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Some criteria for boundedness and compactness of the Hardy operator with some special kernels

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Abstract

We present necessary and sufficient conditions for boundedness and compactness of Hardy operator (1.4) with kernel (1.3) for $1 < p \le q < \infty$.

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1 Introduction

First, let us recall that the weighted Lebesgue space $L^r(w)$, with $r \ge 1$ and w a weight function on $(0,\infty)$, is defined as a set of all functions $f = f(x)$ such that

$$\int_0^\infty |f(x)|^r w(x)\,dx < \infty.$$

We will investigate the *Hardy-type inequality*

$$\left(\int_0^\infty \left|\int_0^x k(x,t)f(t)\,dt\right|^q u(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty |f(x)|^p v(x)\,dx\right)^{\frac{1}{p}} \tag{1.1}$$

with $1 < p, q < \infty$ and u, v weight functions on $(0,\infty)$.

This inequality was investigated by many authors. For $k(x,t) \equiv 1$, we obtain the 'classical' Hardy inequality

$$\left(\int_0^\infty \left|\int_0^x f(t)\,dt\right|^q u(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty |f(x)|^p v(x)\,dx\right)^{\frac{1}{p}}, \tag{1.2}$$

the case $k(x,t) = a(x)b(t)$ can be easily reduced to the classical case with modified weights $U(x) = |a(x)|^q u(x)$ and $V(x) = |b(x)|^{-p} v(x)$ instead of u and v, respectively. Stepanov [1] has investigated 'convolutionary' kernels like $(x-t)^\alpha$, and probably the most general approach is connected with the name of Oinarov who investigated positive kernels k such that

$$k(x,t) \approx k(x,z) + k(z,t), \quad t < z < x$$

(see [2]) and also several more general kernels (see [3, 4]). Also, inequalities with modified or generalized kernels have been investigated; let us mention the recent book [5] where multiple Hardy-type inequalities with the so-called product kernels are considered. See also [6] and [7] for further details.

Here we consider kernels $k(x,t)$ of the type

$$k(x,t) = \sum_{i=1}^{m} a_i(x)b_i(t) \tag{1.3}$$

and we want to find conditions on the weight functions u, v and on the functions a_i, b_i, for which the integral operator

$$(kf)(x) := \int_0^x k(x,t)f(t)\,\mathrm{d}t \tag{1.4}$$

maps the space $L^p(v)$ continuously into $L^q(u)$.

Remark 1.1 The case of kernel (1.3) with special functions $b_i(t) = t^{i-1}$ was investigated by Rychkov [8] for $p = q = 2$. Such kernels appear for general p, q by the investigation of higher-order Hardy inequalities; see [9].

Now, let us denote, for given a_i, b_i, u and v,

$$A_i(x) = a_i(x)u^{1/q}(x), \qquad B_i(t) = b_i(t)v^{-1/p}(t). \tag{1.5}$$

Then we can rewrite inequality (1.1) [for functions f] as the unweighted inequality

$$\left(\int_0^\infty \left|\int_0^x K(x,t)g(t)\,\mathrm{d}t\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty \left|g(x)\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \tag{1.6}$$

[for functions $g(x) = f(x)v^{1/p}(x)$] with

$$K(x,t) = \sum_{i=1}^{m} A_i(x)B_i(t). \tag{1.7}$$

Remark 1.2 If $m = 1$, we have $K(x,t) = A(x)B(t)$, and we can rewrite (1.6) as

$$\left(\int_0^\infty \left|\int_0^x B(t)g(t)\,\mathrm{d}t\right|^q \left|A(x)\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty \left|g(x)\right|^p \mathrm{d}x\right)^{\frac{1}{p}},$$

which is in fact the classical (weighted) Hardy inequality

$$\left(\int_0^\infty \left|\int_0^x h(t)\,\mathrm{d}t\right|^q \left|A(x)\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \le C\left(\int_0^\infty \left|h(x)\right|^p \left|B(x)\right|^{-p} \mathrm{d}x\right)^{\frac{1}{p}} \tag{1.8}$$

[for the function $h(x) = g(x)B(x)$] with the weight functions $|A(x)|^q$ and $|B(x)|^{-p}$.

It is well known (see, *e.g.*, [6] or [7]) that for the case $1 < p \leq q < \infty$, inequality (1.8) holds for all functions $h \geq 0$ if and only if the so-called Muckenhoupt-Bradley condition

$$\sup_{z>0} A_M(z) < \infty$$

is satisfied, where

$$\begin{aligned} A_M(z) &= \left(\int_z^\infty |A(x)|^q \,\mathrm{d}x\right)^{1/q} \left(\int_0^z \left[|B(x)|^{-p}\right]^{1-p'} \,\mathrm{d}x\right)^{1/p'} \\ &= \left(\int_z^\infty |A(x)|^q \,\mathrm{d}x\right)^{1/q} \left(\int_0^z |B(x)|^{p'} \,\mathrm{d}x\right)^{1/p'} \end{aligned} \tag{1.9}$$

with $p' = p/(p-1)$.

It follows from (1.9) that we need the integrability of $|A(x)|^q$ on (z,∞) and of $|B(x)|^{p'}$ on $(0,z)$ for all $z > 0$, *i.e.*,

$$A \in L^q(z,\infty) \quad \text{and} \quad B \in L^{p'}(0,z)$$

[where we denote by $L^r(\alpha,\beta)$ the classical Lebesgue spaces of functions defined on (α,β)].

In accordance with this remark, we suppose throughout the paper that the functions A_i, B_i from (1.7) satisfy

$$A_i \in L^q(z,\infty) \quad \text{and} \quad B_i \in L^{p'}(0,z) \tag{1.10}$$

for all $z > 0$ and $i = 1, 2, \ldots, m$.

2 Sufficient conditions

For $p > 1$ and u, v weight functions on $(0,\infty)$, let us define

$$A_M(x;u,v) = A_M(x) := \left(\int_x^\infty u(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \left(\int_0^x v^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{1}{p'}}. \tag{2.1}$$

Remark 2.1 Let us recall that the condition

$$\sup_{x>0} A_M(x) < \infty$$

is necessary and sufficient for the validity of the classical Hardy inequality (1.2) for $p \leq q$.

It is easy to find a sufficient condition for (1.1) to hold, if we consider the 'partial' operators

$$(k_i f)(x) := \int_0^x a_i(x) b_i(t) f(t)\,\mathrm{d}t \tag{2.2}$$

as operators from $L^p(v)$ into $L^q(u)$.

Theorem 2.2 *Let* $1 < p \leq q < \infty$. *For* $A_M(x)$ *defined by* (2.1), *denote*

$$A_{M,i}(x) := A_M\bigl(x; u|a_i|^q, v|b_i|^{-p}\bigr), \quad i = 1, 2, \ldots, m, \tag{2.3}$$

where a_i, b_i *are the functions in* (1.3). *Then the Hardy-type inequality* (1.1) *with kernel* $k(x,t)$ *from* (1.3) *holds if the weight functions* u, v *satisfy for* $i = 1, 2, \ldots, m$ *the conditions*

$$\sup_{x>0} A_{M,i}(x) < \infty. \tag{2.4}$$

Proof Conditions (2.4) guarantee the validity of the Hardy inequality

$$\left(\int_0^\infty \left|\int_0^x g(t)\,\mathrm{d}t\right|^q u(x)\bigl|a_i(x)\bigr|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \leq C_{1,i}\left(\int_0^\infty \bigl|g(t)\bigr|^p v(t)\bigl|b_i(t)\bigr|^{-p}\,\mathrm{d}t\right)^{\frac{1}{p}}$$

for functions g; if we take $g(t) = f(t)b_i(t)$, we can rewrite the foregoing Hardy inequality as

$$\left(\int_0^\infty \left|\int_0^x a_i(x)b_i(t)f(t)\,\mathrm{d}t\right|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \leq C_{1,i}\left(\int_0^\infty \bigl|f(t)\bigr|^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}}.$$

Now, using (1.3), the Minkowski and the last inequality, we obtain

$$\begin{aligned}
\left(\int_0^\infty \left|\int_0^x k(x,t)f(t)\,\mathrm{d}t\right|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} &= \left(\int_0^\infty \left|\sum_{i=1}^m \int_0^x a_i(x)b_i(t)f(t)\,\mathrm{d}t\right|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\leq \sum_{i=1}^m \left(\int_0^\infty \left|\int_0^x a_i(x)b_i(t)f(t)\,\mathrm{d}t\right|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\leq \sum_{i=1}^m C_{1,i}\left(\int_0^\infty \bigl|f(t)\bigr|^p v(t)\,\mathrm{d}t\right)^{\frac{1}{p}},
\end{aligned}$$

i.e., we have derived inequality (1.1). □

Remark 2.3 Let us mention that the expression $A_{M,i}(x)$ in (2.3) is nothing else than the expression $A_M(x)$ from (1.9), where we replace A, B by A_i, B_i, respectively, with A_i, B_i from (1.5).

3 Necessary conditions

First let us introduce some auxiliary notions.

Definition 3.1 We will say that the system (matrix) $\{D_{i,j}\}_{i,j=1}^m$ of real numbers $D_{i,j}$ satisfies the ellipticity condition if there exists a constant $C_E > 0$ such that

$$\sum_{i,j=1}^m D_{i,j}\xi_i\xi_j \geq C_E \sum_{i=1}^m D_{i,i}\xi_i^2 \quad \text{for all } \xi = \{\xi_i\}_{i=1}^m \in \mathbb{R}^m.$$

Remark 3.2 (i) The ellipticity condition is equivalent to the positive definiteness of the quadratic form

$$\sum_{i,j=1}^{m} \tilde{D}_{i,j}\xi_i\xi_j \quad \text{with } \xi = \{\xi_i\}_{i=1}^m \in \mathbb{R}^m,$$

where

$$\tilde{D}_{i,j} = \begin{cases} (1-c)D_{i,i}, & i = j, \\ D_{i,j}, & i \neq j \end{cases}$$

with some $0 < c < 1$.

(ii) A sufficient condition for the ellipticity to be satisfied is

$$D_{i,j} \leq (1-c)\sqrt{D_{i,i}D_{j,j}} \quad \text{for } i \neq j.$$

Indeed, if we denote $M^- = \{(i,j) : D_{i,j}\xi_i\xi_j < 0, 1 \leq i,j \leq m\}$, then

$$\begin{aligned} \sum_{i,j=1}^{m} D_{i,j}\xi_i\xi_j &\geq \sum_{i=1}^{m} D_{i,i}\xi_i^2 - \sum_{(i,j)\in M^-} |D_{i,j}||\xi_i||\xi_j| \\ &\geq \sum_{i=1}^{m} D_{i,i}\xi_i^2 - \sum_{(i,j)\in M^-} (1-c)\sqrt{D_{i,i}D_{j,j}}|\xi_i||\xi_j| \\ &\geq \sum_{i=1}^{m} D_{i,i}\xi_i^2 - \sum_{(i,j)\in M^-} \frac{(1-c)}{2}\left(D_{i,i}\xi_i^2 + D_{j,j}\xi_j^2\right) \\ &\geq c\sum_{i=1}^{m} D_{i,i}\xi_i^2. \end{aligned}$$

To find necessary conditions for (1.1) to hold, we consider three cases.

(I) The case $p = 2$, $1 < q < \infty$. Denote for this case

$$A_2 := \sup_{z>0}\left(\int_z^\infty \left(\int_0^z k^2(x,t)v^{-1}(t)\,\mathrm{d}t\right)^{\frac{q}{2}} u(x)\,\mathrm{d}x\right)^{\frac{1}{q}}.$$

Theorem 3.3 *Let $p = 2$ and $1 < q < \infty$. Then the following condition*

$$A_2 < \infty \tag{3.1}$$

is necessary for inequality (1.1) *to hold.*

Proof For $z > 0$ and for $\{B_i\}_{i=1}^m$ from (1.5) let $\{B_{i_k}\}_{k=1}^n$ be an arbitrary maximal linearly independent subsystem of $\{B_i\}_{i=1}^m$ in $L^2(0,z)$, which, for simplicity, we denote by $\{B_i\}_{i=1}^n$. Using the Gram-Schmidt method of orthogonalization to the system $\{B_i\}_{i=1}^n$ in $L^2(0,z)$, we obtain a system $\{B_{i,z}\}_{i=1}^n$ such that

$$B_i(t) = \sum_{j=1}^{n} \beta_{i,j}(z)B_{j,z}(t)$$

for $t \in (0,z)$ and $i = 1,2,\dots,m$. Using this we rewrite the kernel in (1.6) in the form

$$\begin{aligned} K(x,t) &= \sum_{i=1}^{m} A_i(x)B_i(t) \\ &= \sum_{i=1}^{m} A_i(x) \sum_{j=1}^{n} \beta_{i,j}(z)B_{j,z}(t) \\ &= \sum_{j=1}^{n} B_{j,z}(t) \sum_{i=1}^{m} \beta_{i,j}(z)A_i(x) \\ &= \sum_{j=1}^{n} A_{j,z}(x)B_{j,z}(t) = K_z(x,t), \end{aligned}$$

for $t \in (0,z)$, where $A_{j,z}(x) = \sum_{i=1}^{m} \beta_{i,j}(z)A_i(x)$.

Let $1 \le j \le n$, then choosing $f_{j,z}(t) = \chi_{(0,z)}(t)B_{j,z}(t)\|B_{j,z}\|^{-1}_{L^2(0,z)}$ and using the orthogonality of the system $\{B_{i,z}\}_{i=1}^{n}$, we estimate the left-hand side of (1.6) as

$$\begin{aligned} \|Kf_{j,z}\|_{L^q} &= \left(\int_0^\infty \left|\int_0^x K(x,t)f_{j,z}(t)\,\mathrm{d}t\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \\ &\ge \left(\int_z^\infty \left|\int_0^z K(x,t)f_{j,z}(t)\,\mathrm{d}t\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \\ &= \left(\int_z^\infty \left|\int_0^z K_z(x,t)B_{j,z}(t)\,\mathrm{d}t\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \|B_{j,z}\|^{-1}_{L^2(0,z)} \\ &= \left(\int_z^\infty \left|\sum_{i=1}^{n} A_{i,z}(x)\int_0^z B_{i,z}(t)B_{j,z}(t)\,\mathrm{d}t\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \|B_{j,z}\|^{-1}_{L^2(0,z)} \\ &= \left(\int_z^\infty \left|A_{j,z}(x)\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \left(\int_0^z \left|B_{j,z}(t)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}. \end{aligned} \tag{3.2}$$

This estimate together with (1.6) implies

$$\begin{aligned} nC = C\sum_{j=1}^{n} \|f_{j,z}\|_{L^2} \ge \sum_{j=1}^{n} \|Kf_{j,z}\|_{L^q} &\\ \ge \sum_{j=1}^{n} \left(\int_z^\infty \left|A_{j,z}(x)\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \left(\int_0^z \left|B_{j,z}(t)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}} &\\ := A_2^*(z). & \end{aligned} \tag{3.3}$$

Using the Minkowski inequality, we estimate $A_2^*(z)$ in the form

$$\begin{aligned} A_2^*(z) &= \sum_{j=1}^{n} \left(\int_z^\infty \left[\left(\int_0^z \left|A_{j,z}(x)B_{j,z}(t)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right]^q \mathrm{d}x\right)^{\frac{1}{q}} \\ &\ge \left(\int_z^\infty \left[\sum_{j=1}^{n} \left(\int_0^z \left|A_{j,z}(x)B_{j,z}(t)\right|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right]^q \mathrm{d}x\right)^{\frac{1}{q}} \end{aligned}$$

$$\geq \left(\int_z^\infty \left(\int_0^z \left|\sum_{j=1}^n A_{j,z}(x)B_{j,z}(t)\right|^2 \mathrm{d}t\right)^{\frac{q}{2}} \mathrm{d}x\right)^{\frac{1}{q}}$$

$$= \left(\int_z^\infty \left(\int_0^z \left|\sum_{j=1}^n \sum_{i=1}^m A_i(x)\beta_{i,j}(z)B_{j,z}(t)\right|^2 \mathrm{d}t\right)^{\frac{q}{2}} \mathrm{d}x\right)^{\frac{1}{q}}$$

$$= \left(\int_z^\infty \left(\int_0^z \left|\sum_{i=1}^m \sum_{j=1}^n A_i(x)\beta_{i,j}(z)B_{j,z}(t)\right|^2 \mathrm{d}t\right)^{\frac{q}{2}} \mathrm{d}x\right)^{\frac{1}{q}}$$

$$= \left(\int_z^\infty \left(\int_0^z \left|\sum_{i=1}^m A_i(x)B_i(t)\right|^2 \mathrm{d}t\right)^{\frac{q}{2}} \mathrm{d}x\right)^{\frac{1}{q}}$$

$$= \left(\int_z^\infty \left(\int_0^z k^2(x,t)v^{-1}(t)\,\mathrm{d}t\right)^{\frac{q}{2}} u(x)\,\mathrm{d}x\right)^{\frac{1}{q}}. \tag{3.4}$$

Since $z > 0$ is arbitrary and inequality (1.1) (*i.e.*, inequality (1.6)) holds, we conclude that C is finite and we get (3.1). □

Moreover, we can show that condition (2.4) is also necessary if we add some assumptions. For this purpose, denote

$$D_{i,j}(z) = \int_0^z B_i(t)B_j(t)\,\mathrm{d}t = \int_0^z b_i(t)b_j(t)v^{-1}(t)\,\mathrm{d}t, \tag{3.5}$$

where $B_i(t) = b_i(t)v^{-1/2}(t)$.

Definition 3.4 We say that the condition E_2 is satisfied if there exists a constant C_E such that for every $z > 0$ the system $\{D_{i,j}(z)\}_{i,j=1}^m$ satisfies the ellipticity condition.

Theorem 3.5 *Let* $p = 2$ *and* $1 < q < \infty$. *Suppose that the system* $\{D_{i,j}(z)\}_{i,j=1}^m$ *from* (3.5) *satisfies the condition* E_2. *Then* (*sufficient*) *condition* (2.4) *is necessary for* (1.1) *to hold*.

Proof Using Theorem 3.3, formula (3.4), the condition E_2 and the Minkowski inequality, we obtain

$$\sum_{j=1}^n \|Kf_{j,z}\|_{L^q} \geq \left(\int_z^\infty \left(\int_0^z k^2(x,t)v^{-1}(t)\,\mathrm{d}t\right)^{\frac{q}{2}} u(x)\,\mathrm{d}x\right)^{\frac{1}{q}}$$

$$= \left(\int_z^\infty \left(\int_0^z \sum_{i,j=1}^m a_i(x)a_j(x)b_i(t)b_j(t)v^{-1}(t)\,\mathrm{d}t\right)^{\frac{q}{2}} u(x)\,\mathrm{d}x\right)^{\frac{1}{q}}$$

$$= \left(\int_z^\infty \left(\sum_{i,j=1}^m D_{i,j}(z)A_i(x)A_j(x)\right)^{\frac{q}{2}} \mathrm{d}x\right)^{\frac{1}{q}}$$

$$\geq \sqrt{C_E}\left(\int_z^\infty \left(\sum_{i=1}^m D_{i,i}(z)A_i(x)^2\right)^{\frac{q}{2}} \mathrm{d}x\right)^{\frac{1}{q}}$$

$$= \sqrt{C_E}\left(\int_z^\infty \left(\sum_{i=1}^m A_i(x)^2 \int_0^z B_i(t)^2\,\mathrm{d}t\right)^{\frac{q}{2}} \mathrm{d}x\right)^{\frac{1}{q}}$$

$$\begin{aligned}
&\geq \sqrt{C_E}\left(\int_z^\infty |A_i(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_0^z B_i(t)^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \\
&= \sqrt{C_E}\left(\int_z^\infty |a_i(x)|^q u(x) \,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_0^z b_i^2(t)v^{-1}(t) \,\mathrm{d}t\right)^{\frac{1}{2}} \\
&= \sqrt{C_E}A_{M,i}(z) \qquad (3.6)
\end{aligned}$$

for arbitrary $i : 1 \leq i \leq m$. Consequently, from (3.3) and (3.6), we get (2.4). The theorem is proved. □

(II) The case $1 < p < 2$, $1 < q < \infty$. Let us denote

$$D_{i,l}^j(z) = \int_0^z \tilde{B}_i^j(t)\tilde{B}_l^j(t) \,\mathrm{d}t, \quad 1 \leq i,j,l \leq m,$$

where $\tilde{B}_i^j(t) := B_i(t)|B_j(t)|^{\frac{p'-2}{2}}$ belongs to $L^2(0,z)$ since

$$\int_0^z |\tilde{B}_i^j(t)|^2 \,\mathrm{d}t \leq \left(\int_0^z |B_i(t)|^{p'} \,\mathrm{d}t\right)^{\frac{2}{p'}}\left(\int_0^z |B_j(t)|^{p'} \,\mathrm{d}t\right)^{\frac{p'-2}{2p'}} < \infty.$$

Definition 3.6 We will say that the condition E_p is satisfied if there exists a constant $C_E > 0$ such that for every $z > 0$ the system $\{D_{i,l}^j(z)\}_{i,l=1}^m$ satisfies the ellipticity condition for $j = 1, 2, \ldots, m$.

Theorem 3.7 *Let* $1 < p < 2$ *and* $1 < q < \infty$. *If the condition* E_p *holds, then* (2.4) *is necessary for inequality* (1.1) *to hold.*

Proof Let $z > 0$ and let $1 \leq j \leq m$. Let $\{\tilde{B}_{i_k}^j\}_{k=1}^n$ be a maximal linearly independent subsystem of $\{\tilde{B}_i^j\}_{i=1}^m$ in $L^2(0,z)$, which, for simplicity, we denote by $\{\tilde{B}_i^j\}_{i=1}^n$. Thus, using the method of orthogonalization, we get an orthogonal system $\{\tilde{B}_{i,z}^j\}_{i=1}^n$ in $L^2(0,z)$ such that

$$\tilde{B}_i^j(t) = \sum_{l=1}^n \beta_{i,l}^j(z)\tilde{B}_{l,z}^j(t), \quad i = 1, 2, \ldots, m. \qquad (3.7)$$

If we denote $g(t) = \tilde{B}_{l,z}^j(t)\|\tilde{B}_{l,z}^j\|_{L^2(0,z)}^{-1}$, $c_j = \|B_j\|_{L^{p'}(0,z)}^{\frac{p-2}{2p-2}}$ and choose the following test function in (1.6) as

$$f_{l,z}^j(t) = \chi_{(0,z)}(t)c_j|B_j(t)|^{\frac{p'-2}{2}}g(t),$$

then the left- and right-hand sides are estimated in the forms:

$$\begin{aligned}
\|f_{l,z}^j\|_{L^p} &= c_j\left(\int_0^z |B_j(t)|^{\frac{(p'-2)p}{2}}|g(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} \\
&\leq c_j\left(\int_0^z |B_j(t)|^{p'} \,\mathrm{d}t\right)^{\frac{2-p}{2p}}\left(\int_0^z |g(t)|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \\
&= 1, \qquad (3.8)
\end{aligned}$$

and

$$\begin{aligned}\|Kf^j_{l,z}\|_{L^q} &= \left(\int_0^\infty \left|\int_0^x K(x,t)f^j_{l,z}(t)\,dt\right|^q dx\right)^{\frac{1}{q}}\\ &\ge \left(\int_z^\infty \left|\int_0^z K(x,t)f^j_{l,z}(t)\,dt\right|^q dx\right)^{\frac{1}{q}}\\ &= c_j\left(\int_z^\infty \left|\int_0^z \tilde{K}^j(x,t)g(t)\,dt\right|^q dx\right)^{\frac{1}{q}}, \end{aligned} \tag{3.9}$$

where $\tilde{K}^j(x,t) = \sum_{i=1}^m A_i(x)\tilde{B}^j_i(t)$.

Using (3.7) we rewrite the kernel $\tilde{K}^j(x,t)$ in the form

$$\begin{aligned}\tilde{K}^j(x,t) &= \sum_{i=1}^m A_i(x)\left[\sum_{l=1}^n \beta^j_{i,l}(z)\tilde{B}^j_{l,z}(t)\right]\\ &= \sum_{l=1}^n A^j_{l,z}(x)\tilde{B}^j_{l,z}(t) = \tilde{K}^j_z(x,t),\end{aligned}$$

where $A^j_{l,z}(x) = \sum_{i=1}^m A_i(x)\beta^j_{i,l}(z)$.

Then we have from (3.8), (3.9) and (1.6) that

$$c_j\left(\int_z^\infty \left|\int_0^z \tilde{K}^j_z(x,t)g(t)\,dt\right|^q dx\right)^{\frac{1}{q}} \le \|Kf^j_{l,z}\|_{L^q} \le C. \tag{3.10}$$

Now, repeating the proofs of the foregoing theorems with respect to (3.10) with the kernel $\tilde{K}^j_z(x,t)$, we can also get a similar estimate as in (3.2) in the form

$$nC \ge \sum_{l=1}^n \|Kf^j_{l,z}\|_{L^q} \ge c_j\sum_{l=1}^n \left(\int_z^\infty |A^j_{l,z}(x)|^q dx\right)^{\frac{1}{q}}\left(\int_0^z |\tilde{B}^j_{l,z}(t)|^2 dt\right)^{\frac{1}{2}}. \tag{3.11}$$

Then supposing the condition E_p, we also obtain the following estimate:

$$\begin{aligned}\sum_{l=1}^n \|Kf^j_{l,z}\|_{L^q} &\ge c_j\sum_{l=1}^n \left(\int_z^\infty |A^j_{l,z}(x)|^q dx\right)^{\frac{1}{q}}\left(\int_0^z |B^j_{l,z}(t)|^2 dt\right)^{\frac{1}{2}}\\ &\ \vdots\\ &\ge c_j\left(\int_z^\infty \left(\sum_{i,l=1}^m D^j_{i,l}(z)A_i(x)A_l(x)\right)^{\frac{q}{2}} dx\right)^{\frac{1}{q}}\\ &\ge c_j\sqrt{C_E}\left(\int_z^\infty \left(\sum_{i=1}^m D^j_{i,i}(z)A_i(x)^2\right)^{\frac{q}{2}} dx\right)^{\frac{1}{q}}\\ &\ge c_j\sqrt{C_E}\left(\int_z^\infty |A_j(x)|^q dx\right)^{\frac{1}{q}}\left(\int_0^z |B_j(t)|^{p'} dt\right)^{\frac{1}{2}},\end{aligned}$$

i.e.,

$$\sum_{l=1}^{n}\|Kf_{l,z}^{j}\|_{L^q} \geq c_j\sqrt{C_E}\left(\int_z^\infty |A_j(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_0^z |B_j(t)|^{p'} \,\mathrm{d}t\right)^{\frac{1}{2}}$$
$$\geq \sqrt{C_E}\left(\int_z^\infty |a_j(x)|^q u(x) \,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_0^z |b_j(t)|^{p'} v^{1-p'}(t) \,\mathrm{d}t\right)^{\frac{1}{p'}}$$
$$= \sqrt{C_E}A_{M,j}(z), \tag{3.12}$$

which holds for $1 \leq j \leq m$ and $z > 0$.

Using this estimate and (3.11), we finally get that

$$\frac{\sqrt{C_E}}{n}\sup_{z>0} A_{M,j}(z) \leq C.$$

The proof is complete. □

(III) The case $1 < p < \infty$, $2 \leq q < \infty$. Another approach how to investigate inequality (1.1) is based on the following lemma (for details, see [7]).

Lemma 3.8 *Let $1 < p,q < \infty$. Then inequality* (1.6) *holds for all $f \in L^p$ if and only if the conjugate inequality*

$$\left(\int_0^\infty |K^*g(x)|^{p'} \,\mathrm{d}x\right)^{\frac{1}{p'}} \leq C\left(\int_0^\infty |g(x)|^{q'} \,\mathrm{d}x\right)^{\frac{1}{q'}} \tag{3.13}$$

holds for all $g \in L^{q'}$, where

$$K^*g(x) := \int_x^\infty K(t,x)g(t) \,\mathrm{d}t \tag{3.14}$$

and $K(t,x) = \sum_{i=1}^m A_i(t)B_i(x)$.

The lemma enables to replace the investigation of inequality (1.6) by the investigation of inequality (3.13).

Denote

$$D_{i,l}^{j}(z) = \int_z^\infty \tilde{A}_i^j(t)\tilde{A}_l^j(t) \,\mathrm{d}t, \quad 1 \leq i,j,l \leq m, \tag{3.15}$$

where $\tilde{A}_l^j(t) := A_l(t)|A_j(t)|^{\frac{q-2}{2}}$ belongs to $L^2(z,\infty)$ since

$$\int_z^\infty |\tilde{A}_i^j(t)|^2 \,\mathrm{d}t \leq \left(\int_z^\infty |A_i(t)|^q \,\mathrm{d}t\right)^{\frac{2}{q}}\left(\int_z^\infty |A_j(t)|^q \,\mathrm{d}t\right)^{\frac{q-2}{2q}} < \infty.$$

Definition 3.9 We will say that the condition $E_{q'}$ is satisfied if there exists a constant $C_E > 0$ and for every $z > 0$ such that the system $\{D_{i,l}^{j}(z)\}_{i,l=1}^m$ satisfies the ellipticity condition for $j = 1,2,\ldots,m$.

Theorem 3.10 *Let* $1 < p < \infty$ *and* $2 \leq q < \infty$. *If the condition* $E_{q'}$ *holds for the system from* (3.15), *then* (2.4) *is necessary for inequality* (1.6) *to hold*.

Proof Let $z > 0$ and let $1 \leq j \leq m$. Let $\{\tilde{A}^j_{i_k}\}_{k=1}^n$ be a maximal linearly independent subsystem of $\{\tilde{A}^j_i\}_{i=1}^m$ in $L^2(z,\infty)$, which, for simplicity, we denote by $\{\tilde{A}^j_i\}_{i=1}^n$. Thus, using the method of orthogonalization, we get an orthogonal system $\{\tilde{A}^j_{i,z}\}_{i=1}^n$ in $L^2(z,\infty)$ such that

$$\tilde{A}^j_i(t) = \sum_{l=1}^n \alpha^j_{i,l}(z)\tilde{A}^j_{l,z}(t), \quad i = 1, 2, \dots, m. \tag{3.16}$$

If we denote $g(t) = \tilde{A}^j_{l,z}(t)\|\tilde{A}^j_{l,z}\|^{-1}_{L^2(z,\infty)}$, $c_j = \|A_j\|^{\frac{q'-2}{2q'-2}}_{L^q(z,\infty)}$ and choose the test function in the form

$$f^j_{l,z}(t) = \chi_{(z,\infty)}(t)c_j\left|A_j(t)\right|^{\frac{q-2}{2}}g(t),$$

then the left- and right-hand sides are similarly estimated in the forms:

$$\left\|f^j_{l,z}\right\|_{L^{q'}} \leq 1, \tag{3.17}$$

and

$$\left\|K^* f^j_{l,z}\right\|_{L^{p'}} \geq c_j\left(\int_0^z\left|\int_z^\infty \tilde{K}^j(t,x)g(t)\,\mathrm{d}t\right|^{p'}\mathrm{d}x\right)^{\frac{1}{p'}}, \tag{3.18}$$

where $\tilde{K}^j(t,x) = \sum_{i=1}^m \tilde{A}^j_i(t)B_i(x)$.

Using (3.7) we can rewrite the kernel $\tilde{K}^j(t,x)$ in the form

$$\begin{aligned}\tilde{K}^j(t,x) &= \sum_{i=1}^m B_i(x)\left[\sum_{l=1}^n \alpha^j_{i,l}(z)\tilde{A}^j_{l,z}(t)\right] \\ &= \sum_{l=1}^n \tilde{A}^j_{l,z}(t)B^j_{l,z}(x) = \tilde{K}^j_z(t,x),\end{aligned}$$

where $B^j_{l,z}(x) = \sum_{i=1}^m B_i(x)\alpha^j_{i,l}(z)$.

Then we have from (3.17), (3.18) and (3.13) that

$$c_j\left(\int_0^z\left|\int_z^\infty \tilde{K}^j_z(t,x)g(t)\,\mathrm{d}t\right|^{p'}\mathrm{d}x\right)^{\frac{1}{p'}} \leq C. \tag{3.19}$$

Now repeating the proof of the foregoing theorem with respect to (3.19) with the kernel $\tilde{K}^j_z(t,x)$, we can also get a similar estimate as in (3.11) in the form

$$nC \geq \sum_{l=1}^n \left\|K^* f^j_{l,z}\right\|_{L^{p'}} \geq c_j \sum_{l=1}^n \left(\int_0^z \left|B^j_{l,z}(t)\right|^{p'}\mathrm{d}t\right)^{\frac{1}{p'}}\left(\int_z^\infty \left|\tilde{A}^j_{l,z}(x)\right|^2\mathrm{d}x\right)^{\frac{1}{2}}. \tag{3.20}$$

Then, supposing that the condition $E_{q'}$ is satisfied, we also obtain the following estimate:

$$\begin{aligned}\sum_{l=1}^{n}\|K^*f_{l,z}^j\|_{L^{p'}} &\geq c_j\sum_{l=1}^{n}\left(\int_0^z\left|B_{l,z}^j(t)\right|^{p'}\,\mathrm{d}t\right)^{\frac{1}{p'}}\left(\int_z^\infty\left|\tilde{A}_{l,z}^j(x)\right|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\\ &\;\vdots\\ &\geq c_j\left(\int_z^\infty\left(\sum_{i,l=1}^{m}D_{i,l}^j(z)B_i(x)B_l(x)\right)^{\frac{p'}{2}}\,\mathrm{d}x\right)^{\frac{1}{p'}}\\ &\geq c_j\sqrt{C_E}\left(\int_z^\infty\left(\sum_{i=1}^{m}D_{i,i}^j(z)\left|B_i(x)\right|^2\right)^{\frac{p'}{2}}\,\mathrm{d}x\right)^{\frac{1}{p'}}\\ &\geq c_j\sqrt{C_E}\left(\int_z^\infty\left|B_j(x)\right|^{p'}\,\mathrm{d}x\right)^{\frac{1}{p'}}\left(\int_0^z\left|A_j(t)\right|^q\,\mathrm{d}t\right)^{\frac{1}{2}},\end{aligned}$$

i.e.,

$$\begin{aligned}\sum_{l=1}^{n}\|K^*f_{l,z}^j\|_{L^{p'}} &\geq c_j\sqrt{C_E}\left(\int_z^\infty\left|A_j(x)\right|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_0^z\left|B_j(t)\right|^{p'}\,\mathrm{d}t\right)^{\frac{1}{2}}\\ &\geq \sqrt{C_E}\left(\int_z^\infty\left|a_j(x)\right|^q u(x)\,\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_0^z\left|b_j(t)\right|^{p'}v^{1-p'}(t)\,\mathrm{d}t\right)^{\frac{1}{p'}}\\ &= \sqrt{C_E}A_{M,j}(z),\end{aligned}\tag{3.21}$$

which holds for all $1 \leq j \leq m$ and $z > 0$.

Using this estimate and (3.20), we finally get that

$$\frac{\sqrt{C_E}}{n}\sup_{z>0}A_{M,j}(z) \leq C.$$

The proof is complete. □

4 Criteria of compactness

As far as the compactness of the imbedding $k : L^p(v) \to L^q(u)$ is concerned, we have the following.

Theorem 4.1 *Let $1 < p \leq q < \infty$. Let us suppose that the functions $A_{M,i}(x)$ from (2.4) satisfy*

$$\lim_{x\to 0+}A_{M,i}(x) = \lim_{x\to\infty}A_{M,i}(x) = 0,\quad i = 1, 2, \dots m.\tag{4.1}$$

Then the operator k from (1.4) maps $L^p(v)$ into $L^q(u)$ compactly.

Proof Conditions (4.1) guarantee that the operators of k_i from $L^p(v)$ into $L^q(u)$ are compact (see, *e.g.*, [7]). Since $k = \sum_{i=1}^m k_i$, the compactness of k follows. □

Theorem 4.2 *Let $1 < p \leq 2$ and $1 < q < \infty$. Let the condition E_p be satisfied. If the operator k from (1.4) is compact from $L^p(v)$ to $L^q(u)$, then conditions (2.4) and (4.1) are satisfied.*

Proof Let us suppose that operator (1.4) is compact. Then it is bounded, and, by using Theorems 3.3 and 3.7, we get (2.4) and also the compactness of the operator

$$Kf(x) = \int_0^x \sum_{i=1}^m A_i(x)B_i(t)f(t)\,\mathrm{d}t \tag{4.2}$$

as an operator from L^p to L^q.

Let $z > 0$ and $1 \le j, l \le m$. Moreover, we choose the function $f^j_{l,z}(t)$ as in the proof of Theorem 3.7. Let $h \in L^{p'}$ be arbitrary, then using (3.8) we have

$$\begin{aligned}\left|\int_0^\infty h(t)f^j_{l,z}(t)\,\mathrm{d}t\right| &\le \left(\int_0^z |h(t)|^{p'}\,\mathrm{d}t\right)^{\frac{1}{p'}}\left(\int_0^z |f^j_{l,z}(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\\ &\le \left(\int_0^z |h(t)|^{p'}\,\mathrm{d}t\right)^{\frac{1}{p'}},\end{aligned}$$

from which it follows that

$$\int_0^\infty h(t)f_{j,z}(t)\,\mathrm{d}t \to 0$$

as $z \to 0+$, *i.e.*, the class of functions $f^j_{l,z}$ weakly converges to zero in L^p as $z \to 0+$.

This and the compactness of operator (4.2) imply that the class of images $Kf^j_{l,z}$ strongly converges to zero in L^q as $z \to 0+$, *i.e.*,

$$\|Kf^j_{l,z}\|_{L^q} \to 0 \quad \text{as } z \to 0+. \tag{4.3}$$

Analogously as in the proof of Theorem 3.7, (3.12) can also be obtained, *i.e.*,

$$\sum_{l=1}^n \|Kf^j_{l,z}\|_{L^q} \ge \sqrt{C_E}A_{M,j}(z), \tag{4.4}$$

which with (4.3) implies that $\lim_{z\to 0+} A_{M,j}(z) = 0$ for all $1 \le j \le m$.

Now we show that $\lim_{z\to\infty} A_{M,j}(z) = 0$ for all $1 \le j \le m$. The compactness of operator (4.2) follows from the compactness of the conjugate operator K^* (3.14) from $L^{q'}$ to $L^{p'}$.

Let $h \in L^{q'}$ and choose $h_z(t) = \chi_{(z,\infty)}(t)h(t)/\|h\|_{L^{q'}(z,\infty)}$. It can be shown as in foregoing cases that the class of functions $\{h_z, z \in (0,\infty)\}$ weakly converges to zero in $L^{q'}$ as $z \to \infty$. Then the class of images K^*h_z strongly converges to zero in $L^{p'}$ as $z \to \infty$, *i.e.*,

$$\|K^*h_z\|_{L^{p'}} \to 0 \quad \text{as } z \to \infty. \tag{4.5}$$

Using the dual principle of $L^{p'}$, we have

$$\begin{aligned}\|K^*h_z\|_{L^{p'}} &= \sup_{\substack{f\in L^p\\ \|f\|_{L^p}\le 1}} \int_0^\infty f(t)K^*h_z(t)\,\mathrm{d}t\\ &= \sup_{\substack{f\in L^p\\ \|f\|_{L^p}\le 1}} \int_0^\infty h_z(x)\left(\int_0^x K(x,t)f(t)\,\mathrm{d}t\right)\mathrm{d}x.\end{aligned} \tag{4.6}$$

Now, choosing $f_{l,z}^j$ as in the proof of Theorem 3.7 instead of f in (4.6), we have

$$\begin{aligned}\|K^*h_z\|_{L^{p'}} &\geq \int_z^\infty h_z(x)\int_0^x \sum_{i=1}^m A_i(x)\tilde{B}_i^j(t)g(t)\,\mathrm{d}t\,\mathrm{d}x\\ &= \int_z^\infty h_z(x)\int_0^z \sum_{i=1}^m A_{i,z}^j(x)\tilde{B}_{i,z}^j(t)g(t)\,\mathrm{d}t\,\mathrm{d}x\\ &\geq c_j\int_z^\infty h_z(x)A_{l,z}^j(x)\,\mathrm{d}x\int_0^z \left|\tilde{B}_{l,z}^j(t)\right|^2\mathrm{d}t\left\|\tilde{B}_{l,z}^j\right\|_{L^2(0,z)}^{-1}\\ &\geq c_j\int_z^\infty h_z(x)A_{l,z}^j(x)\,\mathrm{d}x\left(\int_0^z \left|\tilde{B}_{l,z}^j(t)\right|^2\mathrm{d}t\right)^{\frac{1}{2}}. \end{aligned} \tag{4.7}$$

Since the operator K is bounded from L^p to L^q, *i.e.*, (3.10) is satisfied, from which we will have that $A_{l,z}^j \in L^q(z,\infty)$, $z > 0$. Then, choosing the function $h_{l,z}^j = A_{l,z}^j|A_{l,z}^j|^{q-2}\chi_{(z,\infty)}$, $h_{l,z}^j(t) := h_z(t)$ and from (4.7), we have

$$\left\|K^*h_{l,z}^j\right\|_{L^{p'}} \geq c_j\left(\int_z^\infty \left|A_{l,z}^j(x)\right|^q\mathrm{d}x\right)^{\frac{1}{q}}\left(\int_0^z \left|\tilde{B}_{l,z}^j(t)\right|^2\mathrm{d}t\right)^{\frac{1}{2}}.$$

As in the proof of Theorem 3.7, by estimating the right-hand side, we obtain also that

$$\sum_{l=1}^n \left\|K^*h_{l,z}^j\right\|_{L^{p'}} \geq \sqrt{C_E}A_{M,j}(z).$$

Consequently, from this and (4.5) we have that $\lim_{z\to\infty} A_{M,j}(z) = 0$ for all $1 \leq j \leq m$.

The theorem is proved. □

Theorem 4.3 *Let* $1 < p < \infty$ *and* $2 \leq q < \infty$. *Let the condition* $E_{q'}$ *be satisfied. If operator* (1.4) *is compact from* $L^p(v)$ *to* $L^q(u)$, *then conditions* (2.4) *and* (4.1) *are satisfied.*

Proof First we show that $\lim_{z\to\infty} A_{M,j}(z) = 0$, for which we use the compactness of the dual operator K^* and the proof of Theorem 3.10. Then it can be shown that the class of functions $f_{l,z}^j$ also weakly converges to zero in $L^{q'}$ as $z \to \infty$ and

$$\left\|K^*f_{l,z}^j\right\|_{L^q} \to 0 \quad \text{as } z \to \infty. \tag{4.8}$$

Analogously as in the proof of Theorem 3.10, (3.21) can be obtained, *i.e.*,

$$\sum_{l=1}^n \left\|K^*f_{l,z}^j\right\|_{L^{p'}} \geq \sqrt{C_E}A_{M,j}(z)$$

which with (4.8) implies that $\lim_{z\to\infty} A_{M,j}(z) = 0$ for all $1 \leq j \leq m$.

To prove $\lim_{z\to 0+} A_{M,j}(z) = 0$, we use the duality principle as in the proof of the foregoing theorem. The formulation of the corresponding proof is left to the reader as an exercise. □

Corollary 4.4 *Let* $1 < p \leq 2$ *and* $p \leq q < \infty$. *Let us suppose that the condition* E_p *is satisfied. Then operator* (1.4) *from* $L^p(v)$ *into* $L^q(u)$ *is bounded and compact if and only if* (2.4) *and* (4.1) *are satisfied, respectively.*

Corollary 4.5 *Let* $2 \leq q < \infty$ *and* $1 < p \leq q$. *Let us suppose that the condition* $E_{q'}$ *is satisfied. Then operator* (1.4) *from* $L^p(v)$ *into* $L^q(u)$ *is bounded and compact if and only if* (2.4) *and* (4.1) *are satisfied, respectively.*

Corollary 4.6 *Let* $1 < p \leq q < \infty$. *Let us suppose that the conditions* E_p *and* $E_{q'}$ *are satisfied. Then operator* (1.4) *from* $L^p(v)$ *into* $L^q(u)$ *is bounded and compact if and only if* (2.4) *and* (4.1) *are satisfied, respectively.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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