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Picard iterations for nonexpansive and Lipschitz strongly accretive mappings in a real Banach space

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Abstract

We study the convergence of a more general Picard iterative sequence for nonexpansive and Lipschitz strongly accretive mappings in an arbitrary real Banach space. Our results improve the results of Ćirić *et al.* (Nonlinear Anal. 70(12):4332-4337, 2009).

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1 Introduction and preliminaries

Let E be a real Banach space with dual E^*, and J will denote the normalized duality map from E to 2^{E^*} defined by

$$Jx = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

Let $T : D(T) \to E$ be a mapping, where $D(T)$ stands for the domain of T.

The mapping T is said to be *Lipschitz* if there exists $L > 0$ such that

$$\|Tx - Ty\| \le L \|x - y\| \tag{1.1}$$

for all $x, y \in D(T)$.

If $L = 1$ in inequality (1.1), then T is called *nonexpansive*.

The mapping T is called *strongly pseudocontractive* if there exists $t > 1$ such that

$$\|x - y\| \le \left\| (1 + r)(x - y) - rt(Tx - Ty) \right\| \tag{1.2}$$

for all $x, y \in D(T)$ and $r > 0$.

If $t = 1$ in inequality (1.2), then T is called *pseudocontractive*.

As a consequence of the result of Kato [1], it follows from inequality (1.2) that T is strongly pseudocontractive if and only if there exists $j(x - y) \in J(x - y)$ such that

$$\langle (I - T)x - (I - T)y, j(x - y) \rangle \ge k \|x - y\|^2 \tag{1.3}$$

for all $x, y \in D(T)$, where $k = \frac{t-1}{t} \in (0, 1)$.

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Consequently, it follows easily (again from Kato [1] and inequality (1.3)) that T is strongly pseudocontractive if and only if

$$\|x - y\| \le \big\| x - y + s\big[(I - T - kI)x - (I - T - kI)y\big] \big\| \tag{1.4}$$

for all $x, y \in D(T)$ and $s > 0$.

Closely related to the class of pseudocontractive mappings is the class of *accretive operators*.

Let $A : D(A) \to E$ be a mapping.

The mapping A is called *accretive* if

$$\|x - y\| \le \big\| x - y + s(Ax - Ay) \big\|$$

for all $x, y \in D(A)$ and $s > 0$.

Also, as a consequence of Kato [1], this accretive condition can be expressed in terms of the duality mapping as follows:

For each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \ge 0. \tag{1.5}$$

Consequently, inequality (1.2) with $t = 1$ yields that A is accretive if and only if $T := (I - A)$ is pseudocontractive. Furthermore, setting $A := (I - T)$, it follows from inequality (1.4) that T is *strongly pseudocontractive* if and only if $(A - kI)$ is *accretive*, and using (1.5) this implies that $T(= I - A)$ is *strongly pseudocontractive* if there exists $k \in (0, 1)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \ge k\|x - y\|^2 \tag{1.6}$$

for all $x, y \in D(A)$.

The mapping A satisfying inequality (1.6) is called *strongly accretive*. It is then clear that A is strongly accretive if and only if $T := (I - A)$ is strongly pseudocontractive.

It is worth to mention that considerable research efforts have been devoted, especially within the past long years or so, to developing constructive techniques for the determination of the kernels of accretive operators in Banach spaces (see, *e.g.*, [2–12]). Two well-known iterative schemes, the *Mann iterative scheme* (see, *e.g.*, [13]) and the *Ishikawa iterative scheme* (see, *e.g.*, [14]), have successfully been employed.

In [9], Liu obtained a fixed point of the strictly pseudocontractive mapping as the limit of an iteratively constructed sequence in general Banach spaces.

Theorem 1.1 *Let X be a Banach space and let K be a nonempty closed convex and bounded subset of X. Let $T : K \to K$ be Lipschitz (with constant $L \ge 1$) and strictly pseudocontractive (i.e., T satisfies inequality (1.4) for all $x, y \in K$). Let $F(T) = \{x \in X : Tx = x\} \ne \emptyset$. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ in K by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$ satisfying

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \alpha_n \to 0.$$

Then $\{x_n\}$ converges strongly to $q \in F(T)$ and $F(T)$ is a singleton.

By generalizing the results of Liu [9], Sastry and Babu [11] proved the following results.

Theorem 1.2 *Let X be a Banach space and let K be a nonempty closed convex and bounded subset of X. Let $T : K \to K$ be Lipschitz (with constant $L \geq 0$) and strictly pseudocontractive (i.e., T satisfies inequality (1.4) for all $x, y \in K$). Suppose that $\{\alpha_n\}_{n \in \mathbb{N}}$ is a sequence in $(0,1]$ such that for some $\eta \in (0,k)$ and for all $n \in \mathbb{N}$,*

$$\alpha_n \leq \frac{k - \eta}{(L+1)(L+2-k)}, \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Fix $x_1 \in K$. Define the sequence $\{x_n\}_{n \in \mathbb{N}}$ in K by

$$x_{n+1} := (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \in \mathbb{N}.$$

Then there exists $\{\beta_n\}_{n \in \mathbb{N}}$, a sequence in $(0,1)$ with each $\beta_n \geq \frac{\eta}{1+k}\alpha_n$, such that

$$\|x_{n+1} - q\| \leq \prod_{j=1}^{n}(1 - \beta_j)\|x_1 - q\|, \quad n \in \mathbb{N}.$$

In particular, $\{x_n\}_{n \in \mathbb{N}}$ converges strongly to $q \in K$ and q is the unique fixed point of T.

In [15, 16], Chidume mentioned that the Mann and Ishikawa iteration schemes are global and their rate of convergence is generally of the order $O(n^{-\frac{1}{2}})$. Also, it is well-known that for an operator U, the *classical iterative sequence* $x_{n+1} = Ux_n$, $x_0 \in D(U)$ (called the *Picard iterative sequence*) converges and is preferred in comparison to the Mann or the Ishikawa sequences since it requires less computations; and moreover, its rate of convergence is always at least as fast as that of a geometric progression.

In [15, 16], Chidume proved the following results.

Theorem 1.3 *Let E be an arbitrary real Banach space, let $A : E \to E$ be a Lipschitz (with constant $L > 0$) and strongly accretive mapping with strong accretivity constant $k \in (0,1)$. Let x^* denote a solution of the equation $Ax = 0$. Set $\epsilon := \frac{1}{2}(\frac{k}{1+L(3+L-k)})$ and define $A_\epsilon : E \to E$ by $A_\epsilon x := x - \epsilon Ax$ for each $x \in E$. For arbitrary $x_0 \in E$, define the sequence $\{x_n\}_{n=0}^{\infty}$ in E by*

$$x_{n+1} = A_\epsilon x_n, \quad n \geq 0.$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to x^* with

$$\|x_{n+1} - x^*\| \leq \delta^n \|x_0 - x^*\|,$$

where $\delta = (1 - \frac{1}{2}k\epsilon) \in (0,1)$. Moreover, x^* is unique.

Corollary 1.4 *Let E be an arbitrary real Banach space and let K be a nonempty convex subset of E. Let $T : K \to K$ be Lipschitz (with constant $L > 0$) and strongly pseudocontractive (i.e., T satisfies inequality (1.4) for all $x, y \in K$). Assume that T has a fixed point $x^* \in K$. Set $\epsilon_0 := \frac{1}{2}(\frac{k}{1+L(3+L-k)})$ and define $T_{\epsilon_0} : K \to K$ by $T_{\epsilon_0}x = (1-\epsilon_0)x + \epsilon_0 Tx$ for each $x \in K$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}_{n=0}^{\infty}$ in K by*

$$x_{n+1} = T_{\epsilon_0}x_n, \quad n \geq 0.$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to x^* with

$$\|x_{n+1} - x^*\| \leq \delta^n \|x_0 - x^*\|,$$

where $\delta := (1 - \frac{1}{2}k\epsilon_0) \in (0,1)$. Moreover, x^* is unique.

Recently, Ćirić *et al.* [17] improved the results of Chidume [15, 16], Liu [9] and Sastry and Babu [11].

We study the convergence of a more general Picard iterative sequence for nonexpansive and Lipschitz strongly accretive mappings in an arbitrary real Banach space. Our results improve the results of Ćirić *et al.* [17].

2 Main results

In the following theorems, $L > 0$ will denote the Lipschitz constant of the operator A and $k > 0$ will denote the strong accretivity constant of A (as in inequality (1.6)). Furthermore, in [17], $\epsilon > 0$ is defined by

$$\epsilon := \frac{k - \eta}{L(2 + L)}, \quad \eta \in (0, k).$$

With these notations, we prove the following theorem.

Theorem 2.1 *Let E be an arbitrary real Banach space, let $A' : E \to E$ be nonexpansive and let $A : E \to E$ be a Lipschitz strongly accretive mapping with strong accretivity constant $k \in (0, 1)$. Let x^* denote a solution of the system $A'x = 0 = Ax$. Define $A_\epsilon : E \to E$ by $A_\epsilon x := x - \epsilon Ax - A'(x - \epsilon Ax)$ for each $x \in E$. For arbitrary $x_0 \in E$, define the sequence $\{x_n\}_{n=0}^{\infty}$ in E by*

$$x_{n+1} = A_\epsilon x_n, \quad n \geq 0.$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to x^* with

$$\|x_{n+1} - x^*\| \leq \rho^n \|x_0 - x^*\|,$$

where $\rho = (1 - \frac{k-\eta}{k(k-\eta)+L(2+L)}\eta) \in (0,1)$. Thus, the choice $\eta = \frac{k}{2}$ yields $\rho = 1 - \frac{k^2}{2[k+2L(2+L)]}$. Moreover, x^* is unique.

Proof Let $S = I - A'$ and $T = I - A$, where I denotes the identity mapping on E. Observe that $A'x^* = 0 = Ax^*$ if and only if x^* is a common fixed point of S and T. Moreover, T is

strongly pseudocontractive since A is strongly accretive. Therefore, T satisfies inequality (1.4) for all $x, y \in E$ and $s > 0$. Furthermore, the recursion formula $x_{n+1} = A_\epsilon x_n$ becomes

$$
\begin{cases}
x_{n+1} = Sy_n, \\
y_n = (1 - \epsilon)x_n + \epsilon Tx_n, \quad n \geq 0.
\end{cases}
\tag{2.1}
$$

Observe that

$$
x^* = (1 + \epsilon)x^* + \epsilon(I - T - kI)x^* - (1 - k)\epsilon x^*,
\tag{2.2}
$$

and from recursion formula (2.1) we get

$$
\begin{aligned}
x_n &= (1 + \epsilon)y_n + \epsilon(I - T - kI)y_n - (1 - k)\epsilon y_n \\
&\quad + \epsilon(x_n - y_n) + \epsilon(Ty_n - Tx_n) \\
&= (1 + \epsilon)y_n + \epsilon(I - T - kI)y_n - (1 - k)\epsilon y_n \\
&\quad + \epsilon^2(x_n - Tx_n) + \epsilon(Ty_n - Tx_n),
\end{aligned}
\tag{2.3}
$$

so that

$$
\begin{aligned}
x_n - x^* &= (1 + \epsilon)\left(y_n - x^*\right) + \epsilon\left[(I - T - kI)y_n - (I - T - kI)x^*\right] \\
&\quad - (1 - k)\epsilon\left(y_n - x^*\right) + \epsilon^2(x_n - Tx_n) + \epsilon(Ty_n - Tx_n).
\end{aligned}
$$

This implies, using inequality (1.4) with $s = \frac{\epsilon}{1+\epsilon}$ and $y = x^*$, that

$$
\begin{aligned}
\left\|x_n - x^*\right\| &\geq (1 + \epsilon)\left[\left\|\left(y_n - x^*\right) + \frac{\epsilon}{1 + \epsilon}\left[(I - T - kI)y_n - (I - T - kI)x^*\right]\right\|\right] \\
&\quad - (1 - k)\epsilon\left\|y_n - x^*\right\| - \epsilon^2\|x_n - Tx_n\| - \epsilon\|Ty_n - Tx_n\| \\
&\geq (1 + \epsilon)\left\|y_n - x^*\right\| - (1 - k)\epsilon\left\|y_n - x^*\right\| - \epsilon^2\|x_n - Tx_n\| - \epsilon\|Ty_n - Tx_n\| \\
&= (1 + k\epsilon)\left\|y_n - x^*\right\| - \epsilon^2\|x_n - Tx_n\| - \epsilon\|Ty_n - Tx_n\|.
\end{aligned}
\tag{2.4}
$$

Observe that

$$
\|x_n - Tx_n\| \leq L\left\|x_n - x^*\right\|, \qquad \|Ty_n - Tx_n\| \leq \epsilon L(1 + L)\left\|x_n - x^*\right\|,
$$

so that

$$
\left\|x_n - x^*\right\| \geq (1 + k\epsilon)\left\|y_n - x^*\right\| - \epsilon\left[1 - k + \epsilon L(2 + L)\right]\left\|x_n - x^*\right\|,
$$

which implies that

$$
\left\|y_n - x^*\right\| \leq \frac{1 + \epsilon\left[1 - k + \epsilon L(2 + L)\right]}{1 + k\epsilon}\left\|x_n - x^*\right\|,
\tag{2.5}
$$

and we have

$$
\begin{aligned}
\rho &= \frac{1 + \epsilon[1 - k + \epsilon L(2 + L)]}{1 + k\epsilon} \\
&= 1 - \frac{\epsilon}{1 + k\epsilon}\big[k - \epsilon L(2 + L)\big] \\
&= 1 - \frac{\epsilon}{1 + k\epsilon}\eta \\
&= 1 - \frac{k - \eta}{k(k - \eta) + L(2 + L)}\eta.
\end{aligned}
\tag{2.6}
$$

From (2.5) and (2.6), we get

$$
\|y_n - x^*\| \le \rho\|x_n - x^*\| \le \cdots \le \rho^n\|x_0 - x^*\|.
\tag{2.7}
$$

Hence, $y_n \to x^*$ as $n \to \infty$. Finally, by (2.1) and (2.7), we obtain

$$
\begin{aligned}
\|x_{n+1} - x^*\| &= \|Sy_n - x^*\| \\
&\le \|y_n - x^*\| \le \rho^n\|x_0 - x^*\| \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Uniqueness follows from the strong accretivity property of A. This completes the proof. $\qquad\square$

The following is an immediate corollary of the above theorem.

Corollary 2.2 *Let E be an arbitrary real Banach space and let K be a nonempty closed convex subset of E. Let $S : K \to K$ be nonexpansive and $T : K \to K$ be Lipschitz (with constant $L > 0$) and strongly pseudocontractive (i.e., T satisfies inequality (1.4) for all $x, y \in K$). Assume that S and T have a common fixed point x^* in K. Set $\epsilon_0 := \frac{k-\eta}{L(2+L)}$, $\eta \in (0, k)$ and define $H_{\varepsilon_0} : K \to K$ by $H_{\varepsilon_0}x = S((1 - \varepsilon_0)x + \varepsilon_0 Tx)$ for each $x \in K$. For arbitrary $x_0 \in E$, define the sequence $\{x_n\}_{n=0}^{\infty}$ in E by*

$$
x_{n+1} = H_{\epsilon_0}x_n, \quad n \ge 0.
$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to x^* with

$$
\|x_{n+1} - x^*\| \le \rho^n\|x_0 - x^*\|,
$$

where $\rho = (1 - \frac{k-\eta}{k(k-\eta)+L(2+L)}\eta) \in (0, 1)$. Moreover, x^* is unique.

Proof Observe that x^* is a common fixed point of S and T, then it is a fixed point of H_{ϵ_0}. Furthermore, recursion formula (2.1) simplifies to the formula

$$
\begin{cases}
x_{n+1} = Sy_n, \\
y_n = (1 - \epsilon_0)x_n + \epsilon_0 Tx_n, \quad n \ge 0,
\end{cases}
\tag{2.8}
$$

which is similar to (2.1). Following the method of computations as in the proof of Theorem 2.1, we obtain

$$
\begin{aligned}
\left\| x_{n+1} - x^* \right\| &= \left\| Sy_n - x^* \right\| \\
&\leq \left\| y_n - x^* \right\| \\
&\leq \frac{1 + \epsilon_0 [1 - k + \epsilon_0 L(2 + L)]}{1 + k\epsilon_0} \left\| x_n - x^* \right\| \\
&\leq \left(1 - \frac{k - \eta}{k(k - \eta) + L(2 + L)} \eta \right) \left\| x_n - x^* \right\|.
\end{aligned}
\tag{2.9}
$$

Set $\rho_0 = 1 - \frac{k-\eta}{k(k-\eta)+L(2+L)} \eta$, then from (2.9) we obtain

$$
\left\| x_{n+1} - x^* \right\| \leq \rho_0 \left\| x_n - x^* \right\| \leq \cdots \leq \rho_0^n \left\| x_0 - x^* \right\| \to 0 \quad \text{as } n \to \infty.
$$

This completes the proof. □

From Theorem 2.1 and Corollary 2.2, we reduce recent results in [17] to the following.

Theorem 2.3 *Let E be an arbitrary real Banach space, let $A : E \to E$ be a Lipschitz (with constant $L > 0$) and strongly accretive mapping with strong accretivity constant $k \in (0, 1)$. Let x^* denote a solution of the equation $Ax = 0$. Set $\epsilon := \frac{k-\eta}{L(2+L)}$, $\eta \in (0, k)$ and define $A_\varepsilon : E \to E$ by $A_\varepsilon x := x - \varepsilon A x$ for each $x \in E$. For arbitrary $x_0 \in E$, define the sequence $\{x_n\}_{n=0}^\infty$ in E by*

$$
x_{n+1} = A_\varepsilon x_n, \quad n \geq 0.
$$

Then $\{x_n\}_{n=0}^\infty$ converges strongly to x^* with

$$
\left\| x_{n+1} - x^* \right\| \leq \theta^n \left\| x_0 - x^* \right\|,
$$

where $\theta = (1 - \frac{k-\eta}{k(k-\eta)+L(2+L)} \eta) \in (0, 1)$. Thus the choice $\eta = \frac{k}{2}$ yields $\theta = 1 - \frac{k^2}{2[k^2 + 2L(2+L)]}$. Moreover, x^* is unique.

Corollary 2.4 *Let E be an arbitrary real Banach space and let K be a nonempty convex subset of E. Let $T : K \to K$ be Lipschitz (with constant $L > 0$) and strongly pseudocontractive (i.e., T satisfies inequality (1.4) for all $x, y \in K$). Assume that T has a fixed point $x^* \in K$. Set $\varepsilon_0 := \frac{k-\eta}{L(2+L)}$, $\eta \in (0, k)$ and define $T_{\varepsilon_0} : K \to K$ by $T_{\varepsilon_0} x = (1 - \varepsilon_0) x + \varepsilon_0 T x$ for each $x \in K$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}_{n=0}^\infty$ in K by*

$$
x_{n+1} = T_{\varepsilon_0} x_n, \quad n \geq 0.
$$

Then $\{x_n\}_{n=0}^\infty$ converges strongly to x^* with

$$
\left\| x_{n+1} - x^* \right\| \leq \theta^n \left\| x_0 - x^* \right\|,
$$

where $\theta := (1 - \frac{k-\eta}{k(k-\eta)+L(2+L)} \eta) \in (0, 1)$. Moreover, x^* is unique.

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