Journal of Inequalities and Applications
a SpringerOpen Journal

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Inequality estimates for the boundedness of multilinear singular and fractional integral operators

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Abstract

In this paper, the inequality of boundedness for the multilinear fractional singular integral operators associated to the weighted Lipschitz functions is estimated. The operators include the Calderón-Zygmund singular integral operator and the fractional integral operator.
MSC: 42B20; 42B25

Keywords: multilinear operators; Calderón-Zygmund singular integral operator; fractional integral operator; weighted Lipschitz space; Triebel-Lizorkin space

1 Introduction

As the development of the singular integral operators, their commutators and multilinear operators have been well studied (see [1–5]). In [1–3, 5–7], the authors proved that the commutators and multilinear operators generated by the singular integral operators and *BMO* functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [8]) proved a similar result when singular integral operators are replaced by the fractional integral operators. In [9–12], the boundedness for the commutators and multilinear operators generated by the singular integral operators and Lipschitz functions on $L^p(R^n)$ ($1 < p < \infty$) and Triebel-Lizorkin spaces are obtained. In [13, 14], the weighted boundedness for the commutators generated by the singular integral operators and *BMO* or Lipschitz functions on $L^p(R^n)$ ($1 < p < \infty$) spaces are obtained. The purpose of this paper is to study the weighted boundedness for some multilinear operators associated to the fractional singular integral operators and the weighted Lipschitz functions. As application, the weighted boundedness for the multilinear operators associated to the Calderón-Zygmund singular integral operator and the fractional integral operator is obtained.

2 Notations and theorems

In this paper, we are going to consider some multilinear operators as follows.

Let m_j be positive integers ($j = 1, \ldots, k$), $m_1 + \cdots + m_k = m$, and let b_j be locally integrable functions on R^n ($j = 1, \ldots, k$). Set

$$R_{m_j+1}(b_j; x, y) = b_j(x) - \sum_{|\alpha| \le m_j} \frac{1}{\alpha!} D^\alpha b_j(y)(x - y)^\alpha.$$

Definition 1 Let $T : S \to S'$ be a linear operator, and there exists a locally integrable function $K(x, y)$ on $R^n \times R^n$ such that

$$T(f)(x) = \int_{R^n} K(x, y) f(y) \, dy$$

for every bounded and compactly supported function f, where K satisfies: for fixed $0 \leq \delta < n$ and $0 < \varepsilon \leq 1$,

$$\left| K(x, y) \right| \leq C|x - y|^{-n+\delta}, \tag{1}$$

and

$$\left| K(y, x) - K(z, x) \right| \leq C|y - z|^{\varepsilon} |x - z|^{-n+\delta-\varepsilon} \quad \text{if } 2|y - z| \leq |x - z|. \tag{2}$$

Given bounded and compactly supported functions f defined on R^n, the multilinear operator associated to T is defined by

$$T^b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^k R_{m_j+1}(b_j; x, y)}{|x - y|^{m-k}} K(x, y) f(y) \, dy.$$

Note that when $m = 0$, T^b is just a multilinear commutator of T and b (see [15, 16]), while when $m > 0$, it is non-trivial generalization of the commutators; when $\eta = 0$, T^b is just a multilinear commutator of the singular integral operator, when $0 \leq \eta < n$, T^b is just a multilinear commutator of the fractional integral operator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [1–4, 6, 7, 9, 10, 15, 16]). The purpose of this paper is to study the weighted boundedness properties for the multilinear operator.

Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function f, let $f(Q) = \int_Q f(x) \, dx$, $f_Q = |Q|^{-1} \int_Q f(x) \, dx$ and

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \left| f(y) - f_Q \right| dy.$$

It is well known that (see [17, 18])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q \left| f(y) - c \right| dy.$$

For $1 \leq p < \infty$ and $0 \leq \eta < n$, let

$$M_{\eta,p}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-p\eta/n}} \int_Q \left| f(y) \right|^p dy \right)^{1/p},$$

which is the Hardy-Littlewood maximal function when $p = 1$ and $\eta = 0$.

The A_p weight is defined by (see [17])

$$A_p = \left\{ w : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x) \, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx \right)^{p-1} < \infty \right\}, \quad 1 < p < \infty,$$

$$A_1 = \left\{ w > 0 : M(w)(x) \leq Cw(x), \text{a.e.} \right\},$$

and $A_\infty = \bigcup_{p \geq 1} A_p$. We know, for $w \in A_1$, w satisfies the double condition, that is, for any cube Q,

$$w(2Q) \leq Cw(Q).$$

The $A(p, q)$ weight is defined by (see [19])

$$A(p, q) = \left\{ w > 0 : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)^q \, dx \right)^{1/q} \left(\frac{1}{|Q|} \int_Q w(x)^{-p/(p-1)} \, dx \right)^{(p-1)/p} < \infty \right\},$$
$$1 < p, q < \infty.$$

Given a weight function w. For $1 < p < \infty$, the weighted Lebesgue space $L^p(w)$ is the space of functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.$$

For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta, \infty}(w)$ be the weighted homogeneous Triebel-Lizorkin space. For $0 < \beta < 1$, the weighted Lipschitz space $\mathrm{Lip}_\beta(w)$ is the space of functions f such that

$$\|f\|_{\mathrm{Lip}_\beta(w)} = \sup_Q \frac{1}{w(Q)^{1+\beta/n}} \int_Q |f(y) - f_Q| \, dy < \infty.$$

We shall prove the following theorems in Section 3.

Theorem 1 *Suppose that T^b is the multilinear operator as Definition 1 such that T is bounded from $L^u(R^n)$ to $L^v(R^n)$ for any $0 \leq \eta < n$, $1 < u < n/\eta$ and $1/u - 1/v = \eta/n$. Let $0 < \beta \leq 1$, $\beta + \delta < n$, $w \in A_1$ and $D^\alpha b_j \in \mathrm{Lip}_\beta(w)$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, k$. Then T^b is bounded from $L^p(w)$ to $L^q(w^{1-q(k-(\delta+k)/n)})$ for any $1 < p < n/(\delta + k + k\beta)$ and $1/p - 1/q = (\delta + k + k\beta)/n$.*

Theorem 2 *Suppose that T^b is the multilinear operator as Definition 1 such that T is bounded from $L^u(R^n)$ to $L^v(R^n)$ for any $0 \leq \eta < n$, $1 < u < n/\eta$ and $1/u - 1/v = \eta/n$. Let $0 < \beta < \min(1/k, \varepsilon/k)$, $\beta + \delta < n$, $w \in A_1$ and $D^\alpha b_j \in \mathrm{Lip}_\beta(w)$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, k$. Then T^b is bounded from $L^p(w)$ to $\dot{F}_q^{k\beta, \infty}(w^{1-q(k-(\delta+k-k\beta)/n)})$ for any $1 < p < n/(\delta + k)$ and $1/p - 1/q = (\delta + k)/n$.*

3 Proofs of theorems

We begin with some preliminary lemmas.

Lemma 1 (see [1]) *Let b be a function on R^n and $D^\alpha b \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(b; x, y)| \leq C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x,y)} |D^\alpha b(z)|^q \, dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

Lemma 2 (see [11, 12]) *For $0 < \beta < 1$, $1 < p < \infty$ and $w \in A_\infty$, we have*

$$\|f\|_{\dot{F}_p^{\beta,\infty}(w)} \approx \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q|\, dx \right\|_{L^p(w)}$$

$$\approx \left\| \sup_{Q \ni \cdot} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c|\, dx \right\|_{L^p(w)}.$$

Lemma 3 (see [8]) *Suppose that $1 \le s < p < n/\eta$, $1/q = 1/p - \eta/n$ and $w \in A(p,q)$. Then*

$$\left\| M_{\eta,s}(f) \right\|_{L^q(w^q)} \le C\|f\|_{L^p(w^p)}.$$

Lemma 4 (see [17, 20]) *For $0 < \beta < 1$, $1 \le p \le \infty$ and $w \in A_1$, we have*

$$\|b\|_{\mathrm{Lip}_\beta(w)} \approx \sup_Q \frac{1}{w(Q)^{\beta/n}} \left(\frac{1}{w(Q)} \int_Q |b(x) - b_Q|^p w(y)^{1-p}\, dx \right)^{1/p}.$$

Lemma 5 (see [20]) *For any cube Q, $b \in \mathrm{Lip}_\beta(w)$, $0 < \beta < 1$ and $w \in A_1$, we have*

$$\sup_{x \in Q} |b(x) - b_Q| \le C\|b\|_{\mathrm{Lip}_\beta(w)} w(Q)^{1+\beta/n} |Q|^{-1}.$$

To prove the theorems, we need the following lemmas.

Key Lemma 1 *Suppose that T^b is the multilinear operator as Definition 1 such that T is bounded from $L^u(R^n)$ to $L^v(R^n)$ for any $0 \le \eta < n$, $1 < u < n/\eta$ and $1/u - 1/v = \eta/n$. Let $0 < \beta \le 1$, $\beta + \delta < n$, $w \in A_1$ and $D^\alpha b_j \in \mathrm{Lip}_\beta(w)$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, k$. Then there exists a constant C_0 such that for every $f \in C_0^\infty(R^n)$, $1 < s < n/\delta$ and $\tilde{x} \in R^n$,*

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|} \int_Q |T^b(f)(x) - C_0|\, dx$$

$$\le C \prod_{j=1}^k \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) w(\tilde{x})^{k+k\beta/n} M_{\delta+k\beta,s}(f)(\tilde{x}).$$

Key Lemma 2 *Suppose that T^b is the multilinear operator as Definition 1 such that T is bounded from $L^u(R^n)$ to $L^v(R^n)$ for any $0 \le \eta < n$, $1 < u < n/\eta$ and $1/u - 1/v = \eta/n$. Let $0 < \beta < \min(1/k, \varepsilon/k)$, $\beta + \delta < n$, $w \in A_1$ and $D^\alpha b_j \in \mathrm{Lip}_\beta(w)$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, k$. Then there exists a constant C_0 such that for every $f \in C_0^\infty(R^n)$, $1 < s < n/\delta$ and $\tilde{x} \in R^n$,*

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|^{1+k\beta/n}} \int_Q |T^b(f)(x) - C_0|\, dx$$

$$\le C \prod_{j=1}^k \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) w(\tilde{x})^{k+k\beta/n} M_{\delta+k,s}(f)(\tilde{x}).$$

Proof of Key Lemma 1 Without loss of generality, we may assume $k = 2$. Fix a cube $Q = Q(x_0, d)$ with $Q \ni \tilde{x}$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha b_j)_{\tilde{Q}} x^\alpha$, then

$R_{m+1}(b_j; x, y) = R_{m+1}(\tilde{b}_j; x, y)$ and $D^\alpha \tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We split $f = f\chi_{\tilde{Q}} + f\chi_{R^n \setminus \tilde{Q}} = f_1 + f_2$. Write

$$
T^b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^{m-2}} K(x, y) f_1(y) \, dy
$$

$$
- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^{m-2}} D^{\alpha_1} \tilde{b}_1(y) K(x, y) f_1(y) \, dy
$$

$$
- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^{m-2}} D^{\alpha_2} \tilde{b}_2(y) K(x, y) f_1(y) \, dy
$$

$$
+ \sum_{|\alpha_1|=m_1, \, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)}{|x-y|^{m-2}} K(x, y) f_1(y) \, dy
$$

$$
+ \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^{m-2}} K(x, y) f_2(y) \, dy,
$$

then

$$
\frac{1}{|Q|} \int_Q \left| T^b(f)(x) - T^{\tilde{b}}(f_2)(x_0) \right| dx
$$

$$
\leq \frac{1}{|Q|} \int_Q \left| \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^{m-2}} K(x, y) f_1(y) \, dy \right| dx
$$

$$
+ \frac{C}{|Q|} \int_Q \left| \sum_{|\alpha_1|=m_1} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^{m-2}} D^{\alpha_1} \tilde{b}_1(y) K(x, y) f_1(y) \, dy \right| dx
$$

$$
+ \frac{C}{|Q|} \int_Q \left| \sum_{|\alpha_2|=m_2} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^{m-2}} D^{\alpha_2} \tilde{b}_2(y) K(x, y) f_1(y) \, dy \right| dx
$$

$$
+ \frac{C}{|Q|} \int_Q \left| \sum_{|\alpha_1|=m_1, \, |\alpha_2|=m_2} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)}{|x-y|^{m-2}} K(x, y) f_1(y) \, dy \right| dx
$$

$$
+ \frac{1}{|Q|} \int_Q \left| T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0) \right| dx := I_1 + I_2 + I_3 + I_4 + I_5.
$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, by Lemmas 1 and 5, we get

$$
\left| R_m(\tilde{b}_j; x, y) \right| \leq C |x-y|^m \sum_{|\alpha|=m} \sup_{x \in \tilde{Q}} \left| D^\alpha b_j(x) - (D^\alpha b_j)_{\tilde{Q}} \right|
$$

$$
\leq C |x-y|^m \frac{w(\tilde{Q})^{1+\beta/n}}{|Q|} \sum_{|\alpha|=m} \left\| D^\alpha b_j \right\|_{\mathrm{Lip}_\beta(w)}.
$$

Thus, by the (L^s, L^t)-boundedness of T with $1 < s < n/\delta$ and $1/t = 1/s - \delta/n$, we obtain

$$
I_1 \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{\mathrm{Lip}_\beta(w)} \right) \left(\frac{|Q|^{2/n} w(\tilde{Q})^{2+2\beta/n}}{|Q|^2} \right) \frac{1}{|Q|} \int_Q \left| T(f_1)(x) \right| dx
$$

$$
\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{\mathrm{Lip}_\beta(w)} \right) \left(\frac{|Q|^{2/n} w(\tilde{Q})^{2+2\beta/n}}{|Q|^2} \right) \left(\frac{1}{|Q|} \int_{R^n} \left| T(f_1)(x) \right|^t dx \right)^{1/t}
$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) \left(\frac{|Q|^{2/n} w(\tilde{Q})^{2+2\beta/n}}{|Q|^2} \right) |Q|^{-1/t} \left(\int_{R^n} |f_1(x)|^s \, dx \right)^{1/s}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) \left(\frac{w(\tilde{Q})}{|\tilde{Q}|} \right)^{2+2\beta/n} \left(\frac{1}{|\tilde{Q}|^{1-s(\delta+2+2\beta)/n}} \int_{\tilde{Q}} |f(y)|^s \, dy \right)^{1/s}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2+2\beta,s}(f)(\tilde{x}).$$

For I_2, by the (L^s, L^t)-boundedness of T with $1 < s < n/\delta$, $1/t = 1/s - \delta/n$ and Lemma 5, we get

$$I_2 \leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{\mathrm{Lip}_\beta(w)} \frac{|Q|^{2/n} w(\tilde{Q})^{1+\beta/n}}{|Q|} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|} \int_Q |T(D^{\alpha_1} \tilde{b}_1 f_1)(x)| \, dx$$

$$\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{\mathrm{Lip}_\beta(w)} \frac{|Q|^{2/n} w(\tilde{Q})^{1+\beta/n}}{|Q|}$$

$$\times \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 f_1)(x)|^t \, dx \right)^{1/t}$$

$$\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{\mathrm{Lip}_\beta(w)} \frac{|Q|^{2/n} w(\tilde{Q})^{1+\beta/n}}{|Q|} |Q|^{-1/t}$$

$$\times \sum_{|\alpha_1|=m_1} \left(\int_{R^n} |D^{\alpha_1} b_1(x) - (D^\alpha b_1)_{\tilde{Q}}|^s |f_1(x)|^s \, dx \right)^{1/s}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) \frac{|Q|^{2/n} w(\tilde{Q})^{2+2\beta/n}}{|Q|^2} |Q|^{-1/t} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(y)|^s \, dy \right)^{1/s}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) \left(\frac{w(\tilde{Q})}{|\tilde{Q}|} \right)^{2+2\beta/n} \left(\frac{1}{|\tilde{Q}|^{1-s(\delta+2+2\beta)/n}} \int_{\tilde{Q}} |f(y)|^s \, dy \right)^{1/s}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2+2\beta,s}(f)(\tilde{x}).$$

For I_3, similar to the proof of I_2, we get

$$I_3 \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2+2\beta,s}(f)(\tilde{x}).$$

Similarly, for I_4, we obtain, for $1 < s < n/\delta$ and $1/t = 1/s - \delta/n$,

$$I_4 \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{|Q|^{2/n}}{|Q|} \int_Q |T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1)(x)| \, dx$$

$$\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{2/n} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1)(x)|^t \, dx \right)^{1/t}$$

$$\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} |Q|^{2/n} |Q|^{-1/t} \left(\int_{R^n} \left| D^{\alpha_1}\tilde{b}_1(x) D^{\alpha_2}\tilde{b}_2(x) \right|^s |f_1(x)|^s \, dx \right)^{1/s}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{\mathrm{Lip}_\beta(w)} \right) \frac{|Q|^{2/n} w(\tilde{Q})^{2+2\beta/n}}{|Q|^2} |Q|^{-1/t} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(y)|^s \, dy \right)^{1/s}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{\mathrm{Lip}_\beta(w)} \right) \left(\frac{w(\tilde{Q})}{|\tilde{Q}|} \right)^{2+2\beta/n} \left(\frac{1}{|\tilde{Q}|^{1-s(\delta+2+2\beta)/n}} \int_{\tilde{Q}} |f(y)|^s \, dy \right)^{1/s}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{\mathrm{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2+2\beta,s}(f)(\tilde{x}).$$

For I_5, we write

$$T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0)$$

$$= \int_{R^n} \left(\frac{K(x,y)}{|x-y|^m} - \frac{K(x_0,y)}{|x_0-y|^m} \right) \prod_{j=1}^{2} R_{m_j}(\tilde{b}_j; x, y) f_2(y) \, dy$$

$$+ \int_{R^n} \left(R_{m_1}(\tilde{b}_1; x, y) - R_{m_1}(\tilde{b}_1; x_0, y) \right) \frac{R_{m_2}(\tilde{b}_2; x, y)}{|x_0-y|^m} K(x_0, y) f_2(y) \, dy$$

$$+ \int_{R^n} \left(R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; x_0, y) \right) \frac{R_{m_1}(\tilde{b}_1; x_0, y)}{|x_0-y|^m} K(x_0, y) f_2(y) \, dy$$

$$- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} D^{\alpha_1}\tilde{b}_1(y) f_2(y)$$

$$\times \left[\frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x,y) - \frac{R_{m_2}(\tilde{b}_2; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} K(x_0,y) \right] dy$$

$$- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} D^{\alpha_2}\tilde{b}_2(y) f_2(y)$$

$$\times \left[\frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x,y) - \frac{R_{m_1}(\tilde{b}_1; x_0, y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} K(x_0,y) \right] dy$$

$$+ \sum_{|\alpha_1|=m_1,\,|\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x,y) - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} K(x_0,y) \right]$$

$$\times D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y) f_2(y) \, dy$$

$$= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q} = \bigcup_{l=0}^{\infty}(2^{l+1}\tilde{Q} \setminus 2^l\tilde{Q})$. By Lemmas 1 and 5, we get

$$\left| R_{m_j}(\tilde{b}_j; x, y) \right| \leq C \sum_{|\alpha|=m_j} \left\| D^\alpha b_j \right\|_{\mathrm{Lip}_\beta(w)} \left(\frac{w(2^{l+1}\tilde{Q})^{1+\beta/n}}{|2^{l+1}Q|} \right) |x-y|^{m_j}.$$

Then, by the conditions on K, we obtain

$$
\begin{aligned}
\left|I_5^{(1)}\right| &\le C \int_{R^n \setminus \tilde{Q}} \left(\frac{|x - x_0|}{|x_0 - y|^{m+n-1-\delta}} + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{m+n+\varepsilon-2-\delta}} \right) \prod_{j=1}^{2} \left|R_{m_j}(\tilde{b}_j; x, y)\right| |f(y)| \, dy \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) \sum_{l=0}^{\infty} \frac{w(2^{l+1}\tilde{Q})^{2+2\beta/n}}{|2^{l+1}Q|^2} \\
&\quad \times \int_{2^{l+1}\tilde{Q} \setminus 2^l\tilde{Q}} \left(\frac{|x - x_0|}{|x_0 - y|^{n-1-\delta}} + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{n+\varepsilon-2-\delta}} \right) |f(y)| \, dy \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) \sum_{l=0}^{\infty} \frac{w(2^{l+1}\tilde{Q})^{2+2\beta/n}}{|2^{l+1}Q|^2} \\
&\quad \times \int_{2^{l+1}\tilde{Q} \setminus 2^l\tilde{Q}} \left(\frac{d}{(2^l d)^{n-1-\delta}} + \frac{d^\varepsilon}{(2^l d)^{n+\varepsilon-2-\delta}} \right) |f(y)| \, dy \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) \sum_{l=0}^{\infty} \left(\frac{w(2^{l+1}\tilde{Q})}{|2^{l+1}\tilde{Q}|} \right)^{2+2\beta/n} (2^{-l} + 2^{-l\varepsilon}) \\
&\quad \times \frac{1}{|2^{l+1}\tilde{Q}|^{1-(\delta+2+2\beta)/n}} \int_{2^{l+1}\tilde{Q}} |f(y)| \, dy \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} \sum_{l=0}^{\infty} (2^{-l} + 2^{-l\varepsilon}) \\
&\quad \times \left(\frac{1}{|2^{l+1}\tilde{Q}|^{1-s(\delta+2+2\beta)/n}} \int_{2^{l+1}\tilde{Q}} |f(y)|^s \, dy \right)^{1/s} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2+2\beta,s}(f)(\tilde{x}) \sum_{l=1}^{\infty} (2^{-l} + 2^{-l\varepsilon}) \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2+2\beta,s}(f)(\tilde{x}).
\end{aligned}
$$

For $I_5^{(2)}$, by the formula (see [1])

$$
R_{m_j}(\tilde{b}_j; x, y) - R_{m_j}(\tilde{b}_j; x_0, y) = \sum_{|\eta| < m_j} \frac{1}{\eta!} R_{m_j - |\eta|}(D^\eta \tilde{b}_j; x, x_0)(x - y)^\eta
$$

and Lemma 5, we get

$$
\begin{aligned}
\left|I_5^{(2)}\right| &\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{\mathrm{Lip}_\beta(w)} \right) \\
&\quad \times \sum_{l=0}^{\infty} \frac{w(2^{l+1}Q)^{2+2\beta/n}}{|2^{l+1}Q|^2} \int_{2^{l+1}\tilde{Q} \setminus 2^l\tilde{Q}} \frac{|x - x_0|}{|x_0 - y|^{n-1-\delta}} |f(y)| \, dy \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{\mathrm{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2+2\beta,s}(f)(\tilde{x}).
\end{aligned}
$$

Similarly,

$$
\left|I_5^{(3)}\right| \le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{\mathrm{Lip}_\beta(w)}\right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2+2\beta,s}(f)(\tilde{x}).
$$

For $I_5^{(4)}$, similar to the estimates of $I_5^{(1)}$ and $I_5^{(2)}$, and noticing that for $b \in \mathrm{Lip}_\beta(w)$, $w \in A_1$ and $x \in Q$, by Lemma 4, we have

$$
\left|b_Q - b_{2^l Q}\right| \le C l w(x) w\left(2^l Q\right)^{\beta/n} \|b\|_{\mathrm{Lip}_\beta(w)}.
$$

Thus, we obtain

$$
\begin{aligned}
\left|I_5^{(4)}\right| &\le C \sum_{|\alpha_1|=m_1} \int_{R^n\setminus\tilde{Q}} \left| \frac{(x-y)^{\alpha_1} K(x,y)}{|x-y|^{m-2}} - \frac{(x_0-y)^{\alpha_1} K(x_0,y)}{|x_0-y|^{m-2}} \right| \\
&\quad \times \left|R_{m_2}(\tilde{b}_2;x,y)\right|\left|D^{\alpha_1}\tilde{b}_1(y)\right|\left|f(y)\right| dy \\
&\quad + C \sum_{|\alpha_1|=m_1} \int_{R^n\setminus\tilde{Q}} \left|R_{m_2}(\tilde{b}_2;x,y) - R_{m_2}(\tilde{b}_2;x_0,y)\right| \\
&\quad \times \frac{\left|(x_0-y)^{\alpha_1} K(x_0,y)\right|}{|x_0-y|^{m-2}} \left|D^{\alpha_1}\tilde{b}_1(y)f(y)\right| dy \\
&\le C \sum_{|\alpha_2|=m_2} \left\|D^{\alpha_2} b_2\right\|_{\mathrm{Lip}_\beta(w)} \sum_{|\alpha|=m_1} \sum_{l=0}^{\infty} \int_{2^{l+1}\tilde{Q}\setminus 2^l\tilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{n-1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-2-\delta}} \right) \\
&\quad \times \frac{w(2^{l+1}\tilde{Q})^{1+\beta/n}}{|2^{l+1}\tilde{Q}|} \left(\left|D^{\alpha_1} b_1(y) - \left(D^{\alpha_1} b_1\right)_{2^{l+1}\tilde{Q}}\right| \right.\\
&\quad \left. + \left|\left(D^{\alpha_1} b_1\right)_{2^{l+1}\tilde{Q}} - \left(D^{\alpha_1} b_1\right)_{\tilde{Q}}\right|\right)\left|f(y)\right| dy \\
&\le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{\mathrm{Lip}_\beta(w)}\right) \sum_{l=0}^{\infty} \int_{2^{l+1}\tilde{Q}\setminus 2^l\tilde{Q}} \left(\frac{d}{(2^l d)^{n-1-\delta}} + \frac{d^\varepsilon}{(2^l d)^{n+\varepsilon-2-\delta}} \right) \\
&\quad \times \frac{w(2^{l+1}\tilde{Q})^{1+\beta/n}}{|2^{l+1}\tilde{Q}|} \left[\frac{w(2^{l+1}\tilde{Q})^{1+\beta/n}}{|2^{l+1}\tilde{Q}|} + l w(\tilde{x}) w\left(2^{l+1}\tilde{Q}\right)^{\beta/n} \right]\left|f(y)\right| dy \\
&\le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{\mathrm{Lip}_\beta(w)}\right) \sum_{l=0}^{\infty} w(\tilde{x}) \left(\frac{w(2^{l+1}\tilde{Q})}{|2^{l+1}\tilde{Q}|} \right)^{1+2\beta/n} \\
&\quad \times \left(\frac{1}{|2^{l+1}\tilde{Q}|^{1-s(\delta+2+2\beta)/n}} \int_{2^{l+1}\tilde{Q}} \left|f(y)\right|^s dy \right)^{1/s} l\left(2^{-l} + 2^{-l\varepsilon}\right) \\
&\le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{\mathrm{Lip}_\beta(w)}\right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2+2\beta,s}(f)(\tilde{x}).
\end{aligned}
$$

Similarly,

$$
\left|I_5^{(5)}\right| \le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\left\|D^\alpha b_j\right\|_{\mathrm{Lip}_\beta(w)}\right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2+2\beta,s}(f)(\tilde{x}).
$$

For $I_5^{(6)}$, we get

$$\left|I_5^{(6)}\right| \le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \int_{R^n \setminus \tilde{Q}} \left| \frac{(x-y)^{\alpha_1+\alpha_2}K(x,y)}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1+\alpha_2}K(x_0,y)}{|x_0-y|^m} \right|$$

$$\times \left(\left|D^{\alpha_1}b_1(y) - \left(D^{\alpha_1}b_1\right)_{2^{l+1}\tilde{Q}}\right| + \left|\left(D^{\alpha_1}b_1\right)_{2^{l+1}\tilde{Q}} - \left(D^{\alpha_1}b_1\right)_{\tilde{Q}}\right| \right)$$

$$\times \left(\left|D^{\alpha_2}b_2(y) - \left(D^{\alpha_2}b_2\right)_{2^{l+1}\tilde{Q}}\right| + \left|\left(D^{\alpha_2}b_2\right)_{2^{l+1}\tilde{Q}} - \left(D^{\alpha_2}b_2\right)_{\tilde{Q}}\right| \right)|f(y)|\,dy$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\|D^{\alpha_j}b_j\right\|_{\text{Lip}_\beta(w)} \right) \sum_{l=0}^{\infty} \int_{2^{l+1}\tilde{Q}\setminus 2^l\tilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{n-1-\delta}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-2-\delta}} \right)$$

$$\times \left[\frac{w(2^{l+1}\tilde{Q})^{1+\beta/n}}{|2^{l+1}\tilde{Q}|} + lw(\tilde{x})w(2^{l+1}\tilde{Q})^{\beta/n} \right]^2 |f(y)|\,dy$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\|D^{\alpha_j}b_j\right\|_{\text{Lip}_\beta(w)} \right) \sum_{l=0}^{\infty} \left[w(\tilde{x}) \left(\frac{w(2^{l+1}\tilde{Q})}{|2^{l+1}\tilde{Q}|} \right)^{\beta/n} \right]^2$$

$$\times \left(\frac{1}{|2^{l+1}\tilde{Q}|^{1-s(\delta+2+2\beta)/n}} \int_{2^{l+1}\tilde{Q}} |f(y)|^s\,dy \right)^{1/s} l^2 \left(2^{-l} + 2^{-l\varepsilon}\right)$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\|D^{\alpha_j}b_j\right\|_{\text{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2+2\beta,s}(f)(\tilde{x}).$$

Thus

$$\left|T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0)\right| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \left\|D^\alpha b_j\right\|_{\text{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2+2\beta,s}(f)(\tilde{x})$$

and

$$I_5 \le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \left\|D^{\alpha_j}b_j\right\|_{\text{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2+2\beta,s}(f)(\tilde{x}).$$

This completes the proof of the lemma. $\square$

Proof of Key Lemma 2 Without loss of generality, we may assume $k = 2$. By using the same argument as in the proof of Key Lemma 1, we have

$$\frac{1}{|Q|^{1+2\beta/n}} \int_Q \left|T^b(f)(x) - T^{\tilde{b}}(f_2)(x_0)\right|\,dx$$

$$\le \frac{1}{|Q|^{1+2\beta/n}} \int_Q \left| \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^{m-2}} K(x,y)f_1(y)\,dy \right|\,dx$$

$$+ \frac{C}{|Q|^{1+2\beta/n}} \int_Q \left| \sum_{|\alpha_1|=m_1} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^{m-2}} D^{\alpha_1}\tilde{b}_1(y)K(x,y)f_1(y)\,dy \right|\,dx$$

$$+ \frac{C}{|Q|^{1+2\beta/n}} \int_Q \left| \sum_{|\alpha_2|=m_2} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^{m-2}} D^{\alpha_2}\tilde{b}_2(y)K(x,y)f_1(y)\,dy \right|\,dx$$

$$+ \frac{C}{|Q|^{1+2\beta/n}} \int_Q \left| \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^{m-2}} K(x,y) f_1(y)\, dy \right| dx$$

$$+ \frac{1}{|Q|^{1+2\beta/n}} \int_Q \left| T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0) \right| dx$$

$$:= J_1 + J_2 + J_3 + J_4 + J_5.$$

We obtain

$$J_1 \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) |Q|^{2/n} \left(\frac{w(\tilde{Q})}{|Q|} \right)^{2+2\beta/n} \left(\frac{1}{|Q|} \int_{R^n} |T(f_1)(x)|^t\, dx \right)^{1/t}$$

$$\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) |Q|^{2/n} \left(\frac{w(\tilde{Q})}{|Q|} \right)^{2+2\beta/n} |Q|^{-1/t} \left(\int_{R^n} |f_1(x)|^s\, dx \right)^{1/s}$$

$$\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) \left(\frac{w(\tilde{Q})}{|\tilde{Q}|} \right)^{2+2\beta/n} \left(\frac{1}{|\tilde{Q}|^{1-s(\delta+2)/n}} \int_{\tilde{Q}} |f(y)|^s\, dy \right)^{1/s}$$

$$\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2,s}(f)(\tilde{x}),$$

$$J_2 \leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{\mathrm{Lip}_\beta(w)} \frac{|Q|^{2/n} w(\tilde{Q})^{1+\beta/n}}{|Q|^{1+2\beta/n}} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1}\tilde{b}_1 f_1)(x)|^t\, dx \right)^{1/t}$$

$$\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{\mathrm{Lip}_\beta(w)} \frac{|Q|^{2/n} w(\tilde{Q})^{1+\beta/n}}{|Q|^{1+2\beta/n}} |Q|^{-1/t} \sum_{|\alpha_1|=m_1} \left(\int_{R^n} |D^{\alpha_1}\tilde{b}_1(x)|^s |f_1(x)|^s\, dx \right)^{1/s}$$

$$\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) \left(\frac{w(\tilde{Q})}{|\tilde{Q}|} \right)^{2+2\beta/n} \left(\frac{1}{|\tilde{Q}|^{1-s(\delta+2)/n}} \int_{\tilde{Q}} |f(y)|^s\, dy \right)^{1/s}$$

$$\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2,s}(f)(\tilde{x}),$$

$$J_3 \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2,s}(f)(\tilde{x}),$$

$$J_4 \leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{|Q|^{2/n}}{|Q|^{2\beta/n}} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 f_1)(x)|^t\, dx \right)^{1/t}$$

$$\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{|Q|^{2/n}}{|Q|^{2\beta/n}} |Q|^{-1/t} \left(\int_{R^n} |D^{\alpha_1}\tilde{b}_1(x) D^{\alpha_2}\tilde{b}_2(x)|^s |f_1(x)|^s\, dx \right)^{1/s}$$

$$\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) \left(\frac{w(\tilde{Q})}{|\tilde{Q}|} \right)^{2+2\beta/n} \left(\frac{1}{|\tilde{Q}|^{1-s(\delta+2)/n}} \int_{\tilde{Q}} |f(y)|^s\, dy \right)^{1/s}$$

$$\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{\mathrm{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2,s}(f)(\tilde{x}).$$

For J_5, similar to the proof of I_5 in Key Lemma 1, we obtain

$$\left| T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0) \right|$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \| D^{\alpha_j} b_j \|_{\mathrm{Lip}_\beta(w)} \right) \sum_{l=0}^{\infty} \frac{w(2^{l+1}\tilde{Q})^{2+2\beta/n}}{|2^{l+1}Q|^2}$$

$$\times \int_{2^{l+1}\tilde{Q} \backslash 2^l\tilde{Q}} \left(\frac{d}{(2^l d)^{n-1-\delta}} + \frac{d^\varepsilon}{(2^l d)^{n+\varepsilon-2-\delta}} \right) |f(y)| \, dy$$

$$+ C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \| D^{\alpha} b_j \|_{\mathrm{Lip}_\beta(w)} \right) \sum_{l=0}^{\infty} \frac{w(2^{l+1}Q)^{2+2\beta/n}}{|2^{l+1}Q|^2} \int_{2^{l+1}\tilde{Q} \backslash 2^l\tilde{Q}} \frac{d}{(2^l d)^{n-1-\delta}} |f(y)| \, dy$$

$$+ C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \| D^{\alpha} b_j \|_{\mathrm{Lip}_\beta(w)} \right) \sum_{l=0}^{\infty} \int_{2^{l+1}\tilde{Q} \backslash 2^l\tilde{Q}} \left(\frac{d}{(2^l d)^{n-1-\delta}} + \frac{d^\varepsilon}{(2^l d)^{n+\varepsilon-2-\delta}} \right)$$

$$\times \frac{w(2^{l+1}\tilde{Q})^{1+\beta/n}}{|2^{l+1}\tilde{Q}|} \left[\frac{w(2^{l+1}\tilde{Q})^{1+\beta/n}}{|2^{l+1}\tilde{Q}|} + l w(\tilde{x}) w(2^{l+1}\tilde{Q})^{\beta/n} \right] |f(y)| \, dy$$

$$+ C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \| D^{\alpha_j} b_j \|_{\mathrm{Lip}_\beta(w)} \right) \sum_{l=0}^{\infty} \int_{2^{l+1}\tilde{Q} \backslash 2^l\tilde{Q}} \left(\frac{d}{(2^l d)^{n-1-\delta}} + \frac{d^\varepsilon}{(2^l d)^{n+\varepsilon-2-\delta}} \right)$$

$$\times \left[\frac{w(2^{l+1}\tilde{Q})^{1+\beta/n}}{|2^{l+1}\tilde{Q}|} + l w(\tilde{x}) w(2^{l+1}\tilde{Q})^{\beta/n} \right]^2 |f(y)| \, dy$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \| D^{\alpha_j} b_j \|_{\mathrm{Lip}_\beta(w)} \right) |Q|^{2\beta/n} \sum_{l=0}^{\infty} \left(\frac{w(2^{l+1}\tilde{Q})}{|2^{l+1}Q|} \right)^{2+2\beta/n} \left(2^{l(2\beta-1)} + 2^{l(2\beta-\varepsilon)} \right)$$

$$\times \left(\frac{1}{|2^{l+1}\tilde{Q}|^{1-s(\delta+2)/n}} \int_{2^{l+1}\tilde{Q}} |f(y)|^s \, dy \right)^{1/s}$$

$$+ C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \| D^{\alpha} b_j \|_{\mathrm{Lip}_\beta(w)} \right) |Q|^{2\beta/n} \sum_{l=0}^{\infty} w(\tilde{x}) \left(\frac{w(2^{l+1}\tilde{Q})}{|2^{l+1}\tilde{Q}|} \right)^{1+2\beta/n}$$

$$\times \left(\frac{1}{|2^{l+1}\tilde{Q}|^{1-s(\delta+2)/n}} \int_{2^{l+1}\tilde{Q}} |f(y)|^s \, dy \right)^{1/s} l \left(2^{l(2\beta-1)} + 2^{l(2\beta-\varepsilon)} \right)$$

$$+ C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \| D^{\alpha_j} b_j \|_{\mathrm{Lip}_\beta(w)} \right) |Q|^{2\beta/n} \sum_{l=0}^{\infty} \left[w(\tilde{x}) \left(\frac{w(2^{l+1}\tilde{Q})}{|2^{l+1}\tilde{Q}|} \right)^{\beta/n} \right]^2$$

$$\times \left(\frac{1}{|2^{l+1}\tilde{Q}|^{1-s(\delta+2)/n}} \int_{2^{l+1}\tilde{Q}} |f(y)|^s \, dy \right)^{1/s} l^2 \left(2^{l(2\beta-1)} + 2^{l(2\beta-\varepsilon)} \right)$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \| D^{\alpha_j} b_j \|_{\mathrm{Lip}_\beta(w)} \right) |Q|^{2\beta/n} w(\tilde{x})^{2+2\beta/n} M_{\delta+2,s}(f)(\tilde{x}).$$

Thus

$$J_5 \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \| D^{\alpha_j} b_j \|_{\mathrm{Lip}_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{\delta+2,s}(f)(\tilde{x}).$$

This completes the proof of the lemma. □

Proof of Theorem 1 By Key Lemma 1, we get the sharp function estimate of T^b as follows:

$$\left(T^b(f)\right)^{\#}(\tilde{x}) \le C \prod_{j=1}^{k} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{\mathrm{Lip}_{\beta}(w)} \right) w(\tilde{x})^{k+k\beta/n} M_{\delta+k+k\beta,s}(f)(\tilde{x}).$$

Now, choose $1 < s < p$ in Key Lemma 1, by using Lemma 3 and notice that $w^{1-q(k-(\delta+k)/n)} \in A_{\infty}$ and $w^{1/p} \in A(p,q)$. We get

$$\left\| T^b(f) \right\|_{L^q(w^{1-q(k-(\delta+k)/n)})}$$

$$\le \left\| M\left(T^b(f)\right) \right\|_{L^q(w^{1-q(k-(\delta+k)/n)})}$$

$$\le C \left\| \left(T^b(f)\right)^{\#} \right\|_{L^q(w^{1-q(k-(\delta+k)/n)})}$$

$$\le C \prod_{j=1}^{k} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{\mathrm{Lip}_{\beta}(w)} \right) \left\| M_{\delta+k+k\beta,s}(f) w^{k+k\beta/n} \right\|_{L^q(w^{1-q(k-(\delta+k)/n)})}$$

$$= C \prod_{j=1}^{k} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{\mathrm{Lip}_{\beta}(w)} \right) \left\| M_{\delta+k+k\beta,s}(f) \right\|_{L^q(w^{q/p})}$$

$$\le C \prod_{j=1}^{k} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{\mathrm{Lip}_{\beta}(w)} \right) \|f\|_{L^p(w)}.$$

This completes the proof of the theorem. □

Proof of Theorem 2 Choose $1 < s < p$ in Key Lemma 2, notice that $w^{1-q(k-(\delta+k-k\beta)/n)} \in A_{\infty}$ and $w^{1/p} \in A(p,q)$. By using Lemmas 2 and 3, we obtain

$$\left\| T^b(f) \right\|_{\dot{F}_q^{k\beta,\infty}(w^{1-q(k-(\delta+k-k\beta)/n)})}$$

$$\le C \prod_{j=1}^{k} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{\mathrm{Lip}_{\beta}(w)} \right) \left\| M_{\delta+k,s}(f) w^{k+k\beta/n} \right\|_{L^q(w^{1-q(k-(\delta+k-k\beta)/n)})}$$

$$= C \prod_{j=1}^{k} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{\mathrm{Lip}_{\beta}(w)} \right) \left\| M_{\delta+k,s}(f) \right\|_{L^q(w^{q/p})}$$

$$\le C \prod_{j=1}^{k} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} b_j \right\|_{\mathrm{Lip}_{\beta}(w)} \right) \|f\|_{L^p(w)}.$$

This completes the proof of the theorem. □

4 Applications

In this section we shall apply Theorems 1 and 2 of the paper to some particular operators such as the Calderón-Zygmund singular integral operator and the fractional integral operator.

Application 1 Calderón-Zygmund singular integral operator.

Let T be the Calderón-Zygmund operator (see [17, 18]). The multilinear operator related to T is defined by

$$T^b(f)(x) = \int \frac{R_{m+1}(b; x, y)}{|x - y|^{m-k}} K(x, y) f(y) \, dy.$$

Then Theorem 1 holds for T^b with $\delta = 0$.

Application 2 Fractional integral operator with rough kernel.

For $0 < \delta < n$, let T_δ be the fractional integral operator with rough kernel defined by (see [8, 17, 18])

$$T_\delta f(x) = \int_{R^n} \frac{\Omega(x - y)}{|x - y|^{n-\delta}} f(y) \, dy,$$

which is the fractional integral operator when $\Omega = 1$ (see [8]). The multilinear operator related to T_δ is defined by

$$T_\delta^b f(x) = \int_{R^n} \frac{R_{m+1}(b; x, y)}{|x - y|^{m+n-k-\delta}} \Omega(x - y) f(y) \, dy,$$

where Ω is homogeneous of degree zero on R^n, $\int_{S^{n-1}} \Omega(x') \, d\sigma(x') = 0$ and $\Omega \in \text{Lip}_\varepsilon(S^{n-1})$ for some $0 < \varepsilon \leq 1$, that is, there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq M|x - y|^\varepsilon$. Then Theorem 2 holds for T_δ^b.

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