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Strong convergence theorem for quasi-ϕ-asymptotically nonexpansive mappings in the intermediate sense in Banach spaces

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Abstract

In this paper, we modify Halpern and Mann's iterations for finding a fixed point of an infinite family of quasi-ϕ-asymptotically nonexpansive mappings in the intermediate sense in Banach spaces. We prove a strong convergence theorem of the iterative sequence generated by the proposed iterative algorithm in a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property. The results presented in this paper improve and extend some recent corresponding results.

MSC: 47H09; 47J25

Keywords: quasi-ϕ-asymptotically nonexpansive mappings in the intermediate sense; strong convergence; fixed point; generalized projection

1 Introduction

Let E be a real Banach space with the dual space E^* and let C be a nonempty closed convex subset of E. We denote by R^+ and R the set of all nonnegative real numbers and the set of all real numbers, respectively. Also, we denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. Recall that if E is smooth, then J is single-valued and norm-to-weak* continuous, and that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. We shall denote by J the single-valued duality mapping.

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} \leq 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be uniformly convex if, for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leq 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \geq \varepsilon$. E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in U$. E is said to be uniformly smooth if the above limit exists uniformly in $x, y \in U$.

Remark 1.1 The following basic properties of a Banach space E can be found in [1]:

(i) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E.

(ii) If E is a reflective and strictly convex Banach space, then J^{-1} is norm-to-weak* continuous.

(iii) If E is a smooth, reflective and strictly convex Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is single-valued, one-to-one and surjective.

(iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex. If E is uniformly smooth, then it is smooth and reflective.

(v) Each uniformly convex Banach space E has the Kadec-Klee property, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. See [1, 2] for more details.

(vi) If E is a strictly convex and reflective Banach space with a strictly convex dual E^* and $J^* : E^* \to E$ is the normalized duality mapping in E^*, then $J^{-1} = J^*$, $JJ^* = I_{E^*}$ and $J^*J = I_E$.

Next, we assume that E is a smooth, reflective and strictly convex Banach space. Consider the functional defined as in [3, 4] by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.2}$$

It is clear that in a Hilbert space H, (1.2) reduces to $\phi(x,y) = \|x - y\|^2$, $\forall x, y \in H$.

It is obvious from the definition of ϕ that

$$\big(\|x\| - \|y\|\big)^2 \le \phi(x,y) \le \big(\|x\| + \|y\|\big)^2, \quad \forall x, y \in E, \tag{1.3}$$

and

$$\phi\big(x, J^{-1}\big(\lambda Jy + (1-\lambda)Jz\big)\big) \le \lambda\phi(x,y) + (1-\lambda)\phi(x,z), \quad \forall x, y \in E. \tag{1.4}$$

Following Alber [3], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \operatorname*{arginf}_{y \in C} \phi(y,x), \quad \forall x \in E. \tag{1.5}$$

That is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the unique solution to the minimization problem $\phi(\overline{x}, x) = \inf_{y \in C} \phi(y,x)$.

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J (see, *e.g.*, [1–5]). In a Hilbert space H, $\Pi_C = P_C$.

Let H be a real Hilbert space, let D be a nonempty subset of H, and let $T : D \to D$ be a nonlinear mapping. The symbol $F(T)$ stands for the fixed point set of T. Recall the following. T is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in D. \tag{1.6}$$

T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|p - Ty\| \le \|p - y\|, \quad \forall p \in F(T), \forall y \in D. \tag{1.7}$$

T is said to be asymptotically nonexpansive if there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \le (1 + \mu_n)\|x - y\|, \quad \forall x, y \in D, \forall n \ge 1. \tag{1.8}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [6]. Since 1972, a host of authors have studied the convergence of iterative algorithms for such a class of mappings.

T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\|p - T^n y\| \le (1 + \mu_n)\|p - y\|, \quad \forall p \in F(T), \forall y \in D, \forall n \ge 1. \tag{1.9}$$

Let C be a nonempty closed convex subset of E, and let T be a mapping from C into itself. A point $p \in C$ is called an asymptotically fixed point of T [7] if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$. The set of asymptotical fixed points of T will be denoted by $\widehat{F}(T)$. A point $p \in C$ is said to be a strong asymptotic fixed point of T, if there exists a sequence $\{x_n\} \subset C$ such that $x_n \to p$ and $\|x_n - Tx_n\| \to 0$. The set of strong asymptotical fixed points of T will be denoted by $\widetilde{F}(T)$.

A mapping $T : C \to C$ is said to be relatively nonexpansive [8–10] if $F(T) \neq \emptyset$, $F(T) = \widehat{F}(T)$ and $\phi(p, Tx) \le \phi(p, x)$, $\forall x \in C, p \in F(T)$.

A mapping $T : C \to C$ is said to be relatively asymptotically nonexpansive if

$$\begin{aligned} &F(T) \neq \emptyset, \qquad F(T) = \widehat{F}(T) \quad \text{and} \\ &\phi(p, T^n x) \le (1 + \mu_n)\phi(p, x), \quad \forall x \in C, p \in F(T), \end{aligned} \tag{1.10}$$

where $\{\mu_n\} \subset [0, \infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

A mapping $T : C \to C$ is said to be quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$, $\forall x \in C, p \in F(T)$.

A mapping $T : C \to C$ is said to be quasi-ϕ-asymptotically nonexpansive if $F(T) \neq \emptyset$, and there exists a real sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\phi(p, T^n x) \le (1 + \mu_n)\phi(p, x), \quad \forall n \ge 1, x \in C, p \in F(T). \tag{1.11}$$

Remark 1.2 From the definition, it is easy to know that

(i) Each relatively nonexpansive mapping is closed;

(ii) The class of quasi-ϕ-asymptotically nonexpansive mappings contains properly the class of quasi-ϕ-nonexpansive mappings as a subclass, but the converse is not true;

(iii) The class of quasi-ϕ-nonexpansive mappings contains properly the class of relatively nonexpansive mappings as a subclass, but the converse may be not true. (See [11–15] for more details.)

Asymptotically (quasi-)nonexpansive mappings in the intermediate sense were first considered by Bruck *et al.* [16]. Very recently Qin and Wang [17] introduced the concept of the asymptotically (quasi-)ϕ-nonexpansive mappings in the intermediate sense as follows:

(1) T is said to be asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\| T^n x - T^n y \| - \|x - y\| \right) \leq 0. \tag{1.12}$$

It is worth mentioning that the class of asymptotically nonexpansive in the intermediate sense mappings may not be Lipschitzian continuous; see [16, 18, 19].

(2) T is said to be asymptotically quasi-nonexpansive in the intermediate sense if $F(T) \neq \emptyset$ and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{p\in F(T), y\in C} \left(\| p - T^n y \| - \|p - y\| \right) \leq 0. \tag{1.13}$$

(3) T is said to be an asymptotically ϕ-nonexpansive mapping in the intermediate sense if and only if

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\phi\left(T^n x, T^n y\right) - \phi(x,y) \right) \leq 0. \tag{1.14}$$

(4) $T : C \to C$ is said to be quasi-ϕ-asymptotically nonexpansive mapping in the intermediate sense if and only if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{p\in F(T), x\in C} \left(\phi\left(p, T^n x\right) - \phi(p,x) \right) \leq 0. \tag{1.15}$$

Remark 1.3 The asymptotically (quasi-)ϕ-nonexpansive mapping in the intermediate sense is a generalization of the asymptotically (quasi-)nonexpansive mapping in the intermediate sense in the framework of Banach spaces.

Definition 1.4 An infinite family of mappings $\{T_i\}_{i=1}^{\infty} : C \to C$ is said to be uniformly quasi-ϕ-asymptotically nonexpansive in the intermediate sense if $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ for each $i \geq 1$ and

$$\limsup_{n\to\infty} \sup_{p\in F(T_i), x\in C} \left(\phi\left(p, T_i^n x\right) - \phi(p,x) \right) \leq 0. \tag{1.16}$$

If we define

$$\xi_n = \max\left\{0, \sup_{p\in F(T_i), x\in C} \left(\phi\left(p, T_i^n x\right) - \phi(p,x) \right)\right\},$$

then $\xi_n \to 0$ as $n \to \infty$. It follows that (1.16) is reduced to

$$\phi\left(p, T_i^n x\right) \leq \phi(p,x) + \xi_n, \quad \forall p \in F(T_i), \forall x \in C, \forall n \geq 1. \tag{1.17}$$

Many problems in nonlinear analysis can be reformulated as a problem of finding a fixed point of a nonexpansive mapping. In 1953, Mann [20] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \tag{1.18}$$

where the initial guess $x_1 \in C$ is arbitrary and $\{a_n\}$ is a real sequence in $[0,1]$. It is known that under appropriate settings the sequence $\{x_n\}$ converges weakly to a fixed point of T. However, for nonexpansive mappings, even in a Hilbert space, the Mann iteration may fail to converge strongly; for example, see [21].

Some attempts to construct the iteration method guaranteeing the strong convergence have been made. For example, Halpern [22] proposed the following so-called Halpern iteration:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \tag{1.19}$$

where $u \in C$ is fixed, $x_1 \in C$ is arbitrarily chosen and $\{a_n\}$ is a real sequence in $[0,1]$.

Recently, Nilsrakoo and Saejung [23] modified Halpern and Mann's iterations introduced the following iteration to find a fixed point of the relatively nonexpansive mappings in the Banach space:

$$x_{n+1} = \Pi_C J^{-1}\big(\alpha_n Ju + (1-\alpha_n)\big(\beta_n Jx_n + (1-\beta_n)JTx_n\big)\big). \tag{1.20}$$

They proved that $\{x_n\}$ converges strongly to $\Pi_{F(T)}u$, where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$, $\Pi_{F(T)}$ is the generalized projection from E onto $F(T)$.

Iteration methods for approximating fixed points of asymptotically nonexpansive mappings, quasi-ϕ-nonexpansive mapping, quasi-ϕ-asymptotically nonexpansive mapping have been further studied by authors (see, *e.g.*, [6, 24–29]).

Quite recently, Qin and Wang [17] introduced the following iterative scheme to find a fixed point of the quasi-ϕ-asymptotically nonexpansive mappings in the intermediate sense in a reflective, strictly convex and smooth Banach space such that both E and E^* have the Kadec-Klee property:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_{(1,i)} = C, \\ C_1 = \bigcap_{i\in\Lambda} C_{(1,i)}, \\ x_1 = \Pi_{C_1}x_0, \\ C_{(n+1,i)} = \{u \in C_{(n,i)} : \phi(x_n, T_i^n x_n) \le 2\langle x_n - u, Jx_n - JT_i^n x_n\rangle + \xi_{(n,i)}\}, \\ C_{n+1} = \bigcap_{i\in\Lambda} C_{(n+1,i)}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \ge 0, \end{cases}$$

where

$$\xi_{(n,i)} = \max\Big\{0, \sup_{p\in F(T_i), x\in C}\big(\phi\big(p, T_i^n x\big) - \phi(p,x)\big)\Big\}.$$

They proved that the sequence $\{x_n\}$ converges strongly to $\bar{x} = \Pi_{\bigcap_{i\in\Lambda} F(T_i)}x_0$.

Inspired and motivated by the recent work of Bruck [16], Qin and Wang [17], Nilsrakoo and Saejung [23], Chang *et al.* [24], *etc.*, in this paper, we modify Halpern and Mann's iterations for finding a fixed point of an infinite family of quasi-ϕ-asymptotically nonexpansive mappings in the intermediate sense in Banach spaces. We prove a strong convergence theorem of the iterative sequence generated by the proposed iterative algorithm in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property. The results presented in this paper improve and extend some recent corresponding results.

2 Preliminaries

Throughout this paper, let E be a real Banach space with the dual space E^* and let C be a nonempty closed convex subset of E. We denote the strong convergence, weak convergence of a sequence $\{x_n\}$ to a point $x \in E$ by $x_n \to x$, $x_n \rightharpoonup x$, respectively, and $F(T)$ is the fixed point set of a mapping T.

Lemma 2.1 [30] *Let E be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property, and let C be a nonempty closed convex subset of E. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in C such that $x_n \to p$ and $\phi(x_n, y_n) \to 0$, where ϕ is the functional defined by* (1.2), *then $y_n \to p$.*

Lemma 2.2 [3] *Let E be a smooth, strictly convex and reflective Banach space and let C be a nonempty closed convex subset of E. Then the following conclusions hold*:

(a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y)$, $\forall x \in C$, $y \in E$;
(b) *If $x \in E$ and $z \in C$, then $z = \Pi_C x$ iff $\langle z - y, Jx - Jz\rangle \ge 0$, $\forall y \in C$;*
(c) *For $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.*

Lemma 2.3 [31] *Let E be a uniformly convex Banach space, r be a positive number and $B_r(0)$ be a closed ball of E. Then, for any sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous, strictly increasing and convex function $g : [0, 2r] \to [0, \infty)$, $g(0) = 0$ such that for any positive integer $i \ne 1$, the following holds*:

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \le \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_1 \lambda_i g\big(\|x_1 - x_i\|\big). \tag{2.1}$$

3 Main results

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings in the intermediate sense and for each $i \ge 1$, let T_i be uniformly L_i-Lipschitzian continuous. $\{x_n\}$ is defined by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \qquad C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n) Jz_n), \\ z_n = J^{-1}(\beta_{n,0} Jx_n + \sum_{i=1}^{\infty} \beta_{n,i} JT_i^n x_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \alpha_n \phi(v, x_0) + (1 - \alpha_n)\phi(v, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \ge 0, \end{cases} \tag{3.1}$$

where $\xi_n = \max\{0, \sup_{p \in \bigcap_{i=1}^{\infty} F(T_i), x \in C}(\phi(p, T_i^n x) - \phi(p, x))\}$, $\Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1}, $\{\beta_{n,0}, \beta_{n,i}\}$ and $\{\alpha_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

(1) *for each $n \ge 0$, $\beta_{n,0} + \sum_{i=1}^{\infty} \beta_{n,i} = 1$;*
(2) $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ *for any $i \ge 1$;*
(3) $0 \le \alpha_n \le \alpha < 1$ *for some $\alpha \in (0, 1)$.*

If $\bigcap_{i=1}^{\infty} F(T_i)$ is a nonempty and bounded subset of C, then the sequence $\{x_n\}$ converges strongly to $p \in \bigcap_{i=1}^{\infty} F(T_i)$, where $p = \Pi_{\bigcap_{i=1}^{\infty} F(T_i)} x_0$.

Proof We shall divide the proof into six steps.

Step 1. We show that $\bigcap_{i=1}^{\infty} F(T_i)$ and C_n are closed and convex for each $n \geq 0$.

Using the similar methods given in the proof of Theorem 3.1 by Qin and Wang [17], the conclusion that $F(T_i)$ is closed and convex subset of C for each $i \geq 1$ can be easily obtained. Therefore $\bigcap_{i=1}^{\infty} F(T_i)$ is closed and convex in C.

Again, by the assumption, $C_0 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geq 1$. Since for any $z \in C_n$, we know

$$\begin{aligned} &\phi(z,y_n) \leq \alpha_n \phi(z,x_0) + (1-\alpha_n)\phi(z,x_n) + \xi_n \\ &\quad \Leftrightarrow \quad 2\alpha_n \langle z, Jx_0\rangle + 2(1-\alpha_n)\langle z, Jx_n\rangle - 2\langle z, Jy_n\rangle \\ &\qquad\qquad \leq \alpha_n \|x_0\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_n\|^2 + \xi_n. \end{aligned} \tag{3.2}$$

Hence the set $C_{n+1} = \{z \in C_n : 2\alpha_n\langle z, Jx_0\rangle + 2(1-\alpha_n)\langle z, Jx_n\rangle - 2\langle z, Jy_n\rangle \leq \alpha_n\|x_0\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_n\|^2 + \xi_n\}$ is closed and convex. Therefore $\Pi_{C_n} x_0$ and $\Pi_{\bigcap_{i=1}^{\infty} F(T_i)} x_0$ are well defined.

Step 2. We show that $\bigcap_{i=1}^{\infty} F(T_i) \subset C_n$ for all $n \geq 0$.

It is obvious that $\bigcap_{i=1}^{\infty} F(T_i) \subset C_0 = C$. Suppose that $\bigcap_{i=1}^{\infty} F(T_i) \subset C_n$ for some $n \geq 1$. Since E is uniformly smooth, E^* is uniformly convex. For any given $q \in \bigcap_{i=1}^{\infty} F(T_i) \subset C_n$, we observe that

$$\begin{aligned} \phi(q,y_n) &= \phi\bigl(q, J^{-1}\bigl(\alpha_n Jx_0 + (1-\alpha_n)Jz_n\bigr)\bigr) \\ &= \|q\|^2 - 2\bigl\langle q, \alpha_n Jx_0 + (1-\alpha_n)Jz_n\bigr\rangle + \bigl\|\alpha_n Jx_0 + (1-\alpha_n)Jz_n\bigr\|^2 \\ &\leq \|q\|^2 - 2\alpha_n\langle q, Jx_0\rangle - 2(1-\alpha_n)\langle q, Jz_n\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|z_n\|^2 \\ &= \alpha_n\phi(q,x_0) + (1-\alpha_n)\phi(q,z_n). \end{aligned} \tag{3.3}$$

On the other hand, it follows from Lemma 2.3 that for any positive integer $l > 1$ and for any $q \in \bigcap_{i=1}^{\infty} F(T_i)$, we have

$$\begin{aligned} \phi(q,z_n) &= \phi\left(q, J^{-1}\left(\beta_{n,0}Jx_n + \sum_{i=1}^{\infty}\beta_{n,i}JT_i^n x_n\right)\right) \\ &= \|q\|^2 - 2\left\langle q, \beta_{n,0}Jx_n + \sum_{i=1}^{\infty}\beta_{n,i}JT_i^n x_n\right\rangle + \left\|\beta_{n,0}Jx_n + \sum_{i=1}^{\infty}\beta_{n,i}JT_i^n x_n\right\|^2 \\ &\leq \|q\|^2 - 2\beta_{n,0}\langle q, Jx_n\rangle - 2\sum_{i=1}^{\infty}\beta_{n,i}\bigl\langle q, JT_i^n x_n\bigr\rangle + \beta_{n,0}\|x_n\|^2 \\ &\quad + \sum_{i=1}^{\infty}\beta_{n,i}\bigl\|T_i^n x_n\bigr\|^2 - \beta_{n,0}\beta_{n,l}g\bigl(\bigl\|Jx_n - JT_l^n x_n\bigr\|\bigr) \\ &= \beta_{n,0}\phi(q,x_n) + \sum_{i=1}^{\infty}\beta_{n,i}\phi\bigl(q, T_i^n x_n\bigr) - \beta_{n,0}\beta_{n,l}g\bigl(\bigl\|Jx_n - JT_l^n x_n\bigr\|\bigr) \end{aligned}$$

$$\begin{aligned}
&\le \beta_{n,0}\phi(q,x_n) + \sum_{i=1}^{\infty}\beta_{n,i}\{\phi(q,x_n)+\xi_n\} - \beta_{n,0}\beta_{n,l}g(\|Jx_n - JT_l^n x_n\|) \\
&\le \phi(q,x_n) + \xi_n - \beta_{n,0}\beta_{n,l}g(\|Jx_n - JT_l^n x_n\|). \qquad (3.4)
\end{aligned}$$

Substituting (3.4) into (3.3), we get

$$\begin{aligned}
\phi(q,y_n) &\le \alpha_n\phi(q,x_0) + (1-\alpha_n)\phi(q,z_n) \\
&\le \alpha_n\phi(q,x_0) + (1-\alpha_n)\bigl[\phi(q,x_n) + \xi_n - \beta_{n,0}\beta_{n,l}g(\|Jx_n - JT_l^n x_n\|)\bigr] \\
&\le \alpha_n\phi(q,x_0) + (1-\alpha_n)\phi(q,x_n) + \xi_n. \qquad (3.5)
\end{aligned}$$

This shows that $q \in C_{n+1}$. Further this implies that $\bigcap_{i=1}^{\infty} F(T_i) \subset C_{n+1}$ and hence $\bigcap_{i=1}^{\infty} F(T_i) \subset C_n$ for all $n \ge 0$. Since $\bigcap_{i=1}^{\infty} F(T_i)$ is nonempty, C_n is a nonempty closed convex subset of E and hence Π_{C_n} exists for all $n \ge 0$. This implies that the sequence $\{x_n\}$ is well defined.

Step 3. We show that $\{x_n\}$ is bounded and $\{\phi(x_n,x_0)\}$ is a convergent sequence.

It follows from (3.1) and Lemma 2.2 that

$$\begin{aligned}
\phi(x_n,x_0) &= \phi(\Pi_{C_n}x_0,x_0) \\
&\le \phi(p,x_0) - \phi(p,x_n) \\
&\le \phi(p,x_0), \quad \forall p \in C_{n+1}, \forall n \ge 0. \qquad (3.6)
\end{aligned}$$

From the definition of C_{n+1} that $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0$, we have

$$\phi(x_n,x_0) \le \phi(x_{n+1},x_0), \quad \forall n \ge 0. \qquad (3.7)$$

Therefore, $\{\phi(x_n,x_0)\}$ is nondecreasing and bounded. So, $\{\phi(x_n,x_0)\}$ is a convergent sequence, without loss of generality, we can assume that $\lim_{n\to\infty}\phi(x_n,x_0) = d \ge 0$. In particular, by (1.3), the sequence $\{(\|x_n\| - \|x_0\|)^2\}$ is bounded. This implies $\{x_n\}$ is also bounded.

Step 4. We prove that $\{x_n\}$ converges strongly to some point $p \in C$.

Since $\{x_n\}$ is bounded and E is reflective, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p$ (some point in C). Since C_n is closed and convex and $C_{n+1} \subset C_n$, this implies that C_n is weakly closed and $p \in C_n$ for each $n \ge 0$. From $x_{n_i} = \Pi_{C_{n_i}}x_0$, we have

$$\phi(x_{n_i},x_0) \le \phi(p,x_0), \quad \forall n_i \ge 0. \qquad (3.8)$$

Since the norm $\|\cdot\|$ is weakly lower semi-continuous, we have

$$\begin{aligned}
\liminf_{n_i\to\infty}\phi(x_{n_i},x_0) &= \liminf_{n_i\to\infty}\{\|x_{n_i}\|^2 - 2\langle x_{n_i},Jx_0\rangle + \|x_0\|^2\} \\
&\ge \|p\|^2 - 2\langle p,Jx_0\rangle + \|x_0\|^2 \\
&= \phi(p,x_0), \qquad (3.9)
\end{aligned}$$

and so

$$\phi(p,x_0) \le \liminf_{n_i\to\infty}\phi(x_{n_i},x_0) \le \limsup_{n_i\to\infty}\phi(x_{n_i},x_0) \le \phi(p,x_0). \qquad (3.10)$$

This implies that $\lim_{n_i\to\infty}\phi(x_{n_i},x_0)\to\phi(p,x_0)$, and so $\|x_n\|\to\|p\|$. Since $x_{n_i}\rightharpoonup p$, in view of the Kadec-Klee property of E, it follows that

$$\lim_{n_i\to\infty} x_{n_i} = p. \tag{3.11}$$

Since $\{\phi(x_n,x_0)\}$ is convergent, this together with $\lim_{n_i\to\infty}\phi(x_{n_i},x_0)\to\phi(p,x_0)$, we have $\lim_{n\to\infty}\phi(x_n,x_0)\to\phi(p,x_0)$. If there exists a subsequence $\{x_{n_j}\}\subset\{x_n\}$ such that $x_{n_j}\to q$, then from Lemma 2.2(a), we have that

$$\begin{aligned}\phi(p,q) &= \lim_{n_i,n_j\to\infty}\phi(x_{n_i},x_{n_j})\\ &= \lim_{n_i,n_j\to\infty}\phi(x_{n_i},\Pi_{C_{n_j}}x_0)\\ &\le \lim_{n_i,n_j\to\infty}\big(\phi(x_{n_i},x_0)-\phi(\Pi_{C_{n_j}}x_0,x_0)\big)\\ &= \lim_{n_i,n_j\to\infty}\big(\phi(x_{n_i},x_0)-\phi(x_{n_i},x_0)\big)\\ &= \phi(p,x_0)-\phi(p,x_0)\\ &= 0.\end{aligned} \tag{3.12}$$

This implies that $p=q$ and

$$\lim_{n\to\infty} x_n = p. \tag{3.13}$$

Step 5. We show that $p\in\bigcap_{i=1}^{\infty}F(T_i)$.

Since $x_{n+1}\in C_{n+1}$, it follows from (3.1) and (3.13) that

$$\phi(x_{n+1},y_n)\le\alpha_n\phi(x_{n+1},x_0)+(1-\alpha_n)\phi(x_{n+1},x_n)+\xi_n\to 0\quad(\text{as } n\to\infty). \tag{3.14}$$

Since $x_n\to p$, by Lemma 2.1

$$\lim_{n\to\infty} y_n = p. \tag{3.15}$$

By (3.3) and (3.4), for any $q\in\bigcap_{i=1}^{\infty}F(T_i)$, we have

$$\phi(q,y_n)\le\alpha_n\phi(q,x_0)+(1-\alpha_n)\phi(q,x_n)+\xi_n-(1-\alpha_n)\beta_{n,0}\beta_{n,l}g\big(\|Jx_n-JT_l^nx_n\|\big). \tag{3.16}$$

So, as $n\to\infty$,

$$\begin{aligned}(1-\alpha_n)\beta_{n,0}\beta_{n,l}g\|Jx_n-JT_l^nx_n\| &\le \alpha_n\phi(q,x_0)+(1-\alpha_n)\phi(q,x_n)+\xi_n-\phi(q,y_n)\\ &\to 0.\end{aligned} \tag{3.17}$$

Therefore,

$$\lim_{n\to\infty}(1-\alpha_n)\beta_{n,0}\beta_{n,l}g\|Jx_n-JT_l^nx_n\| = 0. \tag{3.18}$$

In view of the property of g, we have

$$\|Jx_n - JT_l^n x_n\| \to 0 \quad (\text{as } n \to \infty). \tag{3.19}$$

Since $Jx_n \to Jp$, this implies that $\lim_{n\to\infty} JT_l^n x_n = Jp$. Remark 1.1(ii) yields

$$T_l^n x_n \rightharpoonup p \quad (\text{as } n \to \infty). \tag{3.20}$$

Again, since

$$\|T_l^n x_n\| - \|p\| = \|J(T_l^n x_n)\| - \|Jp\| \le \|J(T_l^n x_n) - Jp\| \to 0 \quad (\text{as } n \to \infty),$$

this together with (3.20) and the Kadec-Klee property of E shows that

$$\lim_{n\to\infty} T_l^n x_n = p. \tag{3.21}$$

By the assumption that T_l is uniformly L_l-Lipschitz continuous, we have

$$\begin{aligned}\|T_l^{n+1} x_n - T_l^n x_n\| &\le \|T_l^{n+1} x_n - T_l^{n+1} x_{n+1}\| + \|T_l^{n+1} x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| + \|x_n - T_l^n x_n\| \\ &\le (L_l + 1)\|x_{n+1} - x_n\| + \|T_l^{n+1} x_{n+1} - x_{n+1}\| + \|x_n - T_l^n x_n\|.\end{aligned}$$

This together with (3.21) and $x_n \to p$ shows that $\lim_{n\to\infty} \|T_l^{n+1} x_n - T_l^n x_n\| = 0$ and $\lim_{n\to\infty} T_l^{n+1} x_n = p$, that is, $\lim_{n\to\infty} T_l T_l^n x_n = p$. In view of the closeness of T_l, it follows that $T_l p = p$, that is, $p \in F(T_l)$. By the arbitrariness of $l \ge 1$, we have $p \in \bigcap_{i=1}^{\infty} F(T_i)$.

Step 6. We prove that $x_n \to p = \Pi_{\bigcap_{i=1}^{\infty} F(T_i)} x_0$.

Let $q = \Pi_{\bigcap_{i=1}^{\infty} F(T_i)} x_0$. From $x_n = \Pi_{C_n} x_0$ and $q \in \bigcap_{i=1}^{\infty} F(T_i) \subset C_n$, we have

$$\phi(x_n, x_0) \le \phi(q, x_0), \quad \forall n \ge 0. \tag{3.22}$$

This implies that

$$\phi(p, x_0) = \lim_{n\to\infty} \phi(x_n, x_0) \le \phi(q, x_0). \tag{3.23}$$

By the definition of $p = \Pi_{\bigcap_{i=1}^{\infty} F(T_i)} x_0$, we have $p = q$. Therefore, $x_n \to p = \Pi_{\bigcap_{i=1}^{\infty} F(T_i)} x_0$. This completes the proof. □

In Theorem 3.1, as $T_i = T$ for each $i \in N$, we can obtain the following corollary.

Corollary 3.2 *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $T : C \to C$ be a closed uniformly L-Lipschitzian continuous and uniformly quasi-ϕ-asymptotically nonexpansive* *mapping in the intermediate sense such that $F(T)$ is a nonempty and bounded subset of C. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \qquad C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JT^n x_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \ge 0, \end{cases} \tag{3.24}$$

where $\xi_n = \max\{0, \sup_{p\in F(T), x\in C}(\phi(p, T^n x) - \phi(p, x))\}$, $\Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1}, $\{\alpha_n\}$ is a sequence in $[0, \alpha]$, $\{\beta_n\} \subset (0,1)$ satisfies that $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then the sequence $\{x_n\}$ converges strongly to $p \in F(T)$, where $p = \Pi_{F(T)} x_0$.

Corollary 3.3 *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings in the intermediate sense and for each $i \ge 1$, T_i is uniformly L_i-Lipschitzian continuous. $\{x_n\}$ is defined by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \qquad C_0 = C, \\ z_n = J^{-1}(\beta_{n,0} Jx_n + \sum_{i=1}^{\infty} \beta_{n,i} JT_i^n x_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \phi(v, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \ge 0, \end{cases} \tag{3.25}$$

where $\xi_n = \max\{0, \sup_{p\in\bigcap_{i=1}^{\infty} F(T_i), x\in C}(\phi(p, T_i^n x) - \phi(p, x))\}$, $\Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1}, $\{\beta_{n,0}, \beta_{n,i}\}$ and $\{\alpha_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(1) *for each $n \ge 0$ $\beta_{n,0} + \sum_{i=1}^{\infty} \beta_{n,i} = 1$;*
(2) $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ *for any $i \ge 1$;*
(3) $0 \le \alpha_n \le \alpha < 1$ *for some $\alpha \in (0,1)$.*

If $\bigcap_{i=1}^{\infty} F(T_i)$ is a nonempty and bounded subset of C, then the sequence $\{x_n\}$ converges strongly to $p \in \bigcap_{i=1}^{\infty} F(T_i)$, where $p = \Pi_{\bigcap_{i=1}^{\infty} F(T_i)} x_0$.

Proof Setting $\alpha_n \equiv 0$ in Theorem 3.1, then we get that $y_n = z_n$. Thus, from the method of the proof of Theorem 3.1, we obtain Corollary 3.3 immediately. □

In the Hilbert space, the following corollary can be directly obtained from Theorem 3.1.

Corollary 3.4 *Let C be a nonempty, closed and convex subset of a Hilbert space E. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed and uniformly L_i-Lipschitzian continuous and uniformly asymptotically quasi-nonexpansive mappings in the intermediate sense such that $\bigcap_{i=1}^{\infty} F(T_i)$ is a nonempty and bounded subset of C. Let $\{x_n\}$ be the sequence generated* *by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \qquad C_0 = C, \\ y_n = \alpha_n x_0 + (1-\alpha_n) z_n, \\ z_n = \beta_{n,0} x_n + \sum_{i=1}^{\infty} \beta_{n,i} T_i^n x_n, \\ C_{n+1} = \{v \in C_n : \|v - y_n\|^2 \le \alpha_n \|v - x_0\|^2 + (1-\alpha_n)\|v - x_n\|^2 + \xi_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \ge 0, \end{cases} \tag{3.26}$$

where $\xi_n = \max\{0, \sup_{p \in \bigcap_{i=1}^{\infty} F(T_i), x \in C} (\|p - T_i^n x\|^2 - \|p - x\|^2)\}$, $P_{C_{n+1}}$ *is the metric projection of* E *onto* C_{n+1}, $\{\beta_{n,0}, \beta_{n,i}\}$ *and* $\{\alpha_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions*:

(1) *for each* $n \ge 0$ $\beta_{n,0} + \sum_{i=1}^{\infty} \beta_{n,i} = 1$;

(2) $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ *for any* $i \ge 1$;

(3) $0 \le \alpha_n \le \alpha < 1$ *for some* $\alpha \in (0,1)$.

Then the sequence $\{x_n\}$ *converges strongly to* $p \in \bigcap_{i=1}^{\infty} F(T_i)$, *where* $p = \Pi_{\bigcap_{i=1}^{\infty} F(T_i)} x_0$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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