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# Some improvements of Minkowski's integral inequality on time scales

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**Abstract**
In the paper, we establish some improvements of Minkowski's inequality on time scales via the delta integral, nabla integral and diamond-$\alpha$ dynamic integral, which is defined as a linear combination of the delta and nabla integrals.
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**Keywords:** delta and nabla integrals; diamond-$\alpha$ integral; Minkowski's inequality; Hölder's inequality; time scales

## 1 Introduction

In 1988, Hilger in [1] established the theory of time scales in his doctoral dissertation, which resulted in his seminal paper in [2]. His work aimed to unify and generalize various mathematical concepts according to the theories of discrete and continuous analysis. Much information concerning time scales and dynamic equations on time scales can be found in the literature in [3–12]. Since then many authors have studied some integral inequalities on time scales in [13–16]. In [13, 14], the authors described the delta integral Minkowski's inequality on time scales as follows.

**Theorem 1.1** *Let $f,g,h \in C_{\mathrm{rd}}([a,b],\mathbb{R})$ and $1/p + 1/q = 1$ with $p > 1$. Then*

$$\begin{aligned}&\left(\int_a^b |h(x)||f(x)+g(x)|^p \Delta x\right)^{\frac{1}{p}}\\&\quad\leq \left(\int_a^b |h(x)||f(x)|^p \Delta x\right)^{\frac{1}{p}} + \left(\int_a^b |h(x)||g(x)|^p \Delta x\right)^{\frac{1}{p}}.\end{aligned}\tag{1.1}$$

Nabla and diamond-$\alpha$ integral Minkowski's inequality on time scales was established in [15], which can be stated as follows.

**Theorem 1.2** *Let $f,g,h \in C_{\mathrm{ld}}([a,b],\mathbb{R})$ and $1/p + 1/q = 1$ with $p > 1$. Then*

$$\begin{aligned}&\left(\int_a^b |h(x)||f(x)+g(x)|^p \nabla x\right)^{\frac{1}{p}}\\&\quad\leq \left(\int_a^b |h(x)||f(x)|^p \nabla x\right)^{\frac{1}{p}} + \left(\int_a^b |h(x)||g(x)|^p \nabla x\right)^{\frac{1}{p}}.\end{aligned}\tag{1.2}$$

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**Theorem 1.3** *Let $f, g, h : [a, b] \to \mathbb{R}$ be $\diamond_\alpha$-integrable functions, and $1/p + 1/q = 1$ with $p > 1$. Then*

$$\begin{aligned}&\left(\int_a^b |h(x)||f(x)+g(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \\ &\quad \le \left(\int_a^b |h(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} + \left(\int_a^b |h(x)||g(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}}.\end{aligned} \tag{1.3}$$

Throughout paper, we suppose that $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval $[a, b]$ means the intersection of a real interval with the given time scale. By a time scale $\mathbb{T}$, we mean an arbitrary nonempty closed subset of real numbers. The set of real numbers, integers, natural numbers, and the Cantor set are examples of time scales. But rational numbers, irrational numbers, complex numbers, and the open interval between 0 and 1 are not time scales.

The purpose of this paper is to establish some improvements of Minkowski's inequality by using the delta integral, the nabla integral and the diamond-$\alpha$ integral on time scales.

## 2 Main results

In this section, our main results are stated and proved.

**Theorem 2.1** *Let $f, g, h \in C_{\mathrm{rd}}([a, b], \mathbb{R})$, $p > 0$, $s, t \in \mathbb{R}\backslash\{0\}$, and $s \neq t$. Let $p, s, t \in \mathbb{R}$ be different, such that $s, t > 1$ and $(s - t)/(p - t) > 1$. Then*

$$\begin{aligned}&\int_a^b |h(x)||f(x)+g(x)|^p \Delta x \\ &\quad \le \left[\left(\int_a^b |h(x)||f(x)|^s \Delta x\right)^{\frac{1}{s}} + \left(\int_a^b |h(x)||g(x)|^s \Delta x\right)^{\frac{1}{s}}\right]^{s(p-t)/(s-t)} \\ &\qquad \times \left[\left(\int_a^b |h(x)||f(x)|^t \Delta x\right)^{\frac{1}{t}} + \left(\int_a^b |h(x)||g(x)|^t \Delta x\right)^{\frac{1}{t}}\right]^{t(s-p)/(s-t)}.\end{aligned} \tag{2.1}$$

*Proof* We have $(s - t)/(p - t) > 1$, and in view of

$$\begin{aligned}&\int_a^b |h(x)||f(x)+g(x)|^p \Delta x \\ &\quad = \int_a^b |h(x)|\left(|f(x)+g(x)|^s\right)^{(p-t)/(s-t)}\left(|f(x)+g(x)|^t\right)^{(s-p)/(s-t)} \Delta x,\end{aligned}$$

by using Hölder's inequality in [13, 14] with indices $(s - t)/(p - t)$ and $(s - t)/(s - p)$, we have

$$\begin{aligned}&\int_a^b |h(x)||f(x)+g(x)|^p \Delta x \\ &\quad \le \left(\int_a^b |h(x)||f(x)+g(x)|^s \Delta x\right)^{(p-t)/(s-t)} \left(\int_a^b |h(x)||f(x)+g(x)|^t \Delta x\right)^{(s-p)/(s-t)}.\end{aligned} \tag{2.2}$$

On the other hand, by using Minkowski's inequality (1.1) for $s > 1$ and $t > 1$, respectively, we obtain

$$\begin{aligned}&\left(\int_a^b |h(x)||f(x)+g(x)|^s \Delta x\right)^{\frac{1}{s}}\\&\quad\le \left(\int_a^b |h(x)||f(x)|^s \Delta x\right)^{\frac{1}{s}} + \left(\int_a^b |h(x)||g(x)|^s \Delta x\right)^{\frac{1}{s}},\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}&\left(\int_a^b |h(x)||f(x)+g(x)|^t \Delta x\right)^{\frac{1}{t}}\\&\quad\le \left(\int_a^b |h(x)||f(x)|^t \Delta x\right)^{\frac{1}{t}} + \left(\int_a^b |h(x)||g(x)|^t \Delta x\right)^{\frac{1}{t}}.\end{aligned}\tag{2.4}$$

From (2.2), (2.3) and (2.4), this completes the proof of Theorem 2.1. □

**Remark 2.1** For Theorem 2.1, for $p > 1$, letting $s = p + \varepsilon$, $t = p - \varepsilon$, when $p, s, t$ are different, $s, t > 1$, and letting $\varepsilon \to 0$, we obtain (1.1).

**Theorem 2.2** *Let $f, g, h \in C_{\mathrm{rd}}([a,b], \mathbb{R})$, $p > 0$, $s, t \in \mathbb{R}\backslash\{0\}$, and $s \neq t$. Let $p, s, t \in \mathbb{R}$ be different, such that $s, t < 1$ and $s, t \neq 0$, and $(s-t)/(p-t) < 1$. Then*

$$\begin{aligned}&\int_a^b |h(x)||f(x)+g(x)|^p \Delta x\\&\quad\ge \left[\left(\int_a^b |h(x)||f(x)|^s \Delta x\right)^{\frac{1}{s}} + \left(\int_a^b |h(x)||g(x)|^s \Delta x\right)^{\frac{1}{s}}\right]^{s(p-t)/(s-t)}\\&\qquad\times \left[\left(\int_a^b |h(x)||f(x)|^t \Delta x\right)^{\frac{1}{t}} + \left(\int_a^b |h(x)||g(x)|^t \Delta x\right)^{\frac{1}{t}}\right]^{t(s-p)/(s-t)}.\end{aligned}\tag{2.5}$$

*Proof* We have $(s-t)/(p-t) < 1$, and in view of

$$\begin{aligned}&\int_a^b |h(x)||f(x)+g(x)|^p \Delta x\\&\quad= \int_a^b |h(x)|\left(|f(x)+g(x)|^s\right)^{(p-t)/(s-t)}\left(|f(x)+g(x)|^t\right)^{(s-p)/(s-t)} \Delta x,\end{aligned}$$

by using reverse Hölder's inequality in [13, 14] with indices $(s-t)/(p-t)$ and $(s-t)/(s-p)$, we obtain

$$\begin{aligned}&\int_a^b |h(x)||f(x)+g(x)|^p \Delta x\\&\quad\ge \left(\int_a^b |h(x)||f(x)+g(x)|^s \Delta x\right)^{(p-t)/(s-t)}\left(\int_a^b |h(x)||f(x)+g(x)|^t \Delta x\right)^{(s-p)/(s-t)}.\end{aligned}\tag{2.6}$$

On the other hand, in view of Minkowski's inequality (see [17]) for the cases of $s < 1$ and $t < 1$,

$$\begin{aligned}&\left(\int_a^b |h(x)||f(x)+g(x)|^s \Delta x\right)^{\frac{1}{s}}\\&\quad\geq \left(\int_a^b |h(x)||f(x)|^s \Delta x\right)^{\frac{1}{s}} + \left(\int_a^b |h(x)||g(x)|^s \Delta x\right)^{\frac{1}{s}},\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}&\left(\int_a^b |h(x)||f(x)+g(x)|^t \Delta x\right)^{\frac{1}{t}}\\&\quad\geq \left(\int_a^b |h(x)||f(x)|^t \Delta x\right)^{\frac{1}{t}} + \left(\int_a^b |h(x)||g(x)|^t \Delta x\right)^{\frac{1}{t}}.\end{aligned}\tag{2.8}$$

Combining (2.6), (2.7) and (2.8), this completes the proof of Theorem 2.2. □

**Theorem 2.3** *Let $f,g,h \in C_{\mathrm{ld}}([a,b],\mathbb{R})$, $p > 0$, $s,t \in \mathbb{R}\backslash\{0\}$, and $s \neq t$. Let $p,s,t \in \mathbb{R}$ be different, such that $s,t > 1$ and $(s-t)/(p-t) > 1$. Then*

$$\begin{aligned}&\int_a^b |h(x)||f(x)+g(x)|^p \nabla x\\&\quad\leq \left[\left(\int_a^b |h(x)||f(x)|^s \nabla x\right)^{\frac{1}{s}} + \left(\int_a^b |h(x)||g(x)|^s \nabla x\right)^{\frac{1}{s}}\right]^{s(p-t)/(s-t)}\\&\qquad\times \left[\left(\int_a^b |h(x)||f(x)|^t \nabla x\right)^{\frac{1}{t}} + \left(\int_a^b |h(x)||g(x)|^t \nabla x\right)^{\frac{1}{t}}\right]^{t(s-p)/(s-t)}.\end{aligned}\tag{2.9}$$

*Proof* This proof is similar to the proof of Theorem 2.1, so we omit it here. □

**Remark 2.2** For Theorem 2.3, for $p > 1$, letting $s = p+\varepsilon$, $t = p-\varepsilon$, when $p$, $s$, $t$ are different, $s,t > 1$, and letting $\varepsilon \to 0$, we get (1.2).

**Theorem 2.4** *Let $f,g,h \in C_{\mathrm{ld}}([a,b],\mathbb{R})$, $p > 0$, $s,t \in \mathbb{R}\backslash\{0\}$, and $s \neq t$. Let $p,s,t \in \mathbb{R}$ be different, such that $s,t < 1$ and $s,t \neq 0$, and $(s-t)/(p-t) < 1$. Then*

$$\begin{aligned}&\int_a^b |h(x)||f(x)+g(x)|^p \nabla x\\&\quad\geq \left[\left(\int_a^b |h(x)||f(x)|^s \nabla x\right)^{\frac{1}{s}} + \left(\int_a^b |h(x)||g(x)|^s \nabla x\right)^{\frac{1}{s}}\right]^{s(p-t)/(s-t)}\\&\qquad\times \left[\left(\int_a^b |h(x)||f(x)|^t \nabla x\right)^{\frac{1}{t}} + \left(\int_a^b |h(x)||g(x)|^t \nabla x\right)^{\frac{1}{t}}\right]^{t(s-p)/(s-t)}.\end{aligned}\tag{2.10}$$

*Proof* The proof of Theorem 2.4 is similar to the proof of Theorem 2.2, so we omit it here. □

**Theorem 2.5** *Let $f, g, h : [a, b] \to \mathbb{R}$ be $\diamondsuit_\alpha$-integrable functions, $p > 0$, $s, t \in \mathbb{R}\backslash\{0\}$, and $s \neq t$. Let $p, s, t \in \mathbb{R}$ be different, such that $s, t > 1$ and $(s - t)/(p - t) > 1$. Then*

$$\begin{aligned}
&\int_a^b |h(x)||f(x) + g(x)|^p \diamondsuit_\alpha x \\
&\quad \leq \left[\left(\int_a^b |h(x)||f(x)|^s \diamondsuit_\alpha x\right)^{\frac{1}{s}} + \left(\int_a^b |h(x)||g(x)|^s \diamondsuit_\alpha x\right)^{\frac{1}{s}}\right]^{s(p-t)/(s-t)} \\
&\qquad \times \left[\left(\int_a^b |h(x)||f(x)|^t \diamondsuit_\alpha x\right)^{\frac{1}{t}} + \left(\int_a^b |h(x)||g(x)|^t \diamondsuit_\alpha x\right)^{\frac{1}{t}}\right]^{t(s-p)/(s-t)}.
\end{aligned} \tag{2.11}$$

*Proof* This theorem is a direct extension of Theorem 2.1 and Theorem 2.3, so we omit this proof here. □

**Remark 2.3** For Theorem 2.5, for $p > 1$, letting $s = p + \varepsilon$, $t = p - \varepsilon$, when $p, s, t$ are different, $s, t > 1$, and letting $\varepsilon \to 0$, we get (1.3).

**Theorem 2.6** *Let $f, g, h : [a, b] \to \mathbb{R}$ be $\diamondsuit_\alpha$-integrable functions, $p > 0$, $s, t \in \mathbb{R}\backslash\{0\}$, and $s \neq t$. Let $p, s, t \in \mathbb{R}$ be different, such that $s, t < 1$ and $s, t \neq 0$, and $(s - t)/(p - t) < 1$. Then*

$$\begin{aligned}
&\int_a^b |h(x)||f(x) + g(x)|^p \diamondsuit_\alpha x \\
&\quad \geq \left[\left(\int_a^b |h(x)||f(x)|^s \diamondsuit_\alpha x\right)^{\frac{1}{s}} + \left(\int_a^b |h(x)||g(x)|^s \diamondsuit_\alpha x\right)^{\frac{1}{s}}\right]^{s(p-t)/(s-t)} \\
&\qquad \times \left[\left(\int_a^b |h(x)||f(x)|^t \diamondsuit_\alpha x\right)^{\frac{1}{t}} + \left(\int_a^b |h(x)||g(x)|^t \diamondsuit_\alpha x\right)^{\frac{1}{t}}\right]^{t(s-p)/(s-t)}.
\end{aligned} \tag{2.12}$$

*Proof* This theorem is a direct extension of Theorem 2.2 and Theorem 2.4, so we omit this proof here. □

**Remark 2.4** For $\alpha = 1$, inequality (2.11) reduces to inequality (2.1), inequality (2.12) reduces to inequality (2.5). For $\alpha = 0$, inequality (2.11) reduces to inequality (2.9), inequality (2.12) reduces to inequality (2.10). For $\mathbb{T} = \mathbb{R}$, the main results of this paper reduce to the results in [18].

**Remark 2.5** The results of this paper can be given for more general time-scale integrals, for example, Lebesgue time-scale integrals in [19] or even multiple Lebesgue time-scale integrals in [20].

**Competing interests**
The author declares that he has no competing interests.

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