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Some results of meromorphic solutions of second-order linear differential equations

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Abstract

In this paper, we investigate the growth and the zeros of difference of the first and second derivative of the solutions of the second-order linear differential equations

$$f'' + A_0 e^{a_0 z} f' + (A_1 e^{a_1 z} + A_2 e^{a_2 z}) f = 0$$

and a small function, where $A_j(z)$ ($\not\equiv 0$) ($j = 0, 1, 2$) are meromorphic functions and $a_0 < 0$, $a_1 a_2 \neq 0$, $a_1 \neq a_2$. Our result extended the results of Peng and Chen, Belaïdi and others.

MSC: 34M10; 30D35

Keywords: linear differential equation; meromorphic function; growth order; Nevanlinna theory; small function

1 Introduction and main results

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory of meromorphic functions (see [1–3]). The term 'meromorphic function' will mean meromorphic in the whole complex plane $\mathbb{C}$. In addition, we will use notations $\rho(f)$ to denote the order of growth of a meromorphic function $f(z)$, $\lambda(f)$ to denote the exponents of convergence of the zero-sequence of a meromorphic function $f(z)$, $\overline{\lambda}(f)$ to denote the exponents of convergence of the sequence of distinct zeros of $f(z)$.

In order to give some estimates of fixed points, we recall the following definitions (see [4, 5]).

Definition 1.1 Let f and g be two meromorphic functions satisfying $\rho(g) < \rho(f)$, and let $z_1, z_2, \ldots$ ($|z_j| = r_j$, $0 \leq r_1 \leq r_2 \leq \cdots$) be the sequence of distinct zeros of the meromorphic function $f - g$. Then $\overline{\tau}_g(f)$, the exponent of convergence of the sequence of distinct zeros of $f - g$, is defined by

$$\overline{\tau}_g(f) = \inf\left\{\tau > 0 \,\middle|\, \sum_{j=1}^{\infty} |z_j|^{-\tau} < +\infty\right\}.$$

It is evident that

$$\overline{\tau}_g(f) = \overline{\lim_{r\to\infty}} \frac{\log \overline{N}(r, \frac{1}{f-g})}{\log r}$$

and $\overline{\tau}_g(f) = \overline{\lambda}(f - g)$.

Clearly, if $g(z) = z$, then Definition 1.1 is equivalent to the definition of the exponent of convergence of the sequence of distinct fixed points of $f(z)$. We denote $\overline{\tau}_z(f) = \overline{\tau}(f)$.

For the second-order linear differential equation

$$f'' + e^{-z}f' + B(z)f = 0, \tag{1.1}$$

where $B(z)$ is an entire function of finite order, it is well known that each solution f of (1.1) is an entire function. If f_1 and f_2 are any two linearly independent solutions of (1.1), then at least one of f_1, f_2 must have infinite order [6]. Hence, 'most' solutions of (1.1) will have infinite order.

Thus a natural question is: What condition on $B(z)$ will guarantee that every solution $f \not\equiv 0$ of (1.1) will have infinite order? Frei, Ozawa, Amemiya and Langley, and Gundersen studied the question. For the case that $B(z)$ is a transcendental entire function, Gundersen [7] proved that if $\rho(B) \neq 1$, then for every solution $f \not\equiv 0$ of (1.1) has infinite order.

In 2002, Chen considered the problem and obtained the following result in [8].

Theorem A *Let a, b be nonzero complex numbers and $a \neq b$, $B(z) \not\equiv 0$ be a nonconstant polynomial or $B(z) = h(z)e^{bz}$, where $h(z)$ is a nonzero polynomial. Then every solution f ($\not\equiv 0$) of the equation*

$$f'' + e^{bz}f' + B(z)f = 0$$

has infinite order.

Theorem B *Suppose that $A_j(z)$ ($\not\equiv 0$) ($j = 0, 1$) are entire functions and $\rho(A_j) < 1$, let a, b be complex numbers and $ab \neq 0$ and $a \neq b$, then every solution f ($\not\equiv 0$) of the equation*

$$f'' + A_1e^{az}f' + A_0e^{bz}f = 0$$

has infinite order.

Recently in [9], Peng and Chen have investigated the order and the hyper-order of solutions of some second-order linear differential equations and have proved the following result.

Theorem C *Suppose that $A_j(z)$ ($\not\equiv 0$) ($j = 1, 2$) are entire functions and $\rho(A_j) < 1$, let a_1, a_2 be complex numbers such that $a_1a_2 \neq 0$, and let $a_1 \neq a_2$ (suppose that $|a_1| \leq |a_2|$). If $\arg a_1 \neq \pi$ or $a_1 < -1$, then every solution f ($\not\equiv 0$) of the equation*

$$f'' + e^{-z}f' + \left(A_1e^{a_1z} + A_2e^{a_2z}\right)f = 0$$

has infinite order and $\rho_2(f) = 1$.

In this paper, we extend and improve the above result from entire solutions to meromorphic solutions.

Theorem 1.1 *Suppose that* $A_j(z)$ $(\not\equiv 0)$ $(j = 0,1,2)$ *are meromorphic functions and* $\rho(A_j) < 1$, *and* a_1, a_2 *are two complex numbers such that* $a_1a_2 \neq 0$, $a_1 \neq a_2$ (*suppose that* $|a_1| \leq |a_2|$). *Let* a_0 *be a strictly negative real constant. If* $\arg a_1 \neq \pi$ *or* $a_1 < a_0$, *then every solution* f $(\not\equiv 0)$, *whose poles are of uniformly bounded multiplicities*, *of the equation*

$$f'' + A_0e^{a_0z}f' + \left(A_1e^{a_1z} + A_2e^{a_2z}\right)f = 0 \tag{1.2}$$

has infinite order and $\rho_2(f) = 1$.

Remark 1 It has been shown by Bank [10] that the growth of a meromorphic solution of a linear differential equation with meromorphic coefficients cannot be estimated uniformly in terms of the growth of the coefficients alone. Bank [10] (see also [11]) stated his result in terms of an example. Hence, in addition to the growth of the coefficients, some additional piece of information is needed when estimating the growth of meromorphic solutions. Bank and Laine [12] have shown that the number of zeros of a solution should also be taken into account in the meromorphic coefficients case. Chiang and Hayman ([13], Corollary 2.5) showed that we can estimate the growth of a meromorphic solution, when $\delta(\infty) > 0$, in terms of the Nevanlinna characteristics of the coefficients. The condition $\delta(\infty) > 0$ is sharp because $\delta(\infty) = 0$ in Bank's example. In this paper, we give an additional condition that the solutions whose poles are of uniformly bounded multiplicities can guarantee the growth of the poles of the meromorphic solution less than or equal to the meromorphic coefficients. We can change the condition that the multiplicity of the poles is uniformly bounded to $\delta(\infty) > 0$ when we considered the hyper order by using Lemma 2.7.

Since the beginning of the last four decades, a substantial number of research articles have been written to describe the fixed points of general transcendental meromorphic functions (see [14]). In [4], Chen first studied the problems on the fixed points of solutions of second-order linear differential equations with entire coefficients. Since then, Wang and Yi [15], Laine and Rieppo [16], Chen and Shon [17], Liu and Zhang [18], El Farissi and Belaïdi [19] studied the problems on the fixed points of solutions of second-order linear differential equations with meromorphic coefficients and their derivatives.

The other main purpose of this paper is to study the exponent of convergence of the sequence of distinct fixed points of all solutions of equation (1.2). In fact, inspired by [5, 20–22], we can generalize the fixed-point to the small function.

Theorem 1.2 *Let* $A_j(z)$, a_j *satisfy the additional hypotheses of Theorem* 1.1. *If* φ $(\not\equiv 0)$ *is a meromorphic function whose order is less than* 1, *then every meromorphic solution* $f \not\equiv 0$, *whose poles are of uniformly bounded multiplicities*, *of equation* (1.2) *satisfies*

$$\overline{\lambda}(f - \varphi) = \overline{\lambda}\left(f' - \varphi\right) = \overline{\lambda}\left(f'' - \varphi\right) = \infty.$$

Corollary 1 *Let* $A_j(z)$, a_j *satisfy the additional hypotheses of Theorem* 1.1. *If* $f \not\equiv 0$ *is a meromorphic solution*, *whose poles are of uniformly bounded multiplicities*, *of equation* (1.2), *then* f, f', f'' *all have infinitely fixed points and satisfy*

$$\bar{\tau}(f) = \bar{\tau}\left(f'\right) = \bar{\tau}\left(f''\right) = \infty.$$

2 Lemmas

The following lemma, due to Gross [23], is important in the factorization and uniqueness theory of meromorphic functions, playing an important role in this paper as well. We give a slightly changed form as follows.

Lemma 2.1 ([22]) *Suppose that* $f_1(z), f_2(z), \dots, f_n(z)$ $(n \geq 2)$ *are meromorphic functions and* $g_1(z), g_2(z), \dots, g_n(z)$ *are entire functions satisfying the following conditions*:

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv f_{n+1}$.

(ii) *If* $1 \leq j \leq n+1$, $1 \leq k \leq n$, *the order of* f_j *is less than the order of* $e^{g_k(z)}$. *If* $n \geq 2$, $1 \leq j \leq n+1$, $1 \leq h < k \leq n$, *and the order of* $f_j(z)$ *is less than the order of* $e^{g_h - g_k}$.

Then $f_j(z) \equiv 0$ $(j = 1, 2, \dots, n+1)$.

Lemma 2.2 ([24]) *Let* f *be a transcendental meromorphic function of finite order* ρ. *Let* $\varepsilon > 0$ *be a constant, and let* k *and* j *be integers satisfying* $k > j \geq 0$. *Then there exists a set* $E_1 \subset [-\frac{\pi}{2}, \frac{3\pi}{2})$, *which has linear measure zero, such that if* $\theta \in [-\frac{\pi}{2}, \frac{3\pi}{2}) \backslash E_1$, *then there is a constant* $R = R(\theta) > 0$ *such that for all* z *satisfying* $\arg z = \theta$ *and* $|z| \geq R$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}.$$

Lemma 2.3 ([17]) *Let* $g(z)$ *be a meromorphic function with* $\rho(g) = \beta < \infty$. *Then, for any given* $\varepsilon > 0$, *there exists a set* $E_2 \subset [-\frac{\pi}{2}, \frac{3\pi}{2})$ *that has linear measure zero such that if* $\psi \in [-\frac{\pi}{2}, \frac{3\pi}{2}) \backslash E_2$, *then there is a constant* $R = R(\psi) > 1$ *such that, for all* z *satisfying* $\arg z = \psi$ *and* $|z| = r > R$, *we have*

$$\exp\{-r^{\beta+\varepsilon}\} \leq |g(z)| \leq \exp\{r^{\beta+\varepsilon}\}.$$

Lemma 2.4 ([17]) *Consider* $g(z) = A(z)e^{az}$, *where* $A(z)$ $(\not\equiv 0)$ *is a meromorphic function with* $\rho(A) = \alpha < 1$, a *is a complex constant,* $a = |a|e^{i\varphi}$ $(\varphi \in [0, 2\pi))$. *Set* $E_3 = \{\theta \in [0, 2\pi) : \cos(\varphi + \theta) = 0\}$, *then* E_3 *is a finite set. Then, for any given* ε $(0 < \varepsilon < 1 - \alpha)$, *there is a set* $E_4 \in [0, 2\pi)$ *that has linear measure zero, if* $z = re^{i\theta}$, $\theta \in [0, 2\pi) \backslash (E_3 \cup E_4)$, *then we have when* r *is sufficiently large*:

(i) *If* $\cos(\varphi + \theta) > 0$, *then*

$$\exp\{(1-\varepsilon)r\delta(az, \theta)\} \leq |g(z)| \leq \exp\{(1+\varepsilon)r\delta(az, \theta)\};$$

(ii) *If* $\cos(\varphi + \theta) < 0$, *then*

$$\exp\{(1+\varepsilon)r\delta(az, \theta)\} \leq |g(z)| \leq \exp\{(1-\varepsilon)r\delta(az, \theta)\};$$

where $\delta(az, \theta) = |a|\cos(\varphi + \theta)$.

Lemma 2.5 ([9]) *Suppose that* $n \geq 1$ *is a positive integer. Let* $P_j(z) = a_{jn}z^n + \cdots$ $(j = 1, 2)$ *be nonconstant polynomials, where* a_{jq} $(q = 1, \dots, n)$ *are complex numbers and* $a_{1n}a_{2n} \neq 0$. *Set* $z = re^{i\theta}$, $a_{jn} = |a_{jn}|e^{i\theta_j}$, $\theta_j \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, $\delta(P_j, \theta) = |a_{jn}|\cos(\theta_j + n\theta)$, *then there is a set* $E_5 \subset [-\frac{\pi}{2n}, \frac{3\pi}{2n})$ *that has linear measure zero. If* $\theta_1 \neq \theta_2$, *then there exists a ray* $\arg z = \theta$,

$\theta \in [-\frac{\pi}{2n}, \frac{\pi}{2n}) \backslash (E_5 \cup E_6)$ *such that*

$$\delta(P_1,\theta) > 0, \qquad \delta(P_2,\theta) < 0 \tag{2.1}$$

or

$$\delta(P_1,\theta) < 0, \qquad \delta(P_2,\theta) > 0, \tag{2.2}$$

where $E_6 = \{\theta \in [-\frac{\pi}{2n}, \frac{3\pi}{2n}) : \delta(P_j,\theta) = 0\}$ *is a finite set, which has linear measure zero.*

In Lemma 2.5, if $\theta \in [-\frac{\pi}{2n}, \frac{\pi}{2n}) \backslash (E_5 \cup E_6)$ is replaced by $\theta \in [\frac{\pi}{2n}, \frac{3\pi}{2n}) \backslash (E_5 \cup E_6)$, then we can obtain the same result.

Lemma 2.6 ([25]) *Let* $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ *be finite order meromorphic functions. If* $f(z)$ *is an infinite order meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = F,$$

then f *satisfies* $\lambda(f) = \overline{\lambda}(f) = \rho(f) = \infty$.

Lemma 2.7 ([26]) *Let* $k \geq 2$ *and* $A_0, A_1, \ldots, A_{k-1}$ *be meromorphic functions. Let* $\rho = \max\{\rho(A_j), j = 0, 1, \ldots, k-1\}$ *and all poles of* f *are of uniformly bounded multiplicity. Then every transcendental meromorphic solution of the differential equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = 0 \tag{2.3}$$

satisfies $\rho_2(f) \leq \rho$.

Remark 2 The condition that the multiplicity of poles of the solution f is uniformly bounded can be changed by $\delta(\infty, f) > 0$ for the solution f (see [27]).

Lemma 2.8 (see [24]) *Let* f *be a transcendental meromorphic function. Let* $\alpha > 1$ *be a constant, and let* k *and* j *be integers satisfying* $k > j \geq 0$. *Then there exists a set* $E_7 \subset (1, \infty)$, *which has finite logarithmic measure, and a constant* $C > 0$ *such that for all* z *satisfying* $|z| \notin E_7 \cup [0,1]$, *we have* (*with* $r = |z|$)

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq C\left[\frac{T(\alpha r, f)}{r}(\log r)^{\alpha} \log T(\alpha r, f)\right]^{k-j}. \tag{2.4}$$

Lemma 2.9 (see [2, 28]) *Let* $F(r)$ *and* $G(r)$ *be nondecreasing real-valued functions on* $(0, \infty)$ *such that* $F(r) \leq G(r)$ *for all* r *outside of a set* $E \subset (0, \infty)$ *of finite linear measure or outside of a set* $H \cup [0,1]$, *where* $H \cup [1, \infty)$ *is of finite logarithmic measure. Then, for every constant* $\alpha > 1$, *there exists an* $r_0 > 0$ *such that* $F(r) \leq G(\alpha r)$ *for all* $r > r_0$.

Lemma 2.10 *Let* a_0 *be a constant satisfying* $a_0 < 0$. *If* $\arg a_1 \neq \pi$ *or* $a_1 < a_0$, *then we have* $a_1 \neq c a_0$ $(0 < c \leq 1)$.

The proof is trivial, we omit it here.

3 Proof of Theorem 1.1

First of all we prove that equation (1.2) cannot have a meromorphic solution $f \not\equiv 0$ with $\rho(f) < 1$. Assume a meromorphic solution $f \not\equiv 0$ with $\rho(f) = \rho_1 < 1$ satisfies equation (1.2). Then $\rho(f^{(j)}) = \rho_1 < 1$ $(j = 1, 2)$. Rewrite (1.2) as

$$A_0 f' e^{a_0 z} + A_1 f e^{a_1 z} + A_2 f e^{a_2 z} = -f''. \tag{3.1}$$

We consider two cases:

(1) $a_2 \neq a_0$, note that $a_1 \neq a_0$, $a_1 \neq a_2$, $\rho(A_1 f) < 1$, $\rho(A_2 f) < 1$, $\rho(A_0 f') < 1$, $\rho(-f'') < 1$ and by Lemma 2.1, we have $f \equiv 0$, which is a contradiction.

(2) $a_2 = a_0$, then (3.1) can be rewritten into

$$A_1 f e^{a_1 z} + \left(A_0 f' + A_2 f\right) e^{a_2 z} = -f''.$$

Note that $a_1 \neq a_2$, $\rho(A_1 f) < 1$, $\rho(A_0 f' + A_2 f) < 1$, $\rho(-f'') < 1$ and by Lemma 2.1 again, we have $f \equiv 0$, which is a contradiction.

Therefore, $\rho(f) \geq 1$.

Now assume f is a meromorphic solution of equation (1.2) with $1 \leq \rho(f) = \rho < \infty$. From equation (1.2), we know that the poles of $f(z)$ can occur only at the poles of A_j $(j = 0, 1, 2)$. Note that the multiplicities of pole points of f are uniformly bounded, and thus we have

$$N(r,f) \leq M_1 \overline{N}(r,f) \leq M_1 \sum_{j=0}^{2} \overline{N}(r, A_j) \leq M \max\bigl\{N(r, A_j) : j = 0, 1, 2\bigr\},$$

where M_1 and M are some suitable positive constants. Then we have $\lambda(1/f) \leq \alpha = \max\{\rho(A_j) : j = 0, 1, 2\} < 1$.

Let $f = g/d$, d be the canonical product formed with the nonzero poles of $f(z)$, with $\beta = \rho(d) = \lambda(d) = \lambda(1/f) \leq \alpha < 1$, let g be an entire function and $1 \leq \rho(g) = \rho(f) = \rho < \infty$. Substituting $f = g/d$ into (1.2), we can get

$$\frac{g''}{g} + \left[A_0 e^{a_0 z} - 2\frac{d'}{d}\right]\frac{g'}{g} + 2\left(\frac{d'}{d}\right)^2 - \frac{d''}{d} - A_0 \frac{d'}{d} e^{a_0 z} + A_1 e^{a_1 z} + A_2 e^{a_2 z} = 0. \tag{3.2}$$

For any given ε $(0 < \varepsilon < 1 - \alpha)$, there is a set $E_1 \subset [-\frac{\pi}{2}, \frac{3\pi}{2})$ that has linear measure zero such that if $\theta \in [-\frac{\pi}{2}, \frac{3\pi}{2}) \backslash E_1$, then there is a constant $R = R(\theta) > 1$ such that for all z satisfying $\arg z = \theta$, and $|z| = r > R$, we have by Lemma 2.3

$$\bigl|A_0(z)\bigr| \leq \exp\bigl\{r^{\alpha+\varepsilon}\bigr\}. \tag{3.3}$$

By Lemma 2.2, for any given ε $(0 < \varepsilon < \min\{1 - \alpha, \frac{|a_2| - |a_1|}{|a_2| + |a_1|}\})$, there exists a set $E_2 \subset [-\frac{\pi}{2}, \frac{3\pi}{2})$ that has linear measure zero such that if $\theta \in [-\frac{\pi}{2}, \frac{3\pi}{2}) \setminus E_2$, then there is a constant $R_0 = R_0(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geq R_0$, we have

$$\left|\frac{g^{(j)}(z)}{g(z)}\right| \leq |z|^{j(\rho - 1 + \varepsilon)}, \quad j = 1, 2, \tag{3.4}$$

and

$$\left|\frac{d^{(j)}(z)}{d(z)}\right| \le |z|^{j(\beta-1+\varepsilon)}, \quad j=1,2. \tag{3.5}$$

Setting $z = re^{i\theta}$, $a_1 = |a_1|e^{i\theta_1}$, $a_2 = |a_2|e^{i\theta_2}$, $\theta_1, \theta_2 \in [-\frac{\pi}{2}, \frac{3\pi}{2})$.

Case 1. $\arg a_1 \neq \pi$, which is $\theta_1 \neq \pi$.

Subcase 1.1. Assume that $\theta_1 \neq \theta_2$. By Lemma 2.5, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \backslash (E_1 \cup E_2 \cup E_5 \cup E_6)$ (where E_5 and E_6 are defined as in Lemma 2.5, $E_1 \cup E_2 \cup E_5 \cup E_6$ is of linear measure zero) satisfying $\delta(a_1 z, \theta) > 0$, $\delta(a_2 z, \theta) < 0$ or $\delta(a_1 z, \theta) < 0$, $\delta(a_2 z, \theta) > 0$ for a sufficiently large r.

When $\delta(a_1 z, \theta) > 0$, $\delta(a_2 z, \theta) < 0$ for a sufficiently large r, we have, by Lemma 2.4,

$$\left|A_1 e^{a_1 z}\right| \ge \exp\left\{(1-\varepsilon)\delta(a_1 z, \theta) r\right\}, \tag{3.6}$$

$$\left|A_2 e^{a_2 z}\right| \le \exp\left\{(1-\varepsilon)\delta(a_2 z, \theta) r\right\} < 1. \tag{3.7}$$

By (3.6) and (3.7), we have

$$\begin{aligned} \left|A_1 e^{a_1 z} + A_2 e^{a_2 z}\right| &\ge \left|A_1 e^{a_1 z}\right| - \left|A_2 e^{a_2 z}\right| \ge \exp\left\{(1-\varepsilon)\delta(a_1 z, \theta) r\right\} - 1 \\ &= \left(1 - o(1)\right) \exp\left\{(1-\varepsilon)\delta(a_1 z, \theta) r\right\}. \end{aligned} \tag{3.8}$$

By (3.2), we get

$$\begin{aligned} \left|A_1 e^{a_1 z} + A_2 e^{a_2 z}\right| \le{}& \left|\frac{g''}{g}\right| + \left(\left|A_0 e^{a_0 z}\right| + 2\left|\frac{d'}{d}\right|\right)\left|\frac{g'}{g}\right| \\ &+ 2\left|\frac{d'}{d}\right|^2 + \left|\frac{d''}{d}\right| + |A_0|\left|\frac{d'}{d}\right|\left|e^{a_0 z}\right|. \end{aligned} \tag{3.9}$$

Since $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \backslash (E_1 \cup E_2 \cup E_5 \cup E_6)$, we know that $\cos\theta > 0$, then $|e^{a_0 z}| = e^{-|a_0| r \cos\theta} < 1$. Therefore, by (3.3) we obtain

$$\left|A_0(z) e^{a_0 z}\right| \le \exp\left\{r^{\alpha+\varepsilon}\right\}. \tag{3.10}$$

Substituting (3.4)-(3.5), (3.8) and (3.10) into (3.9), we obtain

$$\left(1 - o(1)\right) \exp\left\{(1-\varepsilon)\delta(a_1 z, \theta) r\right\} \le M_1 r^{k(\rho-1+\varepsilon)} \exp\left\{r^{\alpha+\varepsilon}\right\}, \tag{3.11}$$

where $M_1 > 0$ and $k > 0$ are some constants. By $\delta(a_1 z, \theta) > 0$ and $\alpha + \varepsilon < 1$, we know that (3.11) is a contradiction.

When $\delta(a_1 z, \theta) < 0$, $\delta(a_2 z, \theta) > 0$, using a proof similar to the above, we can get a contradiction.

Subcase 1.2. Assume that $\theta_1 = \theta_2$. By Lemma 2.5, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \backslash (E_1 \cup E_2 \cup E_5 \cup E_6)$ satisfying $\delta(a_1 z, \theta) > 0$. Since $|a_1| \le |a_2|$, $a_1 \neq a_2$ and $\theta_1 = \theta_2$, then $|a_1| < |a_2|$, thus $\delta(a_2 z, \theta) > \delta(a_1 z, \theta) > 0$. For a sufficiently large r, we get, by Lemma 2.4,

$$\left|A_1 e^{a_1 z}\right| \le \exp\left\{(1+\varepsilon)\delta(a_1 z,\theta)r\right\}, \tag{3.12}$$

$$\left|A_2 e^{a_2 z}\right| \ge \exp\left\{(1-\varepsilon)\delta(a_2 z,\theta)r\right\}. \tag{3.13}$$

By (3.12) and (3.13), we get

$$\begin{aligned} \left|A_1 e^{a_1 z} + A_2 e^{a_2 z}\right| &\ge \left|A_2 e^{a_2 z}\right| - \left|A_1 e^{a_1 z}\right| \\ &\ge \exp\left\{(1-\varepsilon)\delta(a_2 z,\theta)r\right\} - \exp\left\{(1+\varepsilon)\delta(a_1 z,\theta)r\right\} \\ &= M_2 \exp\left\{(1+\varepsilon)\delta(a_1 z,\theta)r\right\}, \end{aligned} \tag{3.14}$$

where $M_2 = \exp\{[(1-\varepsilon)\delta(a_2 z,\theta) - (1+\varepsilon)\delta(a_1 z,\theta)]r\} - 1$.

Since $0 < \varepsilon < \min\{1-\alpha, \frac{|a_2|-|a_1|}{|a_2|+|a_1|}\}$, we see that $(1-\varepsilon)\delta(a_2 z,\theta) - (1+\varepsilon)\delta(a_1 z,\theta) > 0$, then $\exp\{[(1-\varepsilon)\delta(a_2 z,\theta) - (1+\varepsilon)\delta(a_1 z,\theta)]r\} > 1$, $M_2 > 0$.

Since $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \backslash (E_1 \cup E_2 \cup E_5 \cup E_6)$, we know that $\cos\theta > 0$, then $|e^{a_0 z}| = e^{-|a_0| r\cos\theta} < 1$. Therefore, by (3.3) we obtain

$$\left|A_0(z) e^{a_0 z}\right| \le \exp\left\{r^{\alpha+\varepsilon}\right\}. \tag{3.15}$$

Substituting (3.4)-(3.5) and (3.14)-(3.15) into (3.9), we obtain

$$M_2 \exp\left\{(1+\varepsilon)\delta(a_1 z,\theta)r\right\} \le M_1 r^{k(\rho-1+\varepsilon)} \exp\left\{r^{\alpha+\varepsilon}\right\}. \tag{3.16}$$

By $\delta(a_1 z,\theta) > 0$, $M_2 > 0$ and $\alpha + \varepsilon < 1$, we know that (3.16) is a contradiction.

Case 2. $a_1 < a_0$, which is $\theta_1 = \pi$.

Subcase 2.1. Assume that $\theta_1 \ne \theta_2$, then $\theta_2 \ne \pi$. By Lemma 2.5, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \backslash (E_1 \cup E_2 \cup E_5 \cup E_6)$ satisfying $\delta(a_2 z,\theta) > 0$. Since $\cos\theta > 0$, we have $\delta(a_1 z,\theta) = |a_1|\cos(\theta_1 + \theta) = -|a_1|\cos\theta < 0$. For a sufficiently large r, we have

$$\left|A_1 e^{a_1 z}\right| \le \exp\left\{(1-\varepsilon)\delta(a_1 z,\theta)r\right\} < 1, \tag{3.17}$$

$$\left|A_2 e^{a_2 z}\right| \ge \exp\left\{(1-\varepsilon)\delta(a_2 z,\theta)r\right\}. \tag{3.18}$$

By (3.17) and (3.18), we get

$$\begin{aligned} \left|A_1 e^{a_1 z} + A_2 e^{a_2 z}\right| &\ge \left|A_2 e^{a_2 z}\right| - \left|A_1 e^{a_1 z}\right| \\ &\ge \exp\left\{(1-\varepsilon)\delta(a_2 z,\theta)r\right\} - 1. \end{aligned} \tag{3.19}$$

Using the same reasoning as in Subcase 1.1, we can get a contradiction.

Subcase 2.2. Assume that $\theta_1 = \theta_2$, then $\theta_1 = \theta_2 = \pi$. By Lemma 2.5, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \backslash (E_1 \cup E_2 \cup E_5 \cup E_6)$, then $\cos\theta < 0$, $\delta(a_1 z,\theta) = |a_1|\cos(\theta_1 + \theta) = -|a_1|\cos\theta > 0$, $\delta(a_2 z,\theta) = |a_2|\cos(\theta_1 + \theta) = -|a_2|\cos\theta > 0$. Since $|a_1| \le |a_2|$, $a_1 \ne a_2$ and $\theta_1 = \theta_2$, then $|a_1| < |a_2|$, thus $\delta(a_2 z,\theta) > \delta(a_1 z,\theta) > 0$. For a sufficiently large r, we get (3.12), (3.13) and (3.14) hold.

Using the same reasoning as in Subcase 1.2, we can get a contradiction.

Concluding the above proof, we obtain $\rho(f) = \rho(g) = +\infty$.

In the following, we prove that if all poles of f are of uniformly bounded multiplicity, then $\rho_2(f) = 1$.

By Lemma 2.7 and $\max\{\rho(A_0e^{a_0z}), \rho(A_1e^{a_1z} + A_2e^{a_2z})\} = 1$, then $\rho_2(f) \le 1$.

By Lemma 2.8, we know that there exists a set $E_7 \subset (1, +\infty)$ with finite logarithmic measure and a constant $C > 0$ such that for all z satisfying $|z| = r \notin [0, 1] \cup E_7$, we get

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le C\big[T(2r,f)\log T(2r,f)\big]^j \le C\big[T(2r,f)\big]^{j+1}, \quad j = 1, 2. \tag{3.20}$$

For Subcases 1.1 and 2.1, we have proved that there exists a ray $\arg z = \theta$ satisfying $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\backslash(E_1 \cup E_2 \cup E_5 \cup E_6)$, for a sufficiently large r, we get that (3.8) or (3.19) hold, that is,

$$\big|A_1e^{a_1z} + A_2e^{a_2z}\big| \ge \exp\{h_1r\}, \tag{3.21}$$

where $h_1 > 0$ is a constant.

By (1.2), we have

$$\frac{f''}{f} + A_0e^{a_0z}\frac{f'}{f} = -\big(A_1e^{a_1z} + A_2e^{a_2z}\big). \tag{3.22}$$

Since $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\backslash(E_1 \cup E_2 \cup E_5 \cup E_6)$, then $\cos\theta > 0$, $e^{-|a_0|r\cos\theta} < 1$. By (3.20), (3.21) and (3.22), we obtain

$$\begin{aligned}\exp\{h_1r\} &\le C\big[T(2r,f)\big]^3 + \exp\big\{r^{\alpha+\varepsilon}\big\}e^{-|a_0|r\cos\theta}C\big[T(2r,f)\big]^2 \\ &\le 2C\exp\big\{r^{\alpha+\varepsilon}\big\}\big[T(2r,f)\big]^3.\end{aligned} \tag{3.23}$$

By $h_1 > 0$, $\alpha + \varepsilon < 1$, (3.23) and Lemma 2.9, we know that for the constant $\alpha = 2$, there exists r_0. When $r > r_0$, we have $\rho_2(f) \ge 1$, then $\rho_2(f) = 1$.

For Subcases 1.2 and 2.2, we have proved that there exists a ray $\arg z = \theta$ such that $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})\backslash(E_1 \cup E_2 \cup E_5 \cup E_6)$. For a sufficiently large r, we get that (3.14) holds, and we also get $\cos\theta < 0$, $\delta(a_1z, \theta) > -|a_0|\cos\theta > 0$.

By (3.14), (3.20) and (3.22), we obtain

$$\begin{aligned}M_2\exp\big\{(1+\varepsilon)\delta(a_1z,\theta)r\big\} &\le C\big[T(2r,f)\big]^3 + \exp\big\{r^{\alpha+\varepsilon}\big\}e^{-|a_0|r\cos\theta}C\big[T(2r,f)\big]^2 \\ &\le 2Ce^{-|a_0|r\cos\theta}\exp\big\{r^{\alpha+\varepsilon}\big\}\big[T(2r,f)\big]^3.\end{aligned} \tag{3.24}$$

By $\delta(a_1z, \theta) > -|a_0|\cos\theta > 0$, $M_2 > 0$, $\alpha + \varepsilon < 1$ and (3.24) and Lemma 2.9, we know that for the constant $\alpha = 2$, there exists r_0, when $r > r_0$, we have $\rho_2(f) \ge 1$, then $\rho_2(f) = 1$.

The proof of Theorem 1.1 is complete.

4 Proof of Theorem 1.2

Assume that f ($\not\equiv 0$) is a meromorphic solution of (1.2); then $\rho(f) = \infty$ by Theorem 1.1. Set $g_0(z) = f(z) - \varphi(z)$, $g_0(z)$ is a meromorphic function and $\rho(g_0) = \rho(f) = \infty$. Substituting $f = g_0 + \varphi$ into (1.2), we have

$$g_0'' + h_{0,1}g_0' + h_{0,0}g_0 = h_0, \tag{4.1}$$

where $h_{0,1} = A_0e^{a_0z}$, $h_{0,0} = A_1e^{a_1z} + A_2e^{a_2z}$ and $h_0 = -(\varphi'' + h_{0,1}\varphi' + h_{0,0}\varphi)$. Obviously, $h_0 \not\equiv 0$. In fact, if $h_0 \equiv 0$, then

$$\varphi'' + h_{0,1}\varphi' + h_{0,0}\varphi = 0.$$

By Theorem 1.1, we have $\rho(\varphi) = \infty$, it is a contradiction. Hence $h_0 \not\equiv 0$. Note that the functions $h_{0,1}$, $h_{0,0}$ and h_0 are of finite order, by Lemma 2.6 and (4.1), we have $\overline{\lambda}(g_0) = \overline{\lambda}(f - \varphi) = \infty$.

Now we prove that $\overline{\lambda}(f' - \varphi) = \infty$. Let $g_1(z) = f'(z) - \varphi(z)$, then $\rho(g_1) = \rho(f') = \rho(f) = \infty$ and $\overline{\lambda}(g_1) = \overline{\lambda}(f' - \varphi)$.

Differentiating both sides of (1.2), we have

$$f''' + h_{0,1}f'' + \left[h'_{0,1} + h_{0,0}\right]f' + h'_{0,0}f = 0. \tag{4.2}$$

By (1.2), we have

$$f = -\frac{1}{h_{0,0}}\left(f'' + h_{0,1}f'\right). \tag{4.3}$$

Substituting (4.3) into (4.2), we have

$$f''' + \left(h_{0,1} - \frac{h'_{0,0}}{h_{0,0}}\right)f'' + \left[h'_{0,1} + h_{0,0} - \frac{h'_{0,0}}{h_{0,0}}h_{0,1}\right]f' = 0. \tag{4.4}$$

Combining $f' = g_1 + \varphi, f'' = g'_1 + \varphi', f''' = g''_1 + \varphi''$ into (4.4), we have

$$g''_1 + h_{1,1}g'_1 + h_{1,0}g_1 = h_1, \tag{4.5}$$

where

$$h_{1,1} = h_{0,1} - \frac{h'_{0,0}}{h_{0,0}},$$

$$h_{1,0} = h'_{0,1} + h_{0,0} - \frac{h'_{0,0}}{h_{0,0}}h_{0,1},$$

$$-h_1 = \varphi'' + \left(h_{0,1} - \frac{h'_{0,0}}{h_{0,0}}\right)\varphi' + \left[h'_{0,1} + h_{0,0} - \frac{h'_{0,0}}{h_{0,0}}h_{0,1}\right]\varphi.$$

Now we prove $h_1 \not\equiv 0$. In fact, if $h_1 \equiv 0$, then

$$\begin{aligned}
&\left[\left(\frac{\varphi'}{\varphi}A_0 + A'_0 + a_0A_0 - a_1A_0\right)A_1 - A_0A'_1\right]e^{(a_0+a_1)z} \\
&\quad + \left[\left(\frac{\varphi'}{\varphi}A_0 + A'_0 + a_0A_0 - a_2A_0\right)A_2 - A_0A'_2\right]e^{(a_0+a_2)z} \\
&\quad + \left[\left(\frac{\varphi''}{\varphi} - a_1\frac{\varphi'}{\varphi}\right)A_1 - \frac{\varphi'}{\varphi}A'_1\right]e^{a_1z} + \left[\left(\frac{\varphi''}{\varphi} - a_2\frac{\varphi'}{\varphi}\right)A_2 - \frac{\varphi'}{\varphi}A'_2\right]e^{a_2z} \\
&\quad + 2A_1A_2e^{(a_1+a_2)z} + A_1^2e^{2a_1z} + A_2^2e^{2a_2z} \equiv 0.
\end{aligned} \tag{4.6}$$

We can rewrite (4.6) in the form

$$f_1e^{(a_0+a_1)z}+f_2e^{(a_0+a_2)z}+f_3e^{a_1z}+f_4e^{a_2z}+f_5e^{(a_1+a_2)z}+f_6e^{2a_1z}+f_7e^{2a_2z}\equiv 0. \tag{4.7}$$

Since $\rho=\rho(\varphi)<1$, $\rho(A_k)<1$ ($k=0,1,2$), then $\rho(f_j)<1$ ($j=1,\dots,7$). Note that $a_1\neq a_2$ and Lemma 2.10, then $2a_1\neq a_1,a_1+a_0,a_1+a_2,2a_2$. In order to apply Lemma 2.1, we need to consider three cases:

(i) If $2a_1\neq a_0+a_2,a_2$. By Lemma 2.1, we get $f_6\equiv 0$, that is, $A_1\equiv 0$, a contradiction.
(ii) If $2a_1=a_0+a_2$, then by Lemma 2.10, $2a_2\neq a_1,2a_1,a_2+a_0,a_1+a_2,a_0+a_1,a_2$. By Lemma 2.1, we get $f_7\equiv 0$, that is, $A_2\equiv 0$, a contradiction.
(iii) If $2a_1=a_2$, then by Lemma 2.10, $2a_2\neq a_1,2a_1,a_2+a_0,a_1+a_2,a_0+a_1,a_2$. By Lemma 2.1, we get $f_7\equiv 0$, that is, $A_2\equiv 0$, a contradiction.

Hence $h_1\not\equiv 0$.

For equation (4.5), since $h_1\not\equiv 0$ and $\rho(g_1)=\infty$, by Lemma 2.6, we have

$$\overline{\lambda}(g_1)=\overline{\lambda}(f'-\varphi)=\rho(g_1)=\rho(f)=\infty.$$

Using a similar way to above, we can easily prove that $h_{1,0}\not\equiv 0$. We omit it here.

In the following, we prove $\overline{\lambda}(f''-\varphi)=\infty$.

Let $g_2(z)=f''-\varphi$, then $\rho(g_2)=\rho(f'')=\rho(f)=\infty$ and $\overline{\lambda}(g_2)=\overline{\lambda}(f''-\varphi)$.

Differentiating the two sides of (1.2), we get

$$f^{(4)}+h_{0,1}f'''+\left[2h'_{0,1}+h_{0,0}\right]f''+\left[h''_{0,1}+2h'_{0,0}\right]f'+h''_{0,0}f=0. \tag{4.8}$$

By (4.4), we have

$$f'=\frac{-1}{h_{1,0}}\left[f'''+h_{1,1}f''\right]. \tag{4.9}$$

From (4.3), (4.8) and (4.9), we obtain

$$f^{(4)}+h_{2,1}f'''+h_{2,0}f''=0, \tag{4.10}$$

where

$$h_{2,1}=h_{0,1}-\frac{\varphi_1}{h_{1,0}}, \tag{4.11}$$

$$h_{2,0}=2h'_{0,1}+h_{0,0}-\frac{h''_{0,0}}{h_{0,0}}-\frac{\varphi_1}{h_{1,0}}\left[h_{0,1}-\frac{h'_{0,0}}{h_{0,0}}\right], \tag{4.12}$$

where

$$\varphi_1=h''_{0,1}+2h'_{0,0}-h_{0,1}\frac{h''_{0,0}}{h_{0,0}}. \tag{4.13}$$

Substituting $f''=g_2+\varphi, f'''=g'_2+\varphi', f^{(4)}=g''_2+\varphi''$ into (4.10), we have

$$g''_2+h_{2,1}g'_2+h_{2,0}g_2=h_2,$$

where

$$h_2 = -\big(\varphi'' + h_{2,1}\varphi' + h_{2,0}\varphi\big). \tag{4.14}$$

Similarly, if one can prove $h_2 \not\equiv 0$, then by Lemma 2.6, we can get $\overline{\lambda}(g_2) = \rho(g_2) = \infty$. Hence $\overline{\lambda}(f''' - \varphi) = \infty$.

Now we prove $h_2 \not\equiv 0$. Note that

$$h_{2,1} = \frac{H_1}{h_{1,0}h_{0,0}}, \tag{4.15}$$

$$h_{2,0} = \frac{H_2}{h_{1,0}h_{0,0}}, \tag{4.16}$$

where

$$\begin{aligned}
H_1 &= h_{0,0}h_{0,1}h'_{0,1} + h^2_{0,0}h_{0,1} - h'_{0,0}h^2_{0,1} - h_{0,0}h''_{0,1} - 2h_{0,0}h'_{0,0} + h_{0,1}h''_{0,0},\\
H_2 &= 2h'^2_{0,1}h_{0,0} + 3h^2_{0,0}h_{0,1} - 2h_{0,0}h_{0,1}h'_{0,1} + h^3_{0,0} - 3h_{0,0}h'_{0,0}h_{0,1} - h'_{0,1}h''_{0,0}\\
&\quad - h''_{0,0}h_{0,0} - h_{0,1}h''_{0,1}h_{0,0} + 2h'^2_{0,0} + h'_{0,0}h''_{0,1} + h^2_{0,1}h''_{0,0}.
\end{aligned}$$

Obviously, H_1, H_2 are meromorphic functions and $\rho(H_j) < 1$ $(j = 1, 2)$. From (4.14), (4.15)-(4.16), we know

$$\frac{h_2}{\varphi} = -\frac{1}{h_{1,0}h_{0,0}}\left[\frac{\varphi''}{\varphi}h_{1,0}h_{0,0} + \frac{\varphi'}{\varphi}H_1 + H_2\right].$$

Let $H = \frac{\varphi''}{\varphi}h_{1,0}h_{0,0} + \frac{\varphi'}{\varphi}H_1 + H_2$. We only need to prove $H \not\equiv 0$.

Note that $\rho(\frac{\varphi''}{\varphi}) < 1$, $\rho(\frac{\varphi'}{\varphi}) < 1$ and $h_{1,0}h_{0,0} = h'_{0,1}h_{0,0} + h^2_{0,0} - h_{0,1}h'_{0,0}$. From this and (4.15)-(4.16), we can write H into the following form:

$$\begin{aligned}
H &= f_1e^{(a_0+a_1)z} + f_2e^{(a_0+a_2)z} + f_3e^{2a_1z} + f_4e^{2a_2z} + f_5e^{(a_1+a_2)z}\\
&\quad + f_6e^{(2a_1+a_2)z} + f_7e^{(a_1+2a_2)z} + f_8e^{(2a_0+a_1)z} + f_9e^{(2a_0+a_2)z} + f_{10}e^{(2a_1+a_0)z}\\
&\quad + f_{11}e^{(2a_2+a_0)z} + f_{12}e^{(a_0+a_1+a_2)z} + f_{13}e^{3a_1z} + f_{14}e^{3a_2z}.
\end{aligned}$$

It is easy to see $f_{13} = A_1^3, f_{14} = A_2^3$, which come from the term $h^3_{0,0}$ in H_2.

We can prove that $\rho(f_i) < 1$ $(1 \le i \le 12)$. Set $\Lambda = \{a_0 + a_1 + a_2, a_0 + a_2, 2a_1, 2a_2, a_1 + a_2, 2a_1 + a_2, a_1 + 2a_2, 2a_0 + a_1, 2a_0 + a_2, 2a_1 + a_0, 2a_2 + a_0, a_1 + a_2, 3a_1, 3a_2\}$. Note that $a_1 \neq a_2$ and Lemma 2.10, then $3a_1 \neq a_1 + a_0, a_1 + 2a_0, 2a_1, 2a_1 + a_0, 2a_1 + a_2, 3a_2, a_1 + 2a_2$ and $3a_2 \neq 2a_2, 3a_1, 2a_1 + a_2, a_1 + 2a_2$. Using the same way as above, we need to consider seven cases:

(i) If $3a_1 \neq a_2 + a_0, a_2 + 2a_0, a_1 + a_2, 2a_2 + a_0, a_0 + a_1 + a_2, 2a_2$. By Lemma 2.1, we get $f_{13} \equiv 0$, that is, $A_1 \equiv 0$, a contradiction.

(ii) If $3a_1 = a_2 + a_0$, then we can conclude that $3a_2 \neq \Lambda - \{3a_2\}$. Hence, by Lemma 2.1, we get $f_{14} \equiv 0$, that is, $A_2 \equiv 0$, a contradiction.

(iii) If $3a_1 = a_2 + 2a_0$, then we can conclude that $3a_2 \neq \Lambda - \{3a_2\}$. Hence, by Lemma 2.1, we get $f_{14} \equiv 0$, that is, $A_2 \equiv 0$, a contradiction.

(iv) If $3a_1 = a_1 + a_2$, then we can conclude that $3a_2 \neq \Lambda - \{3a_2\}$. Hence, by Lemma 2.1, we get $f_{14} \equiv 0$, that is, $A_2 \equiv 0$, a contradiction.

(v) If $3a_1 = 2a_2 + a_0$, then we can conclude that $3a_2 \neq \Lambda - \{3a_2\}$. Hence, by Lemma 2.1, we get $f_{14} \equiv 0$, that is, $A_2 \equiv 0$, a contradiction.

(vi) If $3a_1 = a_0 + a_1 + a_2$, then we can conclude that $3a_2 \neq \Lambda - \{3a_2\}$. Hence, by Lemma 2.1, we get $f_{14} \equiv 0$, that is, $A_2 \equiv 0$, a contradiction.

(vii) If $3a_1 = 2a_2$, then we conclude that $3a_2 \neq \Lambda - \{3a_2\}$. Hence, by Lemma 2.1, we get $f_{14} \equiv 0$, that is, $A_2 \equiv 0$, a contradiction.

Hence, we prove $H \not\equiv 0$ and $h_2 \not\equiv 0$.

This proved the theorem.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors drafted the manuscript, read and approved the final manuscript.

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