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Cesàro summable difference sequence space

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Abstract

The difference sequence spaces $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$ were introduced by Kizmaz (Can. Math. Bull. 24:169-176, 1981). In this paper, we introduce the Cesáro summable difference sequence space $C_1(\Delta)$ which strictly includes the spaces $c_0(\Delta)$ and $c(\Delta)$ but overlaps with $\ell_\infty(\Delta)$. It is shown that the newly introduced space $C_1(\Delta)$ turns out to be an inseparable BK space which does not possess any of the following: AK property, monotonicity, normality and perfectness. The Köthe-Toeplitz duals of $C_1(\Delta)$ are computed and as an application, the matrix classes $(C_1(\Delta), \ell_\infty)$, $(C_1(\Delta), c; P)$ and $(C_1(\Delta), c_0)$ are also characterized.

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Keywords: sequence space; BK space; Schauder basis; Köthe-Toeplitz duals; matrix map

1 Notations and definitions

By s we shall denote the linear space of all complex sequences over $\mathbb{C}$ (the field of complex numbers). ℓ_∞, c and c_0 denote the spaces of all bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$.

Throughout this paper, unless otherwise specified, we write $\sum_k$ for $\sum_{k=1}^\infty$ and $\lim_n$ for $\lim_{n\to\infty}$.

The definitions given below may be conveniently found in [1–3].

A complete metric linear space is called a Frèchet space. Let X be a linear subspace of s such that X is a Frèchet space with continuous coordinate projections. Then we say that X is an FK space. If the metric of an FK space is given by a complete norm, then we say that X is a BK space.

We say that an FK space X has AK, or has the AK property, if (e_k), the sequence of unit vectors, is a Schauder basis for X.

A sequence space X is called

(i) normal (or solid) if $y = (y_k) \in X$ whenever $|y_k| \le |x_k|$, $k \ge 1$, for some $x = (x_k) \in X$,
(ii) monotone if it contains the canonical preimages of all its stepspaces,
(iii) sequence algebra if $xy = (x_k y_k) \in X$ whenever $x = (x_k), y = (y_k) \in X$,
(iv) convergence free when, if $x = (x_k)$ is in X and if $y_k = 0$ whenever $x_k = 0$, then $y = (y_k)$ is in X.

The idea of dual sequence spaces was introduced by Köthe and Toeplitz [4] whose main results concerned α-duals; the α-dual of $X \subset s$ being defined as

$$X^\alpha = \left\{a = (a_k) \in s : \sum_k |a_k x_k| < \infty \text{ for all } x = (x_k) \in X\right\}.$$

In the same paper [4], they also introduced another kind of dual, namely, the β-dual (see [5] also, where it is called the g-dual by Chillingworth) defined as

$$X^\beta = \left\{a = (a_k) \in s : \sum_k a_k x_k \text{ converges for all } x = (x_k) \in X\right\}.$$

Obviously, $\phi \subset X^\alpha \subset X^\beta$, where ϕ is the well-known sequence space of finitely non-zero scalar sequences. Also, if $X \subset Y$, then $Y^\eta \subset X^\eta$ for $\eta = \alpha, \beta$. For any sequence space X, we denote $(X^\delta)^\eta$ by $X^{\delta\eta}$, where $\delta, \eta = \alpha$ or β. It is clear that $X \subset X^{\eta\eta}$, where $\eta = \alpha$ or β.

For a sequence space X, if $X = X^{\alpha\alpha}$ then X is called a Köthe space or a perfect sequence space.

A sequence space $x = (x_k)$ of complex numbers is said to be $(C,1)$ summable (or Cesàro summable of order 1) to $\ell \in \mathbb{C}$ if $\lim_k \sigma_k = \ell$, where $\sigma_k = \frac{1}{k}\sum_{i=1}^k x_i$. By C_1 we shall denote the linear space of all $(C,1)$ summable sequences of complex numbers over $\mathbb{C}$, *i.e.*,

$$C_1 = \left\{x = (x_k) \in s : \left(\frac{1}{k}\sum_{i=1}^k x_i\right) \in c\right\}.$$

It is easy to see that C_1 is a BK space normed by

$$\|x\| = \sup_k \frac{1}{k}\left|\sum_{i=1}^k x_i\right|, \quad x = (x_k) \in C_1.$$

The notion of difference sequence space was introduced by Kizmaz [6] in 1981 as follows:

$$X(\Delta) = \{x = (x_k) \in s : (\Delta x_k) \in X\}$$

for $X = \ell_\infty, c, c_0$; where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$ (the set of natural numbers). For a detailed account of difference sequence spaces, one may refer to [7–18] where many more references can be found.

2 Motivation and introduction

During the last 32 years, a large amount of work has been carried out by many mathematicians regarding various generalizations of difference sequence spaces of Kizmaz. Keeping aside some exceptions (see, for instance, [7, 8]), in most of these works, the underlying spaces have remained the same, *i.e.*, ℓ_∞, c and c_0. In the present work, we take the opportunity to introduce a difference sequence space with underlying space as C_1.

We observe that

(i) $C_1 \nsubseteq c(\Delta)$ as $((-1)^k) \in C_1$ but $((-1)^k) \notin c(\Delta)$,
(ii) $c(\Delta) \nsubseteq C_1$ as $(k) \in c(\Delta)$ but $(k) \notin C_1$, and
(iii) $c \subset c(\Delta) \cap C_1$.

Thus the sequence spaces C_1 and $c(\Delta)$ overlap but do not contain each other. Similarly, C_1 and ℓ_∞ also overlap without containing each other as is clear from the fact that $C_1 \nsubseteq \ell_\infty$, $\ell_\infty \nsubseteq C_1$ and $c \subset C_1 \cap \ell_\infty$. Note that the sequence $((-1)^{k-1}\sqrt{k})$ is $(C,1)$ summable but not bounded, whereas the sequence $x = (x_k)$ given by $x_1 = 1$, $x_2 = 0$ and

$$x_k = \begin{cases} 1, & \text{if } 2^{i-1} < k \leq 3(2^{i-2}) \ (i = 2, 3, \ldots); \\ 0, & \text{otherwise} \end{cases}$$

is bounded but not $(C,1)$ summable. This has motivated the authors to look for a new sequence space which properly includes the spaces C_1, $c(\Delta)$ and ℓ_∞.

We now introduce a sequence space $C_1(\Delta)$, Cesàro summable difference sequence space, as follows:

$$C_1(\Delta) = \left\{x = (x_k) \in s : (\Delta x_k) \in C_1\right\}.$$

The overall picture regarding inclusions among the already existing spaces ℓ_∞, c, c_0, C_1, $\ell_\infty(\Delta)$, $c(\Delta)$, $c_0(\Delta)$ and the newly introduced space $C_1(\Delta)$ is as shown below:

$$\begin{array}{ccccc}
 & & C_1 & \subset & C_1(\Delta) \\
 & & \cup & & \cup \\
c_0 & \subset & c & \subset & \ell_\infty \\
\cap & & \cap & & \cap \\
c_0(\Delta) & \subset & c(\Delta) & \subset & \ell_\infty(\Delta) \\
 & & \cap & & \\
 & & C_1(\Delta) & &
\end{array}$$

In this paper we show that $C_1(\Delta)$ strictly includes the spaces $c_0(\Delta)$ and $c(\Delta)$ but overlaps with $\ell_\infty(\Delta)$. It is shown that the newly introduced space $C_1(\Delta)$ is an inseparable BK space which does not possess any of the following: AK property, monotonicity, normality and perfectness. The Köthe-Toeplitz duals of $C_1(\Delta)$ are computed, and as an application, the matrix classes $(C_1(\Delta),\ell_\infty)$, $(C_1(\Delta),c;P)$ and $(C_1(\Delta),c_0)$ are also characterized.

3 Inclusion theorems and topological properties of $C_1(\Delta)$

We begin with elementary inclusion theorems justifying that $C_1(\Delta)$ is much wider than ℓ_∞, C_1 and $c(\Delta)$.

Theorem 3.1 $\ell_\infty \subset C_1(\Delta)$, *the inclusion being strict.*

Proof Let $x = (x_k) \in \ell_\infty$. Then there exists $M > 0$ such that $|x_1 - x_{k+1}| \leq M$ for all $k \geq 1$, and so $\frac{1}{k}\sum_{i=1}^{k} \Delta x_i \to 0$ as $k \to \infty$. For strict inclusion, observe that $(k) \in C_1(\Delta)$ but $(k) \notin \ell_\infty$. □

Theorem 3.2 $C_1 \subset C_1(\Delta)$, *the inclusion being strict.*

Proof For $x = (x_k) \in C_1$, we have $\lim_k \frac{1}{k} x_k = 0$, and so $\frac{1}{k}\sum_{i=1}^{k} \Delta x_i \to 0$ as $k \to \infty$. Inclusion is strict in view of the example cited in Theorem 3.1. □

Theorem 3.3 $c(\Delta) \subset C_1(\Delta)$, *the inclusion being strict.*

Proof Inclusion is obvious since $c \subset C_1$. To see that the inclusion is strict, consider the sequence $x = (x_k) = (1,2,1,2,1,2,\ldots)$. □

Remark 3.4 Let X and Y be sequence spaces. If $X \not\subseteq Y$, then $X(\Delta) \not\subseteq Y(\Delta)$.

Proof Since $X \nsubseteq Y$, there is a sequence $x = (x_k) \in X$ such that $x \notin Y$. Consider the sequence $y = (y_k) = (0, -x_1, -x_1 - x_2, -x_1 - x_2 - x_3, \ldots)$. Then $y \in X(\Delta)$ but $y \notin Y(\Delta)$. □

Remark 3.5 We have already observed that $C_1 \nsubseteq \ell_\infty$ and $\ell_\infty \nsubseteq C_1$, so by Remark 3.4, it follows that neither $C_1(\Delta) \subseteq \ell_\infty(\Delta)$ nor $\ell_\infty(\Delta) \subseteq C_1(\Delta)$. Also, we have $c(\Delta) \subset C_1(\Delta) \cap \ell_\infty(\Delta)$. In view of this and Theorem 3.3, we can say that $C_1(\Delta)$ strictly includes $c(\Delta)$ and hence $c_0(\Delta)$ but overlaps with $\ell_\infty(\Delta)$.

We now study the linear topological structure of $C_1(\Delta)$.

Theorem 3.6 *$C_1(\Delta)$ is a BK space normed by*

$$\|x\|_\Delta = |x_1| + \sup_k \frac{1}{k}\left|\sum_{i=1}^{k} \Delta x_i\right|, \quad x = (x_k) \in C_1(\Delta).$$

The proof is a routine verification by using 'standard' techniques and hence is omitted.

Theorem 3.7 *$C_1(\Delta)$ is not separable.*

Proof Let A be the set of all sequences $x_a, x_b, \ldots$, where

$$x_a = (k+a)_k = (1+a, 2+a, \ldots), \qquad x_b = (k+b)_k = (1+b, 2+b, \ldots), \qquad \ldots$$

with $|a-b| > \frac{1}{2}$; $a, b \in \mathbb{R}$. Clearly, $A \subset C_1(\Delta)$ and A is uncountable. Let D be any dense set in $C_1(\Delta)$.

Define a map $f : A \to D$ as follows:

Let $x_a \in A \subset C_1(\Delta)$. As D is dense in $C_1(\Delta)$, so there exists some $z_{x_a} \in D$ such that $\|x_a - z_{x_a}\|_\Delta < \frac{1}{4}$.

We set $f(x_a) = z_{x_a}$.

For $x_a, x_b \in A$, we have

$$\begin{aligned} \|x_a - x_b\|_\Delta &= \bigl|(1+a) - (1+b)\bigr| + \sup_k \frac{1}{k}\left|\sum_{i=1}^{k} \Delta(x_a - x_b)_i\right| \\ &\geq |a-b| \\ &> \frac{1}{2}. \end{aligned}$$

Now

$$\begin{aligned} \|z_{x_a} - x_b\|_\Delta &\geq \|x_a - x_b\|_\Delta - \|x_a - z_{x_a}\|_\Delta \\ &> \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \end{aligned}$$

and already we have $\|x_b - z_{x_b}\|_\Delta < \frac{1}{4}$, therefore $z_{x_a} \neq z_{x_b}$. Hence f is one-to-one. As $f(A) \subset D$ so D is uncountable. Thus, $C_1(\Delta)$ has no countable dense set. □

Corollary 3.8 *$C_1(\Delta)$ does not have a Schauder basis.*

The result follows from the fact that if a normed space has a Schauder basis, then it is separable.

Corollary 3.9 *$C_1(\Delta)$ does not have the AK property.*

Theorem 3.10 *$C_1(\Delta)$ is not normal (solid) and hence neither perfect nor convergence free.*

Proof Taking $x = (x_k) = (k-1)$ and $y = (y_k) = ((-1)^k(k-1))$, we see that $x \in C_1(\Delta)$ but $y \notin C_1(\Delta)$ although $|y_k| \leq |x_k|$, $k \geq 1$ and so $C_1(\Delta)$ is not normal. It is well known [1] that every perfect space, and also every convergence free space, is normal and consequently $C_1(\Delta)$ is neither perfect nor convergence free. □

Theorem 3.11 *$C_1(\Delta)$ is neither monotone nor a sequence algebra.*

Proof Take $x = (x_k) = (k) \in C_1(\Delta)$. Consider $y = (y_k)$ where

$$y_k = \begin{cases} x_k, & \text{if } k \text{ is odd;} \\ 0, & \text{if } k \text{ is even} \end{cases}$$

i.e., $y = (1, 0, 3, 0, 5, \ldots)$. Then $(\Delta y_k) = (1, -3, 3, -5, 5, \ldots)$ and so $(\Delta y_k) \notin C_1$, *i.e.*, $(y_k) \notin C_1(\Delta)$ and hence $C_1(\Delta)$ is not monotone. To see that $C_1(\Delta)$ is not a sequence algebra, take $x = y = (k)$ and observe that $x, y \in C_1(\Delta)$ but $xy = (k^2) \notin C_1(\Delta)$. □

4 Köthe-Toeplitz duals of $C_1(\Delta)$

In this section we compute the Köthe-Toeplitz duals of $C_1(\Delta)$ and show that $C_1(\Delta)$ is not perfect.

Theorem 4.1

$$\left[C_1(\Delta)\right]^\alpha = \left\{a = (a_k) : \sum_k k|a_k| < \infty\right\} = D_1.$$

Proof Let $a = (a_k) \in D_1$. For any $x = (x_k) \in C_1(\Delta)$, we have $(\frac{1}{k}\sum_{i=1}^k \Delta x_i) \in c$, *i.e.*, $(\frac{1}{k}(x_1 - x_{k+1})) \in c$ and so there exists some $M > 0$ such that $|x_k| \leq M(k-1) + x_1$ for $k \geq 1$ and hence $\sup_k k^{-1}|x_k| < \infty$, which implies that

$$\sum_k |a_k x_k| = \sum_k \left(k|a_k|\right)\left(k^{-1}|x_k|\right) < \infty.$$

Thus, $a = (a_k) \in [C_1(\Delta)]^\alpha$.

Conversely, let $a = (a_k) \in [C_1(\Delta)]^\alpha$. Then $\sum_k |a_k x_k| < \infty$ for all $x = (x_k) \in C_1(\Delta)$. Taking $x_k = k$ for all $k \geq 1$, we have $x = (x_k) \in C_1(\Delta)$ whence $\sum_k k|a_k| < \infty$. □

Remark 4.2 It is well known [6, 16] that $[c_0(\Delta)]^\alpha = [c(\Delta)]^\alpha = [\ell_\infty(\Delta)]^\alpha = D_1$, so we conclude that $[c_0(\Delta)]^\alpha = [c(\Delta)]^\alpha = [\ell_\infty(\Delta)]^\alpha = [C_1(\Delta)]^\alpha$, *i.e.*, the α-duals of $c_0(\Delta)$, $c(\Delta)$, $\ell_\infty(\Delta)$ and $C_1(\Delta)$ coincide.

Theorem 4.3

$$[C_1(\Delta)]^{\alpha\alpha} = \left\{a = (a_k) : \sup_k k^{-1}|a_k| < \infty\right\} = D_2.$$

Proof Taking $m = 1$ and $X = c$ in [12, Theorem 2.13], we have $[c(\Delta)]^{\alpha\alpha} = \{a = (a_k) : \sup_k k^{-1}|a_k| < \infty\}$ and the result follows in view of Remark 4.2. □

Corollary 4.4 $C_1(\Delta)$ *is not perfect.*

The proof follows at once when we observe that the sequence $((-1)^k(k-1)) \in [C_1(\Delta)]^{\alpha\alpha}$ but does not belong to $C_1(\Delta)$.

Theorem 4.5

$$[C_1(\Delta)]^{\beta} = \left\{a = (a_k) : \sum_k k|a_k| < \infty\right\} = D_3.$$

Proof Let $a = (a_k) \in D_3$ and $x = (x_k) \in C_1(\Delta)$. Then $(\frac{1}{k}\sum_{i=1}^{k} \Delta x_i) \in c$. For $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} a_k x_k = -\sum_{k=2}^{n} (k-1)a_k \left(\frac{1}{k-1}\sum_{i=1}^{k-1} \Delta x_i\right) + x_1 \sum_{k=1}^{n} a_k.$$

Obviously, (a_k) and $((k-1)a_k) \in \ell_1$. We define $y = (y_k)$ by $y_1 = 0$ and $y_k = \frac{1}{k-1}\sum_{i=1}^{k-1} \Delta x_i$ for all $k \geq 2$. Then $y \in c$ and since $c^{\alpha} = \ell_1$, the series $\sum_{k=2}^{\infty}(k-1)a_k(\frac{1}{k-1}\sum_{i=1}^{k-1} \Delta x_i)$ converges absolutely.

Conversely, if $a = (a_k) \in [C_1(\Delta)]^{\beta}$, then $\sum_k a_k x_k$ converges for all $x = (x_k) \in C_1(\Delta)$. In particular, taking $x_k = 1$ for all k, we have $\sum_k a_k$ converges and so $\sum_{k=2}^{\infty}(k-1) \times a_k(\frac{1}{k-1}\sum_{i=1}^{k-1} \Delta x_i)$ converges for all $x = (x_k) \in C_1(\Delta)$. Since $x = (x_k) \in C_1(\Delta)$ if and only if $y = (\frac{1}{k}\sum_{i=1}^{k} \Delta x_i) \in c$, we have $((k-1)a_k) \in c^{\alpha}$. □

Corollary 4.6 $[c_0(\Delta)]^{\alpha} = [c(\Delta)]^{\alpha} = [\ell_{\infty}(\Delta)]^{\alpha} = [C_1(\Delta)]^{\alpha} = [C_1(\Delta)]^{\beta}$.

5 Matrix maps

Finally, we characterize certain matrix classes. For any complex infinite matrix $A = (a_{nk})$, we shall write $A_n = (a_{nk})_{k\in\mathbb{N}}$ for the sequence in the nth row of A. If X, Y are any two sets of sequences, we denote by (X, Y) the class of all those infinite matrices $A = (a_{nk})$ such that the series $A_n(x) = \sum_k a_{nk}x_k$ converges for all $x = (x_k) \in X$ $(n = 1, 2, \ldots)$ and the sequence $Ax = (A_n x)_{n\in\mathbb{N}}$ is in Y for all $x \in X$.

The following theorem is well known.

Theorem 5.1 [3, p.219] *Let X and Y be BK spaces and suppose that $A = (a_{nk})$ is an infinite matrix such that $(\sum_k a_{nk}x_k)_{n\in\mathbb{N}} \in Y$ for each $x \in X$, i.e., $A \in (X, Y)$, then $A : X \to Y$ is a bounded linear operator.*

Theorem 5.2 $A \in (C_1(\Delta), \ell_{\infty})$ *if and only if* $\sup_n \sum_{k=2}^{\infty}(k-1)|a_{nk}| < \infty$.

Proof Suppose that $\sup_n \sum_{k=2}^{\infty}(k-1)|a_{nk}| < \infty$ and $x = (x_k) \in C_1(\Delta)$. Proceeding as in Theorem 4.5, we have $\sum_{k=2}^{\infty} |a_{nk} \sum_{i=1}^{k-1} \Delta x_i| < \infty$.

For $m \in \mathbb{N}$,

$$\sum_{k=1}^{m} a_{nk}x_k = -\sum_{k=1}^{m} a_{nk}\left(\sum_{i=1}^{k-1} \Delta x_i\right) + x_1 \sum_{k=1}^{m} a_{nk},$$

which yields the absolute convergence of $\sum_k a_{nk}x_k$ for each $n \in \mathbb{N}$, and finally we have

$$\left|\sum_k a_{nk}x_k\right| \le \left(\sup_{k\ge 2}\left|\frac{1}{k-1}\sum_{i=1}^{k-1} \Delta x_i\right|\right)\left(\sup_n \sum_{k=2}^{\infty}(k-1)|a_{nk}|\right) + x_1 \sup_n \sum_k (k-1)|a_{nk}|$$

for all $n \in \mathbb{N}$.

Conversely, by Theorem 5.1, we have

$$\left|\sum_k a_{nk}x_k\right| = \left|A_n(x)\right| \le \sup_n \left|A_n(x)\right| = \|Ax\|_\infty \le \|A\| \|x\|_\Delta \tag{5.1}$$

for each $n \in \mathbb{N}$ and $x = (x_k) \in C_1(\Delta)$.

Choose any $n \in \mathbb{N}$ and any $r \in \mathbb{N}$ and define

$$x_k = \begin{cases} (k-1)\operatorname{sgn} a_{nk}, & \text{if } 1 < k \le r; \\ 0, & \text{otherwise.} \end{cases}$$

Then $x = (x_k) \in c \subset C_1(\Delta)$ with $\|x\|_\Delta = 1$. Inserting this value of $x = (x_k)$ in (5.1) , we have

$$\sum_{k=2}^{r}(k-1)|a_{nk}| \le \|A\|. \tag{5.2}$$

Letting $r \to \infty$ and noting that (5.2) holds for every $n \in \mathbb{N}$, we are through. □

Remark 5.3 If $x = (x_k) \in C_1(\Delta)$, then there exists some $\ell \in \mathbb{C}$ such that $\lim_k \frac{1}{k}\sum_{i=1}^{k} \Delta x_i = \ell$. We shall call ℓ the $C_1(\Delta)$ limit of the sequence (x_k) and by $(C_1(\Delta), c; P)$ we shall denote that subset of $(C_1(\Delta), c)$ for which $C_1(\Delta)$ limits are preserved.

Theorem 5.4 *$A \in (C_1(\Delta), c; P)$ if and only if*

(i) $\sup_n \sum_{k=2}^{\infty}(k-1)|a_{nk}| < \infty$,
(ii) $\lim_n \sum_k (k-1)a_{nk} = -1$,
(iii) $\lim_n a_{nk} = 0$ *for each* k,
(iv) $\lim_n \sum_k a_{nk} = 0$.

Proof Let the conditions (i)-(iv) hold and suppose that $x = (x_k) \in C_1(\Delta)$ with $\lim_k \frac{1}{k} \times \sum_{i=1}^{k} \Delta x_i = \ell$. It is implicit in (i) that, for each $n \in \mathbb{N}$, $\sum_k (k-1)|a_{nk}|$ converges. It follows that $\sum_{k=2}^{\infty}(k-1)a_{nk}(\frac{1}{k-1}\sum_{i=1}^{k-1} \Delta x_i)$ converges, whence

$$\sum_k a_{nk}x_k = -\sum_{k=2}^{\infty}(k-1)a_{nk}\left(\frac{1}{k-1}\sum_{i=1}^{k-1} \Delta x_i - \ell\right) - \ell \sum_k (k-1)a_{nk} + x_1 \sum_k a_{nk}. \tag{5.3}$$

Let $\epsilon_k = \frac{1}{k}\sum_{i=1}^{k} \Delta x_i - \ell$, $H = \sup_k |\epsilon_k|$ and $M = \sup_n \sum_k (k-1)|a_{nk}|$.

Then, for any $p \in \mathbb{N}$, we have

$$\left|\sum_{k=2}^{\infty}(k-1)a_{nk}\left(\frac{1}{k-1}\sum_{i=1}^{k-1}\Delta x_i - \ell\right)\right| \le H\sum_{k=2}^{p}(k-1)|a_{nk}| + M\sup_{k>p}|\epsilon_{k-1}|$$

and hence

$$\limsup_{n}\left|\sum_{k=2}^{\infty}(k-1)a_{nk}\left(\frac{1}{k-1}\sum_{i=1}^{k-1}\Delta x_i - \ell\right)\right| \le M\sup_{k>p}|\epsilon_{k-1}|.$$

Letting $p \to \infty$, we have $\sum_{k=2}^{\infty}(k-1)a_{nk}(\frac{1}{k-1}\sum_{i=1}^{k-1}\Delta x_i - \ell) \to 0$ as $n \to \infty$. Making use of this and also of (ii) and (iv) in (5.3), we get the result.

Conversely, let $A \in (C_1(\Delta), c; P)$. Then $(\sum_k a_{nk}x_k)_{n\in\mathbb{N}} \in c$ for all $x = (x_k) \in C_1(\Delta)$. By the same argument as in Theorem 5.2, we have $\sup_n \sum_{k=2}^{\infty}(k-1)|a_{nk}| < \infty$. Taking $x = e_k \in C_1(\Delta)$, we get $(a_{nk})_{n\in\mathbb{N}} \in c$ with $\lim_n a_{nk} = 0$ for each k. Also, for $x = (k-1)$, we have $(\sum_k (k-1)a_{nk})_{n\in\mathbb{N}} \in c$ with $\lim_n \sum_k (k-1)a_{nk} = -1$, and finally $x = (1,1,1,\dots) \in C_1(\Delta)$ yields $\lim_n \sum_k a_{nk} = 0$. □

Theorem 5.5 $A \in (C_1(\Delta), c_0)$ *if and only if*

(i) $\sup_n \sum_{k=2}^{\infty}(k-1)|a_{nk}| < \infty$,
(ii) $\lim_n \sum_k (k-1)a_{nk} = 0$,
(iii) $\lim_n a_{nk} = 0$ *for each* k,
(iv) $\lim_n \sum_k a_{nk} = 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
VKB and SG contributed equally. All authors read and approved the final manuscript.

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