{supp} y)$ and applying again convexity of the modular $I_{\varphi,\omega}$, we obtain

$$\begin{aligned}\int_0^{\alpha} \varphi\left(\left(\frac{x+y}{2}\right)^*(t)\right)\omega(t)\,dt = I_{\varphi,\omega}\left(\left(\frac{x+y}{2}\right)\chi_{e_\alpha}\right) &\leq \frac{1}{2}I_{\varphi,\omega}(x\chi_{e_\alpha}) + \frac{1}{2}I_{\varphi,\omega}(y\chi_{e_\alpha}) \\ &= \frac{1}{2}\int_0^{\alpha(x)} \varphi\big((x\chi_{e_\alpha})^*(t)\big)\omega(t)\,dt \\ &\quad + \frac{1}{2}\int_0^{\alpha(y)} \varphi\big((y\chi_{e_\alpha})^*(t)\big)\omega(t)\,dt.\end{aligned}$$

Simultaneously, by equality (18), we may assume without loss of generality that $\alpha(x) = t_0(x) + m((e_\alpha \setminus e_{t_0}) \cap \operatorname{supp} x) < \alpha$, whence

$$\begin{aligned}\int_0^{\alpha(x)} \varphi\big((x\chi_{e_\alpha})^*(t)\big)\omega(t)\,dt &< \int_0^{\alpha(x)} \varphi\big((x\chi_{e_\alpha})^*(t)\big)\omega(t)\,dt \\ &\quad + \int_{\alpha(x)}^{\alpha(x)+d} \varphi\big((x\chi_{A_{t_0}\setminus e_\alpha})^*\big(t-\alpha(x)\big)\big)\omega(t)\,dt \\ &\le \int_0^{\alpha} \varphi\big(x^*(t)\big)\omega(t)\,dt = 1.\end{aligned}$$

So, we get $I_{\varphi,\omega}(\frac{x+y}{2}) < 1$.

Case 2.2.2. Finally, assume that $(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y)^*(0) = (\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y)^*(\alpha) = (\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y)^*(t) > 0$ for some $t > \alpha$ and define

$$\begin{aligned} A &= \left\{ t \in [0,\gamma) : \frac{1}{2}\varphi\big(x(t)\big) + \frac{1}{2}\varphi\big(y(t)\big) = \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(0)\right\}, \\ A_+ &= \left\{ t \in [0,\gamma) : \varphi \circ \left(\frac{x+y}{2}\right)(t) = \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(0)\right\}, \\ A_- &= \left\{ t \in [0,\gamma) : \varphi \circ \left(\frac{x-y}{2}\right)(t) = \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right)^*(0)\right\}. \end{aligned}$$

Applying convexity of the Orlicz function and the equality in formula (11), we get the conditions $m(A) > \alpha$, $A_+ \subset A$, $A_- \subset A$ and $\min(m(A_+), m(A_-)) \ge \alpha$. Since $\alpha > \frac{\gamma}{2}$, the set $A_{x,y} = A_+ \cap A_- = \{t \in A : \min(|x(t)|, |y(t)|) = 0\}$ has positive measure. If $m(A_{x,y}) \ge \alpha$, we can assume that $e_\alpha \subset A_{x,y}$ (where e_α is defined analogously as in (19)); in the opposite case, we can assume that $A_{x,y} \subset e_\alpha \subset A$. Proceeding analogously as in Case 2.2.1, we obtain $I_{\varphi,\omega}(\frac{x+y}{2}) < 1$. □

Theorem 2.3 *In the case when $\gamma = \infty$, the Orlicz-Lorentz function space $\Lambda_{\varphi,\omega}$ is uniformly non-square if and only if $\varphi \in \Delta_2(\mathbb{R})$, $\psi \in \Delta_2(\mathbb{R})$ and ω is regular.*

Proof Necessity. The necessity of the condition $\varphi \in \Delta_2(\mathbb{R})$ follows from Theorem 2.1. If $\psi \notin \Delta_2(\mathbb{R})$, then $\Lambda_{\varphi,\omega}$ contains an order isomorphic copy of l^1 (see [38, Theorem 7.18] or [29, Theorem 2]), whence it is not reflexive. Finally, suppose that ω is not regular. Then we can find a sequence (t_n) of positive numbers such that

$$\int_0^{2t_n} \omega(t)\,dt \le \left(1 + \frac{1}{n}\right)\int_0^{t_n} \omega(t)\,dt$$

for any $n \in \mathbb{N}$. Since $b_\varphi = \infty$, for every $n \in \mathbb{N}$, there exists a_n satisfying

$$\varphi(a_n)\int_0^{2t_n} \omega(t)\,dt = 1.$$

Define

$$\begin{aligned} x_n &= a_n\chi_{[0,2t_n)}, \\ y_n &= a_n\chi_{[0,t_n)} - a_n\chi_{[t_n,2t_n)}. \end{aligned}$$

Then $I_{\varphi,\omega}(x_n) = I_{\varphi,\omega}(y_n) = 1$ and

$$I_{\varphi,\omega}\left(\frac{x_n + y_n}{2}\right) = I_{\varphi,\omega}\left(\frac{x_n - y_n}{2}\right) = \int_0^{t_n} \varphi(a_n)\omega(t)\,dt \geq \frac{n}{n+1}\int_0^{2t_n} \varphi(a_n)\omega(t)\,dt \to 1,$$

whence we have $\min(\|\frac{x_n-y_n}{2}\|, \|\frac{x_n+y_n}{2}\|) \to 1$.

Sufficiency. Let $x, y \in S(\Lambda_{\varphi,\omega})$. By $\psi \in \Delta_2(\mathbb{R})$ we conclude that there is $\eta \in (0,1)$ such that $\varphi(\frac{u}{2}) \leq \frac{1-\eta}{2}\varphi(u)$ for all $u > 0$ (see [55]). Let us set

$$A_1 = \{t \in (0,\infty) : x(t)y(t) > 0\},$$
$$A_2 = \{t \in (0,\infty) : x(t)y(t) < 0\},$$
$$A_3 = \{t \in (0,\infty) : |x(t)| > 0 \text{ and } y(t) = 0\}.$$

Since $I_{\varphi,\omega}(x) = 1$, we have $\max(I_{\varphi,\omega}(x\chi_{A_1\cup A_3}), I_{\varphi,\omega}(x\chi_{A_2})) \geq 1/2$. Suppose that $I_{\varphi,\omega}(x\chi_{A_1\cup A_3}) \geq 1/2$. Since the inequality

$$\varphi\left(\frac{x(t) - y(t)}{2}\right) \leq \varphi\left(\frac{\max(|x(t)|, |y(t)|)}{2}\right) \leq \frac{1-\eta}{2}\varphi\left(\max\left(|x(t)|, |y(t)|\right)\right)$$
$$\leq \frac{1}{2}\varphi\left(x(t)\right) + \frac{1}{2}\varphi\left(y(t)\right) - \frac{\eta}{2}\varphi\left(x(t)\right)$$

holds for m-a.e. $t \in A_1 \cup A_3$, we get

$$\varphi \circ \left(\frac{x-y}{2}\right) \leq \frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y - \frac{\eta}{2}\varphi \circ x\chi_{A_1\cup A_3}.$$

Hence, by uniform monotonicity of the Lorentz space Λ_ω (see Theorem 1.2), we obtain

$$I_{\varphi,\omega}\left(\frac{x-y}{2}\right) = \left\|\varphi \circ \left(\frac{x-y}{2}\right)\right\|_\omega \leq \left\|\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y - \frac{\eta}{2}\varphi \circ x\chi_{A_1\cup A_3}\right\|_\varphi \leq 1 - \delta\left(\frac{\eta}{4}\right),$$

where $\delta(\frac{\eta}{4})$ is the constant from the definition of uniform monotonicity of the Lorentz space Λ_ω corresponding to $\frac{\eta}{4}$. Analogously, we get $I_{\varphi,\omega}(\frac{x+y}{2}) \leq 1 - \delta(\frac{\eta}{4})$ in the case when $I_{\varphi,\omega}(x\chi_{A_2}) \geq 1/2$. Finally, by Lemma 1.1, we obtain

$$\min\left(\left\|\frac{x-y}{2}\right\|, \left\|\frac{x+y}{2}\right\|\right) \leq 1 - r,$$

where $r = r(\delta(\frac{\eta}{4}))$ depends only on $\delta(\frac{\eta}{4})$. □

Theorem 2.4 *If $\alpha = \gamma < \infty$, then the Orlicz-Lorentz function space $\Lambda_{\varphi,\omega}$ is uniformly non-square if and only if $\varphi \in \Delta_2(\infty)$, $\psi \in \Delta_2(\infty)$, ω is regular and $\int_0^{\gamma_0/2} \varphi(\delta)\omega(t)\,dt < 1$.*

Proof Necessity. The necessity of the conditions $\varphi \in \Delta_2(\infty)$ and $\int_0^{\gamma_0/2} \varphi(\delta)\omega(t)\,dt < 1$ has been shown in Theorem 2.2, whereas the necessity of the conditions $\psi \in \Delta_2(\infty)$ and regularity of ω can be shown analogously as in Theorem 2.3.

Sufficiency. Let $x, y \in S(\Lambda_{\varphi,\omega})$. If we show the inequality

$$\min\left(I_{\varphi,\omega}\left(\frac{x-y}{2}\right), I_{\varphi,\omega}\left(\frac{x+y}{2}\right)\right) \leq 1 - q \tag{20}$$

for some $q > 0$ independent of x and y, then Lemma 1.1 will give the inequality

$$\min\left(\left\|\frac{x-y}{2}\right\|, \left\|\frac{x+y}{2}\right\|\right) \le 1 - r,$$

with some $r > 0$ depending only on q, and the proof will be finished. In order to show (20), we consider three cases.

Case 1. First assume that $\int_0^\gamma \varphi(\delta)\omega(t)\,dt < 1$ (in particular, this holds if $\delta = 0$ or $0 < a_\varphi = \delta$). Then we can find $u_\delta > \delta$ such that $\int_0^\gamma \varphi(u_\delta)\omega(t)\,dt =: a_\delta < 1$. Since for any $u > \delta$ there holds

$$\varphi\left(\frac{u}{2}\right) < \frac{1}{2}\varphi(u),$$

by $\psi \in \Delta_2(\infty)$, there exists $\eta = \eta(u_\delta) \in (0,1)$ such that

$$\varphi\left(\frac{u}{2}\right) \le \frac{1-\eta}{2}\varphi(u) \tag{21}$$

for all $u \ge u_\delta$ (see [55]). Define

$$\begin{aligned}
A &= \{t \in [0,\gamma) : |x(t)| \ge u_\delta\},\\
A_1 &= \{t \in A : x(t)y(t) \ge 0\},\\
A_2 &= \{t \in A : x(t)y(t) < 0\}.
\end{aligned}$$

We have $I_{\varphi,\omega}(x\chi_{[0,\gamma)\setminus A}) < a_\delta$, whence $I_{\varphi,\omega}(x\chi_A) > 1 - a_\delta$ and consequently

$$\max\left(I_{\varphi,\omega}(x\chi_{A_1}), I_{\varphi,\omega}(x\chi_{A_2})\right) > \frac{1-a_\delta}{2}.$$

If $I_{\varphi,\omega}(x\chi_{A_1}) > (1-a_\delta)/2$, analogously as in the proof of Theorem 2.3, we get

$$\varphi \circ \frac{x-y}{2} \le \frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y - \frac{\eta}{2}\varphi \circ x\chi_{A_1}.$$

Hence, by uniform monotonicity of the Lorentz space Λ_ω (see Theorem 1.2), we obtain

$$\begin{aligned}
I_{\varphi,\omega}\left(\frac{x-y}{2}\right) &= \left\|\varphi \circ \left(\frac{x-y}{2}\right)\right\|_\omega \le \left\|\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y - \frac{\eta}{2}\varphi \circ x\chi_{A_1}\right\|_\omega\\
&\le 1 - \delta\left(\frac{\eta(1-a_\delta)}{4}\right),
\end{aligned}$$

where $\delta(\eta(1-a_\delta)/4)$ is the constant from the definition of uniform monotonicity of the Lorentz space Λ_ω corresponding to $\eta(1-a_\delta)/4$. If $I_{\varphi,\omega}(x\chi_{A_2}) > (1-a_\delta)/2$, then we get similarly that $I_{\varphi,\omega}(\frac{x+y}{2}) \le 1 - \delta(\eta(1-a_\delta)/4)$. Therefore, if $\int_0^\gamma \varphi(\delta)\omega(t)\,dt < 1$, we obtain inequality (20) with $q = \delta(\eta(1-a_\delta)/4)$.

Case 2. Now assume that $\int_0^\gamma \varphi(\delta)\omega(t)\,dt \geq 1$ and $\gamma_0 > 0$. Then for

$$c := \frac{1 - \int_0^{\gamma_0/2} \varphi(\delta)\omega(t)\,dt}{8},$$

by the condition $\int_0^{\gamma_0/2} \varphi(\delta)\omega(t)\,dt < 1$, we have $0 < c < \frac{1}{8}$. Moreover, we can find a constant $v_\delta > \delta$ such that

$$\int_0^{\gamma_0/2} \varphi(v_\delta)\omega(t)\,dt = 1 - 4c.$$

Applying again the condition $\psi \in \Delta_2(\infty)$, we get that there exists $\eta = \eta(v_\delta) \in (0,1)$ such that inequality (21) holds for any $u \geq v_\delta$. Denote

$$A_{x,v_\delta} = \{t \in [0,\gamma) : |x(t)| \geq v_\delta\}, \tag{22}$$

$$A_{y,v_\delta} = \{t \in [0,\gamma) : |y(t)| \geq v_\delta\}. \tag{23}$$

Now we divide the proof of this case into several parts.

Case 2.1. If $\max(I_{\varphi,\omega}(x\chi_{A_{x,v_\delta}}), I_{\varphi,\omega}(y\chi_{A_{y,v_\delta}})) \geq c$, then proceeding analogously as in the Case 1, we get

$$\min\left(I_{\varphi,\omega}\left(\frac{x-y}{2}\right), I_{\varphi,\omega}\left(\frac{x+y}{2}\right)\right) \leq 1 - \delta\left(\frac{\eta c}{4}\right), \tag{24}$$

where $\delta(\frac{\eta c}{4})$ is the constant from the definition of uniform monotonicity of the Lorentz space Λ_ω corresponding to $\frac{\eta c}{4}$.

Case 2.2. Now assume that $\max(I_{\varphi,\omega}(x\chi_{A_{x,v_\delta}}), I_{\varphi,\omega}(y\chi_{A_{y,v_\delta}})) < c$ and define $t_0 > 0$ and $u_0 > 0$ by the formulas

$$\int_0^{t_0} \varphi(v_\delta)\omega(t)\,dt = 1 - 2c \quad \text{and} \quad \int_0^{\gamma} \varphi(u_0)\omega(t)\,dt = c.$$

By the definition of v_δ and the inequality $\int_0^\gamma \varphi(\delta)\omega(t)\,dt \geq 1$, we have $t_0 > \frac{\gamma_0}{2}$ and $u_0 < \delta$, respectively.

Now we will show that

$$m(A_{x,u_0}) \geq t_0 \quad \text{and} \quad m(A_{y,u_0}) \geq t_0, \tag{25}$$

where

$$A_{x,u_0} = \{t \in [0,\gamma) : |x(t)| \geq u_0\}, \tag{26}$$

$$A_{y,u_0} = \{t \in [0,\gamma) : |y(t)| \geq u_0\}. \tag{27}$$

Indeed, by the equalities $I_{\varphi,\omega}(x) = I_{\varphi,\omega}(y) = 1$ and the definition of u_0, we have $I_{\varphi,\omega}(x\chi_{A_{x,u_0}}) \geq 1 - c$ and $I_{\varphi,\omega}(y\chi_{A_{y,u_0}}) \geq 1 - c$, whence by $\max(I_{\varphi,\omega}(x\chi_{A_{x,v_\delta}}), I_{\varphi,\omega}(y\chi_{A_{y,v_\delta}})) < c$ and the definition of t_0, we get (25).

Let

$$t_1 = \frac{\min(t_0 - \frac{\gamma_0}{2}, \frac{\gamma_0}{2})}{4}$$

and

$$A^+_{x,y,u_0} = \left\{ t \in [0,\gamma) : \min\left(\left|x(t)\right|, \left|y(t)\right|\right) \geq \frac{u_0}{4} \text{ and } x(t)y(t) > 0 \right\}, \tag{28}$$

$$A^-_{x,y,u_0} = \left\{ t \in [0,\gamma) : \min\left(\left|x(t)\right|, \left|y(t)\right|\right) \geq \frac{u_0}{4} \text{ and } x(t)y(t) < 0 \right\}. \tag{29}$$

Case 2.2.1. First assume that $m(A^+_{x,y,u_0}) \geq t_1$ and define

$$z_1 = \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right) - \varphi \circ \left(\frac{x-y}{2}\right).$$

Denoting by $p(u)$ the right derivative of φ at a point u, we have $p(u) =: p > 0$ for $u \in [0,\delta)$. Note that for m-a.e. $t \in A^+_{x,y,u_0}$, we have

$$\begin{aligned}\left(\frac{1}{2}\varphi\left(x(t)\right) + \frac{1}{2}\varphi\left(y(t)\right)\right) - \varphi\left(\frac{x(t)-y(t)}{2}\right) &\geq \varphi\left(\frac{x(t)+y(t)}{2}\right) - \varphi\left(\frac{x(t)-y(t)}{2}\right) \\ &\geq \int_{\varphi(\frac{x(t)-y(t)}{2})}^{\varphi(\frac{x(t)+y(t)}{2})} p(u)\,du \geq \int_0^{u_0/2} p\,du = \frac{pu_0}{2}.\end{aligned}$$

Hence, by $m(A^+_{x,y,u_0}) \geq t_1$ and $t_1 < \gamma_0$, we get

$$\|z_1\|_\omega \geq \int_0^{t_1} \frac{pu_0}{2}\omega(t)\,dt = \frac{pu_0\omega_0 t_1}{2},$$

where $\omega_0 = \omega(t)$ for any $t \in (0,\gamma_0)$. Analogously, if $m(A^-_{x,y,u_0}) \geq t_1$, for

$$z_2 = \left(\frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y\right) - \varphi \circ \left(\frac{x+y}{2}\right)$$

we obtain

$$\|z_2\|_\omega \geq \int_0^{t_1} \frac{pu_0}{2}\omega(t)\,dt = \frac{pu_0\omega_0 t_1}{2}.$$

Therefore, if $\max(m(A^+_{x,y,u_0}), m(A^-_{x,y,u_0})) \geq t_1$, by uniform monotonicity of the Lorentz space Λ_ω, we have

$$\min\left(I_{\varphi,\omega}\left(\frac{x-y}{2}\right), I_{\varphi,\omega}\left(\frac{x+y}{2}\right)\right) \leq 1 - \delta\left(\frac{pu_0\omega_0 t_1}{2}\right), \tag{30}$$

where $\delta(\frac{pu_0\omega_0 t_1}{2})$ is the constant from the definition of uniform monotonicity of the Lorentz space Λ_ω corresponding to $\frac{pu_0\omega_0 t_1}{2}$.

Case 2.2.2. Finally, suppose that $\max(m(A^+_{x,y,u_0}), m(A^-_{x,y,u_0})) < t_1$. Then for

$$B_{x,u_0} = A_{x,u_0} \backslash \left(A^+_{x,y,u_0} \cup A^-_{x,y,u_0}\right), \tag{31}$$

$$B_{y,u_0} = A_{y,u_0} \backslash \left(A^+_{x,y,u_0} \cup A^-_{x,y,u_0}\right), \tag{32}$$

we have

$$B_{x,u_0} \cap B_{y,u_0} = \emptyset \tag{33}$$

and by (25) and definition of t_1,

$$\min\left(m(B_{x,u_0}), m(B_{y,u_0})\right) \geq t_0 - 2t_1 \geq t_0 - \frac{1}{2}\left(t_0 - \frac{\gamma_0}{2}\right) = \frac{t_0}{2} + \frac{\gamma_0}{4} > \frac{\gamma_0}{2}, \tag{34}$$

whence we get

$$m(B_{x,u_0} \cup B_{y,u_0}) \geq t_0 + \frac{\gamma_0}{2} > \gamma_0. \tag{35}$$

Define

$$a = \min\left(\frac{(t_0 + \frac{\gamma_0}{2}) - \gamma_0}{8}, \frac{\gamma_0}{4}\right) \quad \text{and} \quad t_2 = \gamma_0 + a.$$

Let $e_{\gamma_0} = e_{\gamma_0}(\frac{x+y}{2})$ and $e_{t_2} = e_{t_2}(\frac{x+y}{2})$ be such that $m(e_{\gamma_0}) = \gamma_0$, $m(e_{t_2}) = t_2$,

$$\int_0^{\gamma_0} \left(\frac{x+y}{2}\right)^*(t)\,dt = \int_{e_{\gamma_0}} \left|\frac{x+y}{2}\right|(t)\,dt$$

and

$$\int_0^{t_2} \left(\frac{x+y}{2}\right)^*(t)\,dt = \int_{e_{t_2}} \left|\frac{x+y}{2}\right|(t)\,dt$$

(see [16, Property 7°, p.64]). Moreover, by the proof of Property 7°, we can assume that $e_{\gamma_0} \subset e_{t_2}$. Denoting $A_{\gamma_0} = e_{t_2} \backslash e_{\gamma_0}$ and $A_{t_2} = [0,\gamma) \backslash e_{t_2}$, by Remark 1.1, we have

$$\begin{aligned}
&I_{\varphi,\omega}\left(\frac{x+y}{2}\right) \\
&\quad = \int_0^{\gamma_0} \varphi\left(\left(\frac{x+y}{2}\right)^*(t)\right)\omega(t)\,dt + \int_{\gamma_0}^{t_2} \varphi\left(\left(\frac{x+y}{2}\right)^*(t)\right)\omega(t)\,dt \\
&\qquad + \int_{t_2}^{\gamma} \varphi\left(\left(\frac{x+y}{2}\right)^*(t)\right)\omega(t)\,dt \\
&\quad = \int_0^{\gamma_0} \varphi\left(\left(\left(\frac{x+y}{2}\right)\chi_{e_{\gamma_0}}\right)^*(t)\right)\omega(t)\,dt + \int_{\gamma_0}^{t_2} \varphi\left(\left(\left(\frac{x+y}{2}\right)\chi_{A_{\gamma_0}}\right)^*(t-\gamma_0)\right)\omega(t)\,dt \\
&\qquad + \int_{t_2}^{\gamma} \varphi\left(\left(\left(\frac{x+y}{2}\right)\chi_{A_{t_2}}\right)^*(t-t_2)\right)\omega(t)\,dt \\
&\quad \leq \frac{1}{2}\int_0^{\gamma_0} \varphi\left((x\chi_{e_{\gamma_0}})^*(t)\right)\omega(t)\,dt + \frac{1}{2}\int_0^{\gamma_0} \varphi\left((y\chi_{e_{\gamma_0}})^*(t)\right)\omega(t)\,dt
\end{aligned}$$

$$+\frac{1}{2}\int_{\gamma_0}^{t_2}\varphi\big((x\chi_{A_{\gamma_0}})^*(t-\gamma_0)\big)\omega(t)\,dt+\frac{1}{2}\int_{\gamma_0}^{t_2}\varphi\big((y\chi_{A_{\gamma_0}})^*(t-\gamma_0)\big)\omega(t)\,dt$$
$$+\frac{1}{2}\int_{t_2}^{\gamma}\varphi\big((x\chi_{A_{t_2}})^*(t-t_2)\big)\omega(t)\,dt+\frac{1}{2}\int_{t_2}^{\gamma}\varphi\big((y\chi_{A_{t_2}})^*(t-t_2)\big)\omega(t)\,dt.$$

By formulas (33) and (35), we have

$$m\big((B_{x,u_0}\cup B_{y,u_0})\cap A_{t_2}\big)=m\big((B_{x,u_0}\cup B_{y,u_0})\backslash e_{t_2}\big)\geq t_0+\frac{\gamma_0}{2}-t_2=t_0-\frac{\gamma_0}{2}-a\geq 7a$$

and, in consequence, we can assume without loss of generality that $(x\chi_{A_{t_2}})^*(a)>u_0$. If $(x\chi_{e_{\gamma_0}})^*(\gamma_0-a)\leq\frac{u_0}{4}$, then

$$\begin{aligned}&\int_{\gamma_0-a}^{\gamma_0}\big[\varphi\big((x\chi_{A_{t_2}})^*(t-\gamma_0+a)\big)-\varphi\big((x\chi_{e_{\gamma_0}})^*(t)\big)\big]\omega(t)\,dt\\&\quad-\int_{t_2}^{t_2+a}\big[\varphi\big((x\chi_{A_{t_2}})^*(t-t_2)\big)-\varphi\big((x\chi_{e_{\gamma_0}})^*\big(t-(t_2-\gamma_0+a)\big)\big)\big]\omega(t)\,dt\\&\geq\big(\omega_0-\omega(t_2)\big)\int_{\gamma_0-a}^{\gamma_0}\big(\varphi\big((x\chi_{A_{t_2}})^*(t-\gamma_0+a)\big)-\varphi\big((x\chi_{e_{\gamma_0}})^*(t)\big)\big)\,dt\\&\geq a\left(\varphi(u_0)-\varphi\left(\frac{u_0}{4}\right)\right)\big(\omega_0-\omega(t_2)\big)\geq\frac{3apu_0(\omega_0-\omega(t_2))}{4},\end{aligned}\tag{36}$$

where p denotes as above the right derivative of φ on the interval $[0,\delta)$ and $\omega_0=\omega(t)$ for any $t\in(0,\gamma_0)$; note that by the definition of γ_0, we have $\omega_0-\omega(t_2)>0$. Hence,

$$\begin{aligned}&\int_0^{\gamma_0}\varphi\big((x\chi_{e_{\gamma_0}})^*(t)\big)\omega(t)\,dt+\int_{\gamma_0}^{t_2}\varphi\big((x\chi_{A_{\gamma_0}})^*(t-\gamma_0)\big)\omega(t)\,dt\\&\quad+\int_{t_2}^{\gamma}\varphi\big((x\chi_{A_{t_2}})^*(t-t_2)\big)\omega(t)\,dt\\&\leq\int_0^{\gamma_0-a}\varphi\big((x\chi_{e_{\gamma_0}})^*(t)\big)\omega(t)\,dt+\int_{\gamma_0-a}^{\gamma_0}\varphi\big((x\chi_{A_{t_2}})^*\big(t-(\gamma_0-a)\big)\big)\omega(t)\,dt\\&\quad+\int_{\gamma_0}^{t_2}\varphi\big((x\chi_{A_{\gamma_0}})^*(t-\gamma_0)\big)\omega(t)\,dt\\&\quad+\int_{t_2}^{t_2+a}\varphi\big((x\chi_{e_{\gamma_0}})^*\big(t-(t_2-\gamma_0+a)\big)\big)\omega(t)\,dt+\int_{t_2+a}^{\gamma}\varphi\big((x\chi_{A_{t_2}})^*(t-t_2)\big)\omega(t)\,dt\\&\quad-\frac{3apu_0(\omega_0-\omega(t_2))}{4}\\&\leq\int_0^{\gamma}\varphi\big(x^*(t)\big)\omega(t)\,dt-\frac{3apu_0(\omega_0-\omega(t_2))}{4}=1-\frac{3apu_0(\omega_0-\omega(t_2))}{4}.\end{aligned}\tag{37}$$

Now assume that $(x\chi_{e_{\gamma_0}})^*(\gamma_0-a)>\frac{u_0}{4}$. Then

$$m\big(e_{\gamma_0}\cap\big(A^+_{x,y,u_0}\cup A^-_{x,y,u_0}\cup B_{x,u_0}\big)\big)>\gamma_0-a\geq\frac{3}{4}\gamma_0,$$

whence we get

$$m(e_{\gamma_0}\cap B_{y,u_0})<\frac{1}{4}\gamma_0.\tag{38}$$

Therefore, by the inequality $\max(m(A^+_{x,y,u_0}), m(A^-_{x,y,u_0})) < t_1 \le \frac{1}{8}\gamma_0$, we obtain

$$m\big(e_{\gamma_0} \cap \big(A^+_{x,y,u_0} \cup A^-_{x,y,u_0} \cup B_{y,u_0}\big)\big) < \frac{1}{2}\gamma_0 < \gamma_0 - a,$$

and, in consequence, $(y\chi_{e_{\gamma_0}})^*(\gamma_0 - a) < \frac{u_0}{4}$. Simultaneously, by formulas (34) and (38) and the equality $t_2 = \gamma_0 + a$, we have

$$m(B_{y,u_0} \cap A_{t_2}) > \frac{t_0}{2} + \frac{\gamma_0}{4} - \frac{\gamma_0}{4} - a > \frac{t_0}{2} - \frac{\gamma_0}{4} - a \ge 3a.$$

Thus, $(y\chi_{A_{t_2}})^*(a) > u_0$, which gives a possibility to repeat the investigations from (36) and (37) for y. In consequence, we have

$$I_{\varphi,\omega}\left(\frac{x+y}{2}\right) \le 1 - \frac{3apu_0(\omega_0 - \omega(t_2))}{8}. \tag{39}$$

Recapitulating Case 2, by inequalities (24), (30) and (39), we get inequality (20) for

$$q = \min\left(\delta\left(\frac{\eta c}{4}\right), \delta\left(\frac{pu_0\omega_0 t_1}{2}\right), \frac{3apu_0(\omega_0 - \omega(t_2))}{8}\right).$$

Case 3. Finally, assume that $\int_0^\gamma \varphi(\delta)\omega(t)\,dt \ge 1$ and $\gamma_0 = 0$. For arbitrary fixed $v_\delta > \delta$, we define the sets A_{x,v_δ} and A_{y,v_δ} by formulas (22) and (23). If $\max(I_{\varphi,\omega}(x\chi_{A_{x,v_\delta}}), I_{\varphi,\omega}(y\chi_{A_{y,v_\delta}})) \ge \frac{1}{8}$, then proceeding analogously as in Case 2, we get inequality (24) with the constant $\delta(\frac{\eta}{32})$.

If $\max(I_{\varphi,\omega}(x\chi_{A_{x,v_\delta}}), I_{\varphi,\omega}(y\chi_{A_{y,v_\delta}})) < \frac{1}{8}$, then we define $t_0 > 0$ and $u_0 > 0$ by the equalities

$$\int_0^{t_0} \varphi(v_\delta)\omega(t)\,dt = \frac{3}{4} \quad \text{and} \quad \int_0^{\gamma} \varphi(u_0)\omega(t)\,dt = \frac{1}{8}.$$

We have $t_0 < \gamma$, $u_0 < \delta$ and $\min(m(A_{x,u_0}), m(A_{y,u_0})) \ge t_0$, where the sets A_{x,u_0} and A_{x,u_0} are defined by formulas (26) and (27). By the assumption $\gamma_0 = 0$, we can find two positive constants t_2 and t_3 such that $0 < t_3 < t_2 < \frac{t_0}{2}$ and $\omega(t_3) > \omega(t_2)$. Let

$$t_1 = \frac{t_3}{8} \quad \text{and} \quad \omega_1 = \int_0^{t_1} \omega(t)\,dt.$$

If $m(A^+_{x,y,u_0}) \ge t_1$ or $m(A^-_{x,y,u_0}) \ge t_1$, where the sets A^+_{x,y,u_0} and A^-_{x,y,u_0} are defined by formulas (28) and (29), then analogously as in Case 2, we obtain inequality (30) with the constant $\delta(\frac{pu_0\omega_1}{2})$.

In the case when $\max(m(A^+_{x,y,u_0}), m(A^-_{x,y,u_0})) < t_1$, we define the sets B_{x,u_0} and B_{y,u_0} by formulas (31) and (32). We have

$$\min\big(m(B_{x,u_0}), m(B_{y,u_0})\big) \ge t_0 - 2t_1 \ge \frac{7}{8}t_0.$$

Defining $a = \frac{\min(t_3, \frac{t_0}{2} - t_2)}{4}$ and repeating the procedure from Case 2, putting t_3 in place of γ_0, we get inequality (39) with the constant $\frac{3apu_0(\omega(t_3)-\omega(t_2))}{8}$.

Summarizing Case 3, we get inequality (20) with

$$q = \min\left(\delta\left(\frac{\eta}{32}\right), \delta\left(\frac{pu_0\omega_1}{2}\right), \frac{3apu_0(\omega(t_3) - \omega(t_2))}{8}\right). \qquad \square$$

Theorem 2.5 *Let $0 < \alpha < \gamma < \infty$ and $0 \le a_\varphi = \delta$. Then the Orlicz-Lorentz function space $\Lambda_{\varphi,\omega}$ is uniformly non-square if and only if $\varphi \in \Delta_2(\infty)$, $\psi \in \Delta_2(\infty)$, ω is regular and $\alpha \in (\frac{\gamma}{2}, \gamma)$.*

Proof Necessity. Condition $\alpha \in (\frac{\gamma}{2}, \gamma)$ follows from Theorem 2.2, while the necessity of remaining conditions can be proved as in Theorem 2.4.

Sufficiency. Analogously as in Theorem 2.4, it is enough to show that there exists $q > 0$ such that inequality (20) holds for any $x, y \in S(\Lambda_{\varphi,\omega})$.

First note that the space $\Lambda_{\varphi,\omega}([0,\alpha))$, in opposite to the space $\Lambda_{\varphi,\omega} = \Lambda_{\varphi,\omega}([0,\gamma))$, is uniformly monotone (see Theorem 1.2). Hence, by [52, Theorem 6], for all $\delta > 0$ there exists $p(\delta) > 0$ such that for any $u \in B(\Lambda_{\varphi,\omega}([0,\alpha)))$ and any $v \in \Lambda_{\varphi,\omega}([0,\alpha))$ with $m\{\operatorname{supp} u \cap \operatorname{supp} v\} = 0$ and $\|v\| \ge \delta$, we have

$$\|u + v\| \ge \big(1 + p(\delta)\big)\|u\|. \tag{40}$$

Now, for any fixed $x, y \in (\Lambda_{\varphi,\omega})$, we denote

$$A_{x,y} = \big\{t \in [0,\gamma) : \max\big\{\big|x(t)\big|, \big|y(t)\big|\big\} > a_\varphi\big\}.$$

In order to show (20), we will consider two cases.

Case 1. If $m(A_{x,y}) \le \alpha$, then we define

$$\widetilde{x} = x \circ \sigma \quad \text{and} \quad \widetilde{y} = y \circ \sigma,$$

where $\sigma : [0, m(A_{x,y})) \to A_{x,y}$ is a measure preserving transformation (see [54, Theorem 17, p.410]). Obviously $\varphi \circ \widetilde{x}$, $\varphi \circ \widetilde{y}$, $\varphi \circ \frac{\widetilde{x}+\widetilde{y}}{2}$ and $\varphi \circ \frac{\widetilde{x}-\widetilde{y}}{2}$ are equimeasurable with $\varphi \circ x\chi_{A_{x,y}}$, $\varphi \circ y\chi_{A_{x,y}}$, $\varphi \circ \frac{x+y}{2}\chi_{A_{x,y}}$ and $\varphi \circ \frac{x+y}{2}\chi_{A_{x,y}}$, respectively. Therefore, by Theorem 2.4, there exists $q(\alpha) > 0$ independent of x and y such that

$$\begin{aligned}\min\left(I_{\varphi,\omega}\left(\frac{x-y}{2}\right), I_{\varphi,\omega}\left(\frac{x+y}{2}\right)\right) &= \min\left(I_{\varphi,\omega}\left(\left(\frac{x-y}{2}\right)\chi_{A_{x,y}}\right), I_{\varphi,\omega}\left(\left(\frac{x+y}{2}\right)\chi_{A_{x,y}}\right)\right) \\ &= \min\left(I_{\varphi,\omega}\left(\frac{\widetilde{x}-\widetilde{y}}{2}\right), I_{\varphi,\omega}\left(\frac{\widetilde{x}+\widetilde{y}}{2}\right)\right) \le 1 - q(\alpha).\end{aligned}$$

Case 2. Let now $m(A_{x,y}) > \alpha$. Denote by $m_0 \in \mathbb{N}$ the smallest possible number satisfying $m_0(\alpha - \gamma/2) \ge \alpha$ and let $p(1/2m_0)$ be the constant from inequality (40) for $\delta = 1/2m_0$. Fix $\varepsilon > 0$ satisfying

$$\left(1 + p\left(\frac{1}{2m_0}\right)\right)(1 - 6\varepsilon) > 1 \quad \text{and} \quad \varepsilon < \frac{1}{12}. \tag{41}$$

Since $\psi \in \Delta_2(\infty)$, for a_ε satisfying the equality

$$\int_0^\alpha \varphi(a_\varepsilon)\omega(t)\,dt = \varepsilon,$$

analogously as in Case 1 of Theorem 2.4, we can find $\eta = \eta(a_\varepsilon) \in (0,1)$ such that

$$\varphi\left(\frac{u}{2}\right) < \frac{1-\eta}{2}\varphi(u) \tag{42}$$

for all $u \geq a_\varepsilon$. We may assume without loss of generality that

$$\min\left(I_{\varphi,\omega}\left(\frac{x-y}{2}\right), I_{\varphi,\omega}\left(\frac{x+y}{2}\right)\right) \geq 1-\varepsilon. \tag{43}$$

Applying [16, Property 7°, p.64], we can find sets $e_\alpha(+) = e_\alpha(\varphi \circ (\frac{x+y}{2}))$ and $e_\alpha(-) = e_\alpha(\varphi \circ (\frac{x-y}{2}))$ of measure α such that

$$\int_0^\alpha \left(\varphi \circ \left(\frac{x+y}{2}\right)\right)^*(t)\,dt = \int_{e_\alpha(+)} \varphi \circ \left(\frac{x+y}{2}\right)(t)\,dt,$$
$$\int_0^\alpha \left(\varphi \circ \left(\frac{x-y}{2}\right)\right)^*(t)\,dt = \int_{e_\alpha(-)} \varphi \circ \left(\frac{x-y}{2}\right)(t)\,dt.$$

Let us define the sets

$$A^+ = \{t \in e_\alpha(+) : \max(|x(t)|, |y(t)|) \geq a_\varepsilon\},$$
$$A_1^+ = \{t \in A^+ : x(t)y(t) > 0\},$$
$$A_2^+ = \{t \in A^+ : x(t)y(t) \leq 0\}$$

and

$$A^- = \{t \in e_\alpha(-) : \max(|x(t)|, |y(t)|) \geq a_\varepsilon\},$$
$$A_1^- = \{t \in A^- : x(t)y(t) \geq 0\},$$
$$A_2^- = \{t \in A^- : x(t)y(t) < 0\}.$$

From [15, Theorem 2.6, p.49] it follows that there are functions u_+ and u_- both equimeasurable with $\omega\chi_{[0,\alpha]}$ and satisfying the equalities

$$\int_0^\alpha \left(\varphi \circ \left(\frac{x+y}{2}\right)\right)^*(t)\omega(t)\,dt = \int_{e_\alpha(+)} \varphi \circ \left(\frac{x+y}{2}\right)(t)u_+(t)\,dt, \tag{44}$$
$$\int_0^\alpha \left(\varphi \circ \left(\frac{x-y}{2}\right)\right)^*(t)\omega(t)\,dt = \int_{e_\alpha(-)} \varphi \circ \left(\frac{x-y}{2}\right)(t)u_-(t)\,dt. \tag{45}$$

By the Hardy-Littlewood inequality, we have

$$\begin{aligned}&\int_{e_\alpha(+)\setminus A^+} \varphi \circ \left(\frac{x+y}{2}\right)(t)u_+(t)\,dt \\ &\quad \leq \int_0^\alpha \left(\varphi \circ \left(\left(\frac{x+y}{2}\right)\chi_{e_\alpha(+)\setminus A^+}\right)\right)^*(t)(u_+\chi_{e_\alpha(+)\setminus A^+})^*(t)\,dt < \int_0^\alpha \varphi(a_\varepsilon)\omega(t)\,dt = \varepsilon,\end{aligned}$$

whence by (43), we conclude that

$$\int_{A^+} \varphi \circ \left(\frac{x+y}{2}\right)(t)u_+(t)\,dt \geq 1-2\varepsilon. \tag{46}$$

Similarly, we get

$$\int_{A^-} \varphi \circ \left(\frac{x-y}{2}\right)(t) u_-(t)\,dt \geq 1 - 2\varepsilon. \tag{47}$$

The remaining part of the proof of Case 2 will be divided into three subcases.

Case 2.1. Suppose $\int_{A_2^+} \varphi \circ (\frac{x+y}{2})(t) u_+(t)\,dt \geq \varepsilon$. Then

$$\varphi\left(\left(\frac{x+y}{2}\right)(t)\right) \leq \varphi\left(\frac{\max(|x(t)|, |y(t)|)}{2}\right) \leq \frac{1-\eta}{2}\left\{\varphi\big(x(t)\big) + \varphi\big(y(t)\big)\right\}$$

for m-a.e. $t \in A_2^+$. Hence, by equality (44), we get

$$\begin{aligned}
I_{\varphi,\omega}\left(\frac{x+y}{2}\right) &= \int_0^\alpha \varphi\left(\left(\frac{x+y}{2}\right)^*(t)\right)\omega(t)\,dt = \int_{e_\alpha(+)} \varphi \circ \left(\frac{x+y}{2}\right)(t) u_+(t)\,dt \\
&\leq \frac{1}{2}\int_{e_\alpha(+)\setminus A_2^+} \left\{\varphi\big(x(t)\big) + \varphi\big(y(t)\big)\right\} u_+(t)\,dt \\
&\quad + \frac{1-\eta}{2}\int_{A_2^+} \left\{\varphi\big(x(t)\big) + \varphi\big(y(t)\big)\right\} u_+(t)\,dt \\
&\leq \frac{1}{2}\left\{\int_{e_\alpha(+)} \varphi\big(x(t)\big) u_+(t)\,dt + \int_{e_\alpha(+)} \varphi\big(y(t)\big) u_+(t)\,dt\right\} - \eta\varepsilon \\
&\leq 1 - \eta\varepsilon.
\end{aligned} \tag{48}$$

Case 2.2. If $\int_{A_1^-} \varphi \circ (\frac{x-y}{2})(t) u_-(t)\,dt \geq \varepsilon$, then analogously as above, we can show that

$$I_{\varphi,\omega}\left(\frac{x-y}{2}\right) \leq 1 - \eta\varepsilon.$$

Case 2.3. Finally, we will prove that the remaining case

$$\int_{A_2^+} \varphi \circ \left(\frac{x+y}{2}\right)(t) u_+(t)\,dt < \varepsilon \quad \text{and} \quad \int_{A_1^-} \varphi \circ \left(\frac{x-y}{2}\right)(t) u_-(t)\,dt < \varepsilon$$

is not possible. In the opposite case, by (46) and (47), we get

$$\int_{A_1^+} \varphi \circ \left(\frac{x+y}{2}\right)(t) u_+(t)\,dt \geq 1 - 3\varepsilon \quad \text{and} \quad \int_{A_2^-} \varphi \circ \left(\frac{x+y}{2}\right)(t) u_-(t)\,dt \geq 1 - 3\varepsilon.$$

Since $A_1^+ \cap A_2^- = \emptyset$, we can assume without loss of generality that $m(A_1^+) \leq \gamma/2$. Moreover, by the Hardy-Littlewood inequality and convexity of the modular $I_{\varphi,\omega}$, we obtain

$$\begin{aligned}
1 - 3\varepsilon &\leq \int_{A_1^+} \varphi \circ \left(\frac{x+y}{2}\right)(t) u_+(t)\,dt \\
&\leq \int_0^{m(A_1^+)} \left(\varphi \circ \left(\left(\frac{x+y}{2}\right)\chi_{A_1^+}\right)\right)^*(t) (u_+\chi_{A_1^+})^*(t)\,dt \\
&\leq \int_0^{m(A_1^+)} \left(\varphi \circ \left(\left(\frac{x+y}{2}\right)\chi_{A_1^+}\right)\right)^*(t)\omega(t)\,dt
\end{aligned}$$

$$= I_{\varphi,\omega}\left(\left(\frac{x+y}{2}\right)\chi_{A_1^+}\right) \le \frac{1}{2} I_{\varphi,\omega}(x\chi_{A_1^+}) + \frac{1}{2} I_{\varphi,\omega}(y\chi_{A_1^+})$$
$$= \frac{1}{2}\int_0^{m(A_1^+)} (\varphi \circ x\chi_{A_1^+})^*(t)\omega(t)\,dt + \frac{1}{2}\int_0^{m(A_1^+)} (\varphi \circ y\chi_{A_1^+})^*(t)\omega(t)\,dt.$$

Since $I_{\varphi,\omega}(x) = I_{\varphi,\omega}(y) = 1$, so

$$\int_0^{m(A_1^+)} (\varphi \circ x\chi_{A_1^+})^*(t)\omega(t)\,dt \ge 1 - 6\varepsilon \quad \text{and} \quad \int_0^{m(A_1^+)} (\varphi \circ y\chi_{A_1^+})^*(t)\omega(t)\,dt \ge 1 - 6\varepsilon. \tag{49}$$

Similarly,

$$\int_0^{m(A_2^-)} (\varphi \circ x\chi_{A_2^-})^*(t)\omega(t)\,dt \ge 1 - 6\varepsilon \quad \text{and} \quad \int_0^{m(A_2^-)} (\varphi \circ y\chi_{A_2^-})^*(t)\omega(t)\,dt \ge 1 - 6\varepsilon. \tag{50}$$

Let $e_{(\alpha-\gamma/2)} = e_{(\alpha-\gamma/2)}(\varphi \circ x\chi_{A_2^-}) \subset A_2^-$ be such that $m(e_{(\alpha-\gamma/2)}) = \alpha - \gamma/2$ and

$$\int_0^{\alpha-\gamma/2} (\varphi \circ x\chi_{A_2^-})^*(t)\,dt = \int_{e_{(\alpha-\gamma/2)}} \varphi \circ x\chi_{A_2^-}(t)\,dt = \int_{e_{(\alpha-\gamma/2)}} \varphi \circ x\chi_{e_{(\alpha-\gamma/2)}}(t)\,dt.$$

Then, by the definition of m_0, the first inequality in (50) and the second inequality in (41), we get

$$\|\varphi \circ x\chi_{e_{(\alpha-\gamma/2)}}\|_\omega = \int_0^{\alpha-\gamma/2} (\varphi \circ x\chi_{A_2^-})^*(t)\omega(t)\,dt \ge \frac{1-6\varepsilon}{m_0} \ge \frac{1}{2m_0}.$$

Consequently, by (40) (note that $m(\operatorname{supp}(\varphi \circ x\chi_{A_1^+} + \varphi \circ x\chi_{e_{(\alpha-\gamma/2)}})) \le \gamma/2 + \alpha - \gamma/2 = \alpha$) and first inequalities of formulas (49) and (41), we obtain

$$1 = \int_0^\alpha \varphi\bigl(x^*(t)\bigr)\omega(t)\,dt \ge \int_0^\alpha (\varphi \circ x\chi_{A_1^+} + \varphi \circ x\chi_{e_{(\alpha-\gamma/2)}})^*(t)\omega(t)\,dt$$
$$= \|\varphi \circ x\chi_{A_1^+} + \varphi \circ x\chi_{e_{(\alpha-\gamma/2)}}\|_\omega \ge \left(1 + p\left(\frac{1}{2m_0}\right)\right)\|\varphi \circ x\chi_{A_1^+}\|_\omega$$
$$\ge \left(1 + p\left(\frac{1}{2m_0}\right)\right)(1 - 6\varepsilon) > 1,$$

which is a contradiction.

Summarizing both cases, we get inequality (20) with $q = \min(q(\alpha), \eta\varepsilon)$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed essentially in writing this paper. However, the contribution of PF was the biggest. All authors read and approved the final manuscript.

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Acknowledgements
The authors thank the referees for valuable comments and suggestions. All authors are supported partially by the State Committee for Scientific Research, Poland, Grant No. N N201 362236.

Received: 2 October 2012 Accepted: 8 January 2013 Published: 25 January 2013

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doi:10.1186/1029-242X-2013-32
Cite this article as: Foralewski et al.: **Non-squareness properties of Orlicz-Lorentz function spaces.** *Journal of Inequalities and Applications* 2013 **2013**:32.