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Integral mean estimates for the polar derivative of polynomials whose zeros are within a circle

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Abstract

For a polynomial $p(z)$ of degree n, having all zeros in $|z| \leq k$, where $k \leq 1$, Dewan *et al.* (Southeast Asian Bull. Math. 34:69-77, 2010) proved that for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$ and for each $r > 0$,

$$n(|\alpha| - k)\left\{\int_0^{2\pi} \left|p(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} \left|1 + ke^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}} \max_{|z|=1}\left|D_\alpha p(z)\right|.$$

In this paper we improve and extend the above inequality. Our result generalizes certain well-known polynomial inequalities.

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1 Introduction and statement of results

Let $p(z)$ be a polynomial of degree n. Then according to Bernstein's inequality [1] on the derivative of a polynomial, we have

$$\max_{|z|=1}\left|p'(z)\right| \leq n \max_{|z|=1}\left|p(z)\right|. \tag{1.1}$$

This result is best possible and equality holds for a polynomial that has all zeros at the origin.

If we restrict to the class of polynomials which have all zeros in $|z| \leq 1$, then it has been proved by Turán [2] that

$$\max_{|z|=1}\left|p'(z)\right| \geq \frac{n}{2} \max_{|z|=1}\left|p(z)\right|. \tag{1.2}$$

The inequality (1.2) is sharp and equality holds for a polynomial that has all zeros on $|z| = 1$.

As an extension to (1.2), Malik [3] proved that if $p(z)$ has all zeros in $|z| \leq k$, where $k \leq 1$, then

$$\max_{|z|=1}\left|p'(z)\right| \geq \frac{n}{1+k} \max_{|z|=1}\left|p(z)\right|. \tag{1.3}$$

This result is best possible and equality holds for $p(z) = (z - k)^n$.

On the other hand, Malik [4] obtained a generalization of (1.2) in the sense that the right-hand side of (1.2) is replaced by a factor involving the integral mean of $p(z)$ on $|z| = 1$. In fact he proved that if $p(z)$ has all its zeros in $|z| \leq 1$, then for each $r > 0$,

$$n\left[\int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^r d\theta\right]^{\frac{1}{r}} \leq \left[\int_0^{2\pi} \left|1 + e^{i\theta}\right|^r d\theta\right]^{\frac{1}{r}} \max_{|z|=1}\left|p'(z)\right|. \tag{1.4}$$

As an extension of (1.4), Aziz [5] proved that if $p(z)$ has all its zeros in $|z| \leq k \leq 1$, then for each $r > 0$,

$$n\left[\int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^r d\theta\right]^{\frac{1}{r}} \leq \left[\int_0^{2\pi} \left|1 + ke^{i\theta}\right|^r d\theta\right]^{\frac{1}{r}} \max_{|z|=1}\left|p'(z)\right|. \tag{1.5}$$

As a generalization of (1.5), Aziz and Shah [6] proved that if $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree n, having all its zeros in $|z| \leq k \leq 1$, then for each $r > 0$,

$$n\left[\int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^r d\theta\right]^{\frac{1}{r}} \leq \left[\int_0^{2\pi} \left|1 + k^{\mu} e^{i\theta}\right|^r d\theta\right]^{\frac{1}{r}} \max_{|z|=1}\left|p'(z)\right|. \tag{1.6}$$

Let $D_\alpha p(z)$ denote the polar derivative of the polynomial $p(z)$ of degree n with respect to $\alpha \in \mathbb{C}$. Then $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$. The polynomial $D_\alpha p(z)$ is of degree at most $n - 1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty}\left[\frac{D_\alpha p(z)}{\alpha}\right] = p'(z).$$

Shah [7] extended (1.2) to the polar derivative of $p(z)$ and proved that if all zeros of the polynomial $p(z)$ lie in $|z| \leq 1$, then for every α with $|\alpha| \geq 1$, we have

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \geq \frac{n}{2}\left(|\alpha| - 1\right) \max_{|z|=1}\left|p(z)\right|. \tag{1.7}$$

This result is best possible and equality holds for $p(z) = (z - 1)^n$ with $\alpha \geq 1$.

Aziz and Rather [8] extended the inequality (1.3) to the polar derivative of a polynomial. In fact, they proved that if all zeros of $p(z)$ lie in $|z| \leq k$, $k \leq 1$, then for every α with $|\alpha| \geq k$, we get

$$\max_{|z|=1}\left|D_\alpha p(z)\right| \geq \frac{n}{1+k}\left(|\alpha| - k\right) \max_{|z|=1}\left|p(z)\right|. \tag{1.8}$$

This result is best possible and equality holds for $p(z) = (z - k)^n$ with $\alpha \geq k$.

Recently Dewan *et al.* [9] generalized the inequalities (1.5) and (1.8). They proved that if $p(z)$ has all its zeros in $|z| \leq k \leq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$ and for each $r > 0$,

$$n\left(|\alpha| - k\right)\left\{\int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} \left|1 + ke^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}} \max_{|z|=1}\left|D_\alpha p(z)\right|. \tag{1.9}$$

In the limiting case, when $r \to \infty$, the above inequality is sharp and equality holds for the polynomial $p(z) = (z - k)^n$ with $\alpha \geq k$.

The following result which we prove is a generalization as well as a refinement of inequalities (1.9) and (1.8). In a precise set up, we have the following.

Theorem 1.1 *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, *is a polynomial of degree* n, *having all its zeros in* $|z| \le k \le 1$ *and* $m = \min_{|z|=k} |p(z)|$, *then for* $\lambda, \alpha \in \mathbb{C}$ *with* $|\lambda| \le 1$, $|\alpha| \ge s_\mu$ *and* $r > 0$, $d > 1$, $q > 1$, *with* $\frac{1}{d} + \frac{1}{q} = 1$, *we have*

$$n\big(|\alpha| - s_\mu\big)\left[\int_0^{2\pi} \big|p\big(e^{i\theta}\big) + \lambda m\big|^r \, d\theta\right]^{\frac{1}{r}} \le \left[\int_0^{2\pi} \big|1 + s_\mu e^{i\theta}\big|^{dr} \, d\theta\right]^{\frac{1}{dr}} \left[\int_0^{2\pi} \big|D_\alpha p\big(e^{i\theta}\big)\big|^{qr} \, d\theta\right]^{\frac{1}{qr}}, \tag{1.10}$$

where $s_\mu = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}$. *In the limiting case, when* $r \to \infty$, *the above inequality is sharp and equality holds for the polynomial* $p(z) = (z-k)^n$ *with* $\alpha \ge k$.

Letting $q \to \infty$ (so that $d \to 1$) in (1.10), we have the following.

Corollary 1.2 *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, *is a polynomial of degree* n, *having all its zeros in* $|z| \le k \le 1$ *and* $m = \min_{|z|=k} |p(z)|$, *then for* $\lambda, \alpha \in \mathbb{C}$ *with* $|\lambda| \le 1$, $|\alpha| \ge s_\mu$ *and* $r > 0$,

$$n\big(|\alpha| - s_\mu\big)\left[\int_0^{2\pi} \big|p\big(e^{i\theta}\big) + \lambda m\big|^r \, d\theta\right]^{\frac{1}{r}} \le \left[\int_0^{2\pi} \big|1 + s_\mu e^{i\theta}\big|^{r} \, d\theta\right]^{\frac{1}{r}} \max_{|z|=1} \big|D_\alpha p(z)\big|, \tag{1.11}$$

where s_μ *is defined as in Theorem* 1.1.

Remark 1.3 Since by Lemma 2.3, $s_\mu \le k$, the inequality (1.11) provides a refinement and generalization of the inequality (1.9).

If we divide both sides of the inequality (1.11) by $|\alpha|$ and make $|\alpha| \to \infty$, we obtain the following refinement and generalization of the inequality (1.6).

Corollary 1.4 *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, *is a polynomial of degree* n, *having all its zeros in* $|z| \le k \le 1$ *and* $m = \min_{|z|=k} |p(z)|$, *then for every* $\lambda \in \mathbb{C}$ *with* $|\lambda| \le 1$ *and* $r > 0$,

$$n\left[\int_0^{2\pi} \big|p\big(e^{i\theta}\big) + \lambda m\big|^r \, d\theta\right]^{\frac{1}{r}} \le \left[\int_0^{2\pi} \big|1 + s_\mu e^{i\theta}\big|^{r} \, d\theta\right]^{\frac{1}{r}} \max_{|z|=1} \big|p'(z)\big|, \tag{1.12}$$

where s_μ *is defined as in Theorem* 1.1.

Letting $r \to \infty$ in (1.10) and choosing the argument of λ suitably with $|\lambda| = 1$, we have the following result.

Corollary 1.5 *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, *is a polynomial of degree* n, *having all its zeros in* $|z| \le k \le 1$, *then for every* $\alpha \in \mathbb{C}$ *with* $|\alpha| \ge s_\mu$,

$$\frac{n(|\alpha| - s_\mu)}{1 + s_\mu}\Big[\max_{|z|=1}\big|p(z)\big| + \min_{|z|=k}\big|p(z)\big|\Big] \le \max_{|z|=1}\big|D_\alpha p(z)\big|, \tag{1.13}$$

where s_μ *is defined as in Theorem* 1.1.

2 Lemmas

For the proof of the theorem, the following lemmas are needed. The first lemma is due to Laguerre [10].

Lemma 2.1 *If all the zeros of an nth degree polynomial $p(z)$ lie in a circular region C and w is any zero of $D_\alpha p(z)$, then at most one of the points w and α may lie outside C.*

Lemma 2.2 *If $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$; $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k \le 1$ and $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$, then on $|z| = 1$*

$$\left|q'(z)\right| \le s_\mu \left|p'(z)\right| \tag{2.1}$$

and

$$\frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right| \le k^\mu, \tag{2.2}$$

where $s_\mu = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}$.

The above lemma is due to Aziz and Rather [8].

Lemma 2.3 *If $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, has all its zeros in $|z| \le k$, $k \le 1$, then*

$$s_\mu \le k^\mu, \tag{2.3}$$

where s_μ is same as above.

Proof By using Lemma 2.2, we have

$$\frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right| \le k^\mu, \tag{2.4}$$

or

$$\mu|a_{n-\mu}| \le n|a_n|k^\mu,$$

or equivalently,

$$\mu|a_{n-\mu}| - n|a_n|k^\mu \le 0.$$

Since $k \le 1$ and $\mu \ge 1$, the above inequality implies

$$\left(k^{\mu-1} - k^\mu\right)\left(\mu|a_{n-\mu}| - n|a_n|k^\mu\right) \le 0,$$

that is,

$$\mu|a_{n-\mu}|k^{\mu-1} - n|a_n|k^\mu k^{\mu-1} - \mu|a_{n-\mu}|k^\mu + n|a_n|k^{2\mu} \le 0,$$

which is equivalent to

$$\mu|a_{n-\mu}|k^{\mu-1} + n|a_n|k^{2\mu} \le \bigl(n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|\bigr)k^{\mu},$$

which implies

$$s_\mu = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|} \le k^{\mu}.$$

□

Lemma 2.4 *If $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, is a polynomial of degree n, having all zeros in the closed disk $|z| \le k$, $k \le 1$, then for every real or complex number α with $|\alpha| \ge s_\mu$ and $|z| = 1$, we have that*

$$\bigl|D_\alpha p(z)\bigr| \ge \bigl(|\alpha| - s_\mu\bigr)\bigl|p'(z)\bigr|, \tag{2.5}$$

where s_μ is same as above.

Proof Let $q(z) = z^n\overline{p(1/\overline{z})}$, then $|q'(z)| = |np(z) - zp'(z)|$ on $|z| = 1$. Thus on $|z| = 1$, we get

$$\begin{aligned}\bigl|D_\alpha p(z)\bigr| &= \bigl|np(z) + (\alpha - z)p'(z)\bigr| = \bigl|\alpha p'(z) + np(z) - zp'(z)\bigr| \\ &\ge \bigl|\alpha p'(z)\bigr| - \bigl|np(z) - zp'(z)\bigr|,\end{aligned}$$

which implies

$$\bigl|D_\alpha p(z)\bigr| \ge |\alpha|\bigl|p'(z)\bigr| - \bigl|q'(z)\bigr|. \tag{2.6}$$

By combining (2.1) and (2.6), we obtain

$$\bigl|D_\alpha p(z)\bigr| \ge \bigl(|\alpha| - s_\mu\bigr)\bigl|p'(z)\bigr|.$$

□

3 Proof of the theorem

Proof of Theorem 1.1 If $k = 0$, then $p(z)$ has all its zeros at the origin, therefore $p(z) = a_n z^n$. In this case $m = 0$, $s_\mu = 0$ and $D_\alpha P(z) = n\alpha a_n z^{n-1}$, therefore on the left-hand side of (1.10), we have

$$n\bigl(|\alpha| - s_\mu\bigr)\left[\int_0^{2\pi} \bigl|p\bigl(e^{i\theta}\bigr) + \lambda m\bigr|^r \, d\theta\right]^{\frac{1}{r}} = n|\alpha||a_n|(2\pi)^{\frac{1}{r}},$$

and on the right-hand side of (1.10) we have

$$\begin{aligned}&\left[\int_0^{2\pi} \bigl|1 + s_\mu e^{i\theta}\bigr|^{dr} \, d\theta\right]^{\frac{1}{dr}} \left[\int_0^{2\pi} \bigl|D_\alpha p\bigl(e^{i\theta}\bigr)\bigr|^{qr} \, d\theta\right]^{\frac{1}{qr}} \\ &\quad = (2\pi)^{\frac{1}{dr}} n|\alpha||a_n|(2\pi)^{\frac{1}{qr}} = n|\alpha||a_n|(2\pi)^{\frac{1}{r}(\frac{1}{d}+\frac{1}{q})} = n|\alpha||a_n|(2\pi)^{\frac{1}{r}}.\end{aligned}$$

Therefore, in the case $k = 0$, Theorem 1.1 is true. So, we suppose that $k > 0$, which implies $s_\mu > 0$. Let $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then on $|z| = 1$, we have

$$|p'(z)| = |nq(z) - zq'(z)|. \tag{3.1}$$

Let $m = \min_{|z|=k} |p(z)|$. Now $m \leq |p(z)|$ for $|z| = k$, therefore, if λ is any real or complex number such that $|\lambda| < 1$, then

$$|\lambda m| < |p(z)| \quad \text{for } |z| = k.$$

Since all the zeros of $p(z)$ lie in $|z| \leq k$, it follows by Rouche's theorem that all the zeros of

$$F(z) = p(z) - \lambda m$$

also lie in $|z| \leq k$. If $G(z) = z^n\overline{F(\frac{1}{\bar{z}})} = q(z) + \bar{\lambda} m z^n$, then by applying Lemma 2.2 to $F(z)$, we have

$$|G'(z)| \leq s_\mu |F'(z)| \quad \text{for } |z| = 1, \tag{3.2}$$

that is,

$$|q'(z) + \bar{\lambda} n m z^{n-1}| \leq s_\mu |p'(z)|.$$

Now using (3.1) in the above inequality, we get

$$|q'(z) + \bar{\lambda} n m z^{n-1}| \leq s_\mu |nq(z) - zq'(z)|. \tag{3.3}$$

Since $p(z)$ has all its zeros in $|z| \leq k \leq 1$, by the Gauss-Lucas theorem all the zeros of $p'(z)$ also lie in $|z| \leq k \leq 1$. This implies that the polynomial

$$z^{n-1}\overline{p'\left(\frac{1}{\bar{z}}\right)} = nq(z) - zq'(z) \tag{3.4}$$

has all its zeros in $|z| \geq \frac{1}{k} \geq 1$.

Therefore, it follows from (3.3) and (3.4) that the function

$$w(z) = \frac{z(q'(z) + \bar{\lambda} n m z^{n-1})}{s_\mu(nq(z) - zq'(z))} \tag{3.5}$$

is analytic for $|z| \leq 1$, and $|w(z)| \leq 1$ for $|z| \leq 1$. Furthermore, $w(0) = 0$. Thus the function

$$1 + s_\mu w(z)$$

is subordinate to the function

$$1 + s_\mu z$$

for $|z| \leq 1$.

Hence by a well-known property of subordination [11], we have for each $r > 0$ and $0 \leq \theta \leq 2\pi$,

$$\int_0^{2\pi} \left|1 + s_\mu w\left(e^{i\theta}\right)\right|^r d\theta \leq \int_0^{2\pi} \left|1 + s_\mu e^{i\theta}\right|^r d\theta. \tag{3.6}$$

Also from (3.5), we have

$$1 + s_\mu w(z) = \frac{n(q(z) + \overline{\lambda} m z^n)}{nq(z) - zq'(z)}.$$

Therefore

$$n\left|q(z) + \overline{\lambda} m z^n\right| = \left|1 + s_\mu w(z)\right|\left|nq(z) - zq'(z)\right|. \tag{3.7}$$

Since $|q(z) + \overline{\lambda} m z^n| = |p(z) + \lambda m|$ for $|z| = 1$, we get from (3.7) and (3.1)

$$n\left|p(z) + \lambda m\right| = \left|1 + s_\mu w(z)\right|\left|p'(z)\right| \quad \text{for } |z| = 1. \tag{3.8}$$

From (2.5) and (3.8), we have

$$n\left(|\alpha| - s_\mu\right)\left|p(z) + \lambda m\right| \leq \left|1 + s_\mu w(z)\right|\left|D_\alpha p(z)\right| \quad \text{for } |z| = 1. \tag{3.9}$$

By combining (3.6) and (3.9), for each $r > 0$, we get

$$\begin{aligned} &\left(n\left(|\alpha| - s_\mu\right)\right)^r \int_0^{2\pi} \left|p\left(e^{i\theta}\right) + \lambda m\right|^r d\theta \\ &\quad \leq \int_0^{2\pi} \left|1 + s_\mu e^{i\theta}\right|^r \left|D_\alpha p\left(e^{i\theta}\right)\right|^r d\theta. \end{aligned} \tag{3.10}$$

Now applying Holder's inequality for $d > 1$, $q > 1$, with $\frac{1}{d} + \frac{1}{q} = 1$ to (3.10), we get

$$\begin{aligned} &n\left(|\alpha| - s_\mu\right)\left[\int_0^{2\pi} \left|p\left(e^{i\theta}\right) + \lambda m\right|^r d\theta\right]^{\frac{1}{r}} \\ &\quad \leq \left[\int_0^{2\pi} \left|1 + s_\mu e^{i\theta}\right|^{dr} d\theta\right]^{\frac{1}{dr}} \left[\int_0^{2\pi} \left|D_\alpha p\left(e^{i\theta}\right)\right|^{qr} d\theta\right]^{\frac{1}{qr}}, \end{aligned} \tag{3.11}$$

which is the desired result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that they have no competing interests. All authors read and approved the final manuscript.

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