Mishra et al. *Journal of Inequalities and Applications* 2013, **2013**:300
http://www.journalofinequalitiesandapplications.com/content/2013/1/300

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On trigonometric approximation of $W(L^p, \xi(t))$ ($p \geq 1$) function by product $(C,1)(E,1)$ means of its Fourier series

Vishnu Narayan Mishra[1*], Vaishali Sonavane[1] and Lakshmi Narayan Mishra[2]

Dedicated to Professor Hari M Srivastava

*Correspondence: vishnunarayanmishra@gmail.com
[1]Department of Applied Mathematics and Humanities, Sardar Vallabhbhai National Institute of Technology, Ichchhanath Mahadev Road, Surat, Gujarat 395007, India
Full list of author information is available at the end of the article

Abstract

In the present paper, we generalize a theorem of Lal and Singh (Indian J. Pure Appl. Math. 33(9):1443-1449, 2002) on the degree of approximation of a function belonging to the weighted $W(L^p, \xi(t))$ ($p \geq 1$)-class using product $(C,1)(E,1)$ means of its Fourier series. We have used here the modified definition of the weighted $W(L^p, \xi(t))$ ($p \geq 1$)-class of functions in view of Khan (Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 31:123-127, 1982) and rectified some errors appearing in the paper of Lal and Singh (Indian J. Pure Appl. Math. 33(9):1443-1449, 2002). A few applications of approximation of functions will also be highlighted.

MSC: 40C99; 40G99; 41A10; 42B05; 42B08

Keywords: Fourier series; product $(C,1)(E,1)$ means; $W(L^p, \xi(t))$ ($p \geq 1$)-class; degree of approximation

1 Introduction

Approximation by trigonometric polynomials is at the heart of approximation theory. The most important trigonometric polynomials used in the approximation theory are obtained by linear summation methods of Fourier series of 2π-periodic functions on the real line (*i.e.* Cesàro means, Nörlund means, Euler means and Product Cesàro-Nörlund means, Cesàro-Euler means *etc.*). Much of the advance in the theory of trigonometric approximation is due to the periodicity of the functions. Various investigators such as Khan [1–3], Qureshi [4], Chandra [5], Leindler [6], Mittal *et al.* [7], Mittal, Rhoades and Mishra [8], Mishra [9], Rhoades *et al.* [10] have determined the degree of approximation of 2π-periodic functions belonging to different classes $\mathrm{Lip}\,\alpha$, $\mathrm{Lip}(\alpha, p)$, $\mathrm{Lip}(\xi(t), p)$ and $W(L^p, \xi(t))$ of functions through trigonometric Fourier approximation (TFA) using different summability matrices. Recently, Mittal *et al.* [11] have obtained the degree of approximation of functions belonging to the $\mathrm{Lip}(\alpha, p)$-class by a general summability matrix, which generalizes the results of Chandra [5]. In this paper, we determine the degree of approximation of functions belonging to the $W(L^p, \xi(t))$ ($p \geq 1$)-class by using $(C,1)(E,1)$ means of its Fourier series, which in turn generalizes the result of Lal and Singh [12]. We also note some errors appearing in the paper of Lal and Singh [12] and rectify them in the light of observations of Khan [13].

Let $f(x)$ be a 2π-periodic and Lebesgue integrable function. The Fourier series of $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} u_n(x) \tag{1.1}$$

with nth partial sum $s_n(f;x)$ called trigonometric polynomial of degree (order) n of the first $n+1$ terms of the Fourier series of f.

A function $f(x) \in \operatorname{Lip}\alpha$ if

$$f(x+t) - f(x) = O(|t^\alpha|) \quad \text{for } 0 < \alpha \le 1, t > 0.$$

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$f(x) \in \operatorname{Lip}(\alpha, p)$, for $a \le x \le b$, if

$$\left(\int_a^b |f(x+t) - f(x)|^p \, dx\right)^{1/p} \le A(|t|^\alpha), \quad 0 < \alpha \le 1, p \ge 1, t > 0.$$

A function $f(x) \in \operatorname{Lip}(\alpha, p)$ for $a \le x \le b$ if

$$\left(\int_a^b |f(x+t) - f(x)|^p \, dx\right)^{1/p} \le A(|t|^\alpha)$$

(Definition 5.38 of Mc Fadden [14]). Given a positive increasing function $\xi(t)$ and an integer $p \ge 1$, $f(x) \in \operatorname{Lip}(\xi(t), p)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p \, dx\right)^{1/p} = O(\xi(t)), \quad t > 0.$$

A function $f \in W(L^p, \xi(t))$ [13] if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p \sin^{\beta p}(x/2) \, dx\right)^{1/p} = O(\xi(t)), \quad \beta \ge 0, p \ge 1, t > 0. \tag{1.2}$$

We note that if $\beta = 0$, then the weighted class $W(L^p, \xi(t))$ coincides with the class $\operatorname{Lip}(\xi(t), p)$ and if $\xi(t) = t^\alpha$, then the $\operatorname{Lip}(\xi(t), p)$ class coincides with the class $\operatorname{Lip}(\alpha, p)$. The class $\operatorname{Lip}(\alpha, r) \to \operatorname{Lip}\alpha$ for $r \to \infty$.

Also, we observe that

$$\operatorname{Lip}\alpha \subseteq \operatorname{Lip}(\alpha, p) \subseteq \operatorname{Lip}(\xi(t), p) \subseteq W(L^p, \xi(t)) \quad \text{for } 0 < \alpha \le 1, p \ge 1.$$

The L_p-norm of a function is defined by

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p \, dx\right)^{1/p}, \quad p \ge 1.$$

The L_∞-norm of a function $f : R \to R$ is defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in R\},$$

and the degree of approximation $E_n(f,x)$ is given by Zygmund ([15], p.114)

$$E_n(f,x) = \underset{n}{\operatorname{Min}}\|f(x) - \tau_n(f;x)\|_p, \tag{1.3}$$

in terms of n, where $\tau_n(f;x) = \sum_{k=0}^{n} a_{n,k}s_k(f;x)$ is a trigonometric polynomial of degree n. This method of approximation is called trigonometric Fourier approximation (tfa).

$$\|\tau_n(f,x) - f(x)\|_\infty = \sup_{x\in R}\{|\tau_n(f,x) - f(x)|\}.$$

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of nth partial sums $\{s_n\}$. If

$$E_k^1 = \frac{1}{2^k}\sum_{r=0}^{k}\binom{k}{r}s_r \to s \quad \text{as } n \to \infty, \tag{1.4}$$

then an infinite series $\sum_{n=0}^{\infty} u_n$ with the partial sums s_n is said to be $(E,1)$ summable to the definite number s (Hardy [16]).

An infinite series $\sum_{k=0}^{\infty} u_k$ is said to be $(C,1)$ summable to s if

$$(C,1) = \frac{1}{(n+1)}\sum_{k=0}^{n} s_k \to s \quad \text{as } n \to \infty.$$

The $(C,1)$ transform of the $(E,1)$ transform E_n^1 defines the $(C,1)(E,1)$ transform of the partial sums s_n of the series $\sum_{n=0}^{\infty} u_n$, *i.e.*, the product summability $(C,1)(E,1)$ is obtained by superimposing $(C,1)$ summability on $(E,1)$ summability.

Thus, if

$$(CE)_n^1 = \frac{1}{(n+1)}\sum_{k=0}^{n} E_k^1 = \frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{2^k}\sum_{r=0}^{k}\binom{k}{r}s_r \to s \quad \text{as } n \to \infty, \tag{1.5}$$

where E_n^1 denotes the $(E,1)$ transform of s_n, then the series $\sum_{n=0}^{\infty} u_n$ with the partial sums s_n is said to be summable $(C,1)(E,1)$ to the definite number s, and we can write

$$(CE)_n^1 \to s[(C,1)(E,1)] \quad \text{as } n \to \infty.$$

Therefore, we have

$$\begin{aligned} s_n \to s \quad &\Rightarrow \quad E_n^1(s_n) = \tau_n = \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}s_r \to s \quad \text{as } n \to \infty, E_n^1 \text{ method is regular,} \\ &\Rightarrow \quad C_n^1(E_n^1(s_n)) = C_n^1E_n^1 \to s \quad \text{as } n \to \infty, C_n^1 \text{ method is regular,} \\ &\Rightarrow \quad C_n^1E_n^1 \text{ method is regular.} \end{aligned}$$

We note that E_n^1 and $(CE)_n^1$ are also trigonometric polynomials of degree (or order) n.

The product transform $(C,1)(E,1)$ plays an important role in signal theory as a double digital filter.

The Riemann-Lebesgue theorem states that if $f(x)$ is integrable over (a,b), then as $\lambda \to \infty$, we have $\int_a^b f(x)\cos\lambda x \to 0$ and $\int_a^b f(x)\sin\lambda x \to 0$.

We shall use the following notation throughout the paper:

$$\phi(t) = \phi_x(t) = f(x+t) + f(x-t) - 2f(x).$$

2 Known theorem

Lal and Singh [12] have obtained a theorem on the degree of approximation of a function belonging to the class $\mathrm{Lip}(\xi(t),p)$ by $(C,1)(E,1)$ means of its Fourier series. They proved the following theorem.

Theorem 2.1 *$f: R \to R$ is a 2π-periodic function belonging to* $\mathrm{Lip}(\xi(t),p)$, *then the degree of approximation of f by $(C,1)(E,1)$ means of its Fourier series satisfies*

$$\left\|(CE)_n^1 - f(x)\right\|_p = O\left(\xi\left(\frac{1}{n+1}\right)\cdot(n+1)^{1/p}\right)$$

provided $\xi(t)$satisfy the following conditions:

$$\left\{\int_0^{1/(n+1)}\left(\frac{t\phi(t)}{\xi(t)}\right)^p dt\right\}^{1/p} = O\left(\frac{1}{n+1}\right), \tag{2.1}$$

$$\left\{\int_{1/(n+1)}^{\pi}\left(\frac{t^{-\delta}\phi(t)}{\xi(t)}\right)^p dt\right\}^{1/p} = O\left((n+1)^{\delta}\right), \tag{2.2}$$

where δ is an arbitrary number such that $q(1-\delta)-1>0$, conditions (2.1) *and* (2.2) *hold uniformly in x and $(CE)_n^1$ are $(C,1)(E,1)$ means of series* (1.1).

Remark 1 The proof proceeds by estimating $(CE)_n^1 - f(x)$, which is represented in terms of an integral. The domain of integration is divided into two parts - from $[0,\frac{1}{n+1}]$ and $[\frac{1}{n+1},\pi]$. Referring to the second integral as I_2 and using Hölder's inequality, the authors [12] obtain

$$\begin{aligned} I_2 &\le \frac{1}{n+1}\left\{\int_{1/n+1}^{\pi}\left(\frac{t^{-\delta}\phi(t)}{\xi(t)}\right)^p dt\right\}^{1/p} \\ &\quad\times\left\{\int_{1/n+1}^{\pi}\left(\frac{\xi(t)(1-\cos^{n+1}(t/2)\cos((n+1)t/2))}{t^2 t^{-\delta}}\right)^q dt\right\}^{1/q} \\ &= O\left((n+1)^{\delta-1}\right)O\left\{\int_{1/n+1}^{\pi}\left(\frac{\xi(t)}{t^2 t^{-\delta}}\right)^q dt\right\}^{1/q}. \end{aligned}$$

The authors then make the substitution $y = 1/t$ to obtain

$$= O\left((n+1)^{\delta-1}\right)O\left[\int_{1/\pi}^{n+1}\left(\frac{\xi(1/y)}{y^{-2}y^{\delta}}\right)^q \frac{dy}{y^2}\right]^{1/q}.$$

In the next step $\xi(1/y)$ is removed from the integrand by replacing it with $O(\xi(\frac{1}{n+1}))$. While $\xi(t)$ is an increasing function, $\xi(1/y)$ is now a decreasing function. Therefore, from the second mean value theorem of integrals, this step is invalid.

Remark 2 There is a fatal error in the proof of the main theorem of Lal and Singh [12, p.1447]. In the calculation of I_1, the authors [12] obtain

$$\int_{\epsilon}^{1/(n+1)} \frac{dt}{t^q} = \left[\frac{t^{-q+1}}{-q+1}\right]_{\epsilon}^{1/(n+1)} \quad \text{for some } 0 < \epsilon < \frac{1}{n+1},$$

note that $-q+1 < 0$. Therefore one has $\frac{1}{q-1}[\frac{1}{\epsilon^{q-1}} - (n+1)^{q-1}]$, which need not be $O((n+1)^{q-1})$ since ϵ might be $O(1/n^{\gamma})$ for some $\gamma > 1$.

3 Main result

The observation of Remark 1 motivated us to determine a proper set of conditions to extend Theorem 2.1 on the degree of approximation of functions f of the weighted $W(L^p, \xi(t))$ $(p \geq 1)$-class by product $(C,1)(E,1)$ means of its Fourier series. More precisely, we prove the following.

Theorem 3.1 *If $f : R \to R$ is 2π-periodic, Lebesgue integrable and belonging to the weighted $W(L^p, \xi(t))$ $(p \geq 1)$-class, then the degree of approximation of $f(x)$ by $(C,1)(E,1)$ means of its Fourier series is given by*

$$\left\|(CE)_n^1(f;x) - f(x)\right\|_p = O\left(n^{\beta+1/p}\xi(1/n)\right) \quad \forall n > 0, \tag{3.1}$$

provided a positive increasing function $\xi(t)$ satisfies the following conditions:

$$\left\{\int_0^{\pi/n} \left(\frac{|\phi_x(t)| \sin^{\beta}(t/2)}{\xi(t)}\right)^p dt\right\}^{1/p} = O(1), \tag{3.2}$$

$$\left\{\int_{\pi/n}^{\pi} \left(\frac{t^{-\delta}|\phi_x(t)|}{\xi(t)}\right)^p dt\right\}^{1/p} = O\left(n^{\delta}\right), \tag{3.3}$$

and

$$\frac{\xi(t)}{t} \text{ is non-increasing in } t, \tag{3.4}$$

where δ is an arbitrary number such that $q(\beta - \delta) - 1 > 0$, q is the conjugate index of p, $p^{-1} + q^{-1} = 1$, $1 \leq p \leq \infty$, conditions (3.2), (3.3) *hold uniformly in x, and $(CE)_n^1$ are $(C,1)(E,1)$ means of Fourier series* (1.1).

Note 1 Using condition (3.4), we get $\xi(\frac{\pi}{n}) \leq \pi\xi(\frac{1}{n})$ for $\frac{\pi}{n} \geq \frac{1}{n}$.

Note 2 Conditions (2.1) and (2.2) of Theorem 2.1 and (3.2) and (3.3) of Theorem 3.1 are derived from the theorem of Khan [17].

4 Lemma

For the proof of our theorem, we need the following lemma.

Lemma 4.1 *For $0 \leq t \leq \pi/n$, we have*

$$1 - \cos^n(t/2)\cos(nt/2) = O\left(n^2t^2\right).$$

Proof of Lemma 4.1 For $0 \le t \le \pi/n$, we have

$$\begin{aligned}
1-\cos^n(t/2)\cos(nt/2) &= 1-\left[1-\frac{t^2}{8}+\frac{t^4}{384}-\cdots\right]^n\left[1-\frac{n^2t^2}{8}+\frac{n^4t^4}{384}-\cdots\right] \\
&= 1-\left[1-\frac{nt^2}{8}\right]\left[1-\frac{n^2t^2}{8}\right] \\
&= 1-\left[1-\frac{n^2t^2}{8}-\frac{nt^2}{8}+\frac{n^3t^4}{64}\right] \\
&= \frac{n^2t^2}{8}\left(1+\frac{1}{n}\right) = O\left(n^2t^2\right).
\end{aligned}$$

This completes the proof of Lemma 4.1. □

5 Proof of Theorem 3.1

Following Titchmarsh [18, p.403] and using the Riemann-Lebesgue theorem, the nth partial sum s_n of Fourier series (1.1) at $t = x$ may be written as

$$s_n(f;x) - f(x) = \frac{1}{\pi}\int_0^\pi \frac{\phi_x(t)}{t}\sin nt\,dt + O(1),$$

so that the $(E,1)$ transform E_n^1 of $s_n(f,x)$ is given by

$$\begin{aligned}
E_n^1(f;x) - f(x) &= \frac{1}{2^n\pi}\int_0^\pi \frac{\phi_x(t)}{t}\left\{\sum_{k=1}^n \binom{n}{k}\sin kt\right\}dt \\
&= \frac{1}{2^n\pi}\int_0^\pi \frac{\phi_x(t)}{t}\,\text{I.P. of}\left\{\sum_{k=1}^n \binom{n}{k}e^{ikt}\right\}dt \\
&= \frac{1}{\pi}\int_0^\pi \frac{\phi_x(t)}{t}\cos^n(t/2)\sin(nt/2)\,dt.
\end{aligned}$$

Now, the $(C,1)(E,1)$ transform of $s_n(f,x)$ is given by

$$\begin{aligned}
(CE)_n^1(f;x)7 &= \frac{1}{n}\sum_{k=1}^n E_k^1 \quad (n = 1,2,\ldots) \\
&= f(x) + \frac{1}{n\pi}\int_0^\pi \frac{\phi_x(t)}{t}\left\{\sum_{k=1}^n \cos^k(t/2)\sin(kt/2)\right\}dt.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(CE)_n^1(f;x) - f(x) &= \frac{1}{n\pi}\int_0^\pi \frac{\phi_x(t)}{t}\,\text{I.P. of}\left\{\sum_{k=1}^n \cos^k(t/2)e^{ikt/2}\right\}dt \\
&= \frac{1}{n\pi}\int_0^\pi \frac{\phi_x(t)}{t}\,\text{I.P. of}\left\{\frac{e^{it/2}\cos(t/2)\{1-(e^{it/2}\cos(t/2))^n\}}{1-e^{it/2}\cos(t/2)}\right\}dt \\
&= \frac{1}{n\pi}\int_0^\pi \frac{\phi_x(t)}{t}\,\text{I.P. of}\left\{\left(\cos(t/2)\left(e^{it/2}-\cos t/2-e^{i(n+1)t/2}\cos^n t/2\right.\right.\right. \\
&\quad \left.\left.\left.+\,e^{int/2}\cos^{n+1} t/2\right)\right)/\left(\sin^2(t/2)\right)\right\}dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n\pi}\int_0^{\pi} \frac{\phi_x(t)}{t}\left\{\frac{\cos(t/2)\sin(t/2)(1-\cos^n(t/2)\cos(nt/2))}{\sin^2(t/2)}\right\}dt \\
&= \frac{1}{n\pi}\int_0^{\pi} \frac{\phi_x(t)}{t\tan(t/2)}\bigl(1-\cos^n(t/2)\cos(nt/2)\bigr)\,dt \\
&\le \frac{1}{n}\int_0^{\pi} \frac{\phi_x(t)}{t^2}\bigl(1-\cos^n(t/2)\cos(nt/2)\bigr)\,dt \\
&= \frac{1}{n}\left[\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right]\int_0^{\pi} \frac{\phi_x(t)}{t^2}\bigl(1-\cos^n(t/2)\cos(nt/2)\bigr)\,dt \\
&= I_1 + I_2, \quad \text{say.}
\end{aligned}
\tag{5.1}
$$

Clearly,

$$
\bigl|\phi(x+t)-\phi(x)\bigr| \le \bigl|f(u+x+t)-f(u+x)\bigr| + \bigl|f(u-x-t)-f(u-x)\bigr|.
$$

Hence, by Minkowski's inequality,

$$
\begin{aligned}
&\left\{\int_0^{2\pi}\bigl|\bigl(\phi(x+t)-\phi(x)\bigr)\sin^{\beta}(x/2)\bigr|^p\,dx\right\}^{1/p} \\
&\quad\le \left\{\int_0^{2\pi}\bigl|\bigl(f(u+x+t)-f(u+x)\bigr)\sin^{\beta}(x/2)\bigr|^p\,dx\right\}^{1/p} \\
&\qquad + \left\{\int_0^{2\pi}\bigl|\bigl(f(u-x-t)-f(u-x)\bigr)\sin^{\beta}(x/2)\bigr|^p\,dx\right\}^{1/p} \\
&\quad= O\bigl(\xi(t)\bigr).
\end{aligned}
$$

Then $f \in W(L^p,\xi(t)) \Rightarrow \phi \in W(L^p,\xi(t))$.

Using Hölder's inequality, the fact that $\phi(t) \in W(L^p,\xi(t))$, condition (3.2), $(\sin t/2)^{-1} \le \pi/t$, for $0 < t \le \pi$, $p^{-1}+q^{-1}=1$, $1 \le p \le \infty$, Lemma 4.1, Note 1 and the second mean value theorem for integrals, we have

$$
\begin{aligned}
|I_1| &\le \frac{1}{n}\left[\int_0^{\pi/n}\left(\frac{|\phi_x(t)|\sin^{\beta}(t/2)}{\xi(t)}\right)^p dt\right]^{1/p} \\
&\quad\times\left[\int_0^{\pi/(n+1)}\left(\frac{\xi(t)(1-\cos^n(t/2)\cos(nt/2))}{t^2\sin^{\beta}(t/2)}\right)^q dt\right]^{1/q} \\
&= O\left(\frac{1}{n}\right)\left[\int_0^{\pi/n}\left(\frac{\xi(t)n^2t^2}{t^2\sin^{\beta}(t/2)}\right)^q dt\right]^{1/q} \\
&= O\left\{n\xi\left(\frac{\pi}{n}\right)\right\}\left\{\lim_{h\to 0}\int_h^{\pi/n} t^{-\beta q}\,dt\right\}^{1/q} \\
&= O\left\{n\xi\left(\frac{1}{n}\right)\right\}\left\{\left(\frac{t^{-\beta q+1}}{-\beta q+1}\right)_h^{\pi/n}\right\}^{1/q}, \quad h\to 0, \\
&= O\left\{n\xi\left(\frac{1}{n}\right)\right\}O\bigl(n^{\beta-1/q}\bigr) = O\left\{n^{\beta+1-1/q}\xi\left(\frac{\pi}{n}\right)\right\} = O\left\{n^{\beta+1/p}\xi\left(\frac{\pi}{n}\right)\right\}.
\end{aligned}
\tag{5.2}
$$

Again applying Hölder's inequality, $|\sin(t/2)| \le 1$, $(\sin t/2)^{-1} \le \pi/t$, for $0 < t \le \pi$, conditions (3.3), (3.4), Note 1 and the second mean value theorem for integrals, we have

$$\begin{aligned}
|I_2| &\le \frac{1}{n}\left[\int_{\pi/n}^{\pi}\left(\frac{t^{-\delta}|\phi_x(t)|\sin^{\beta}(t/2)}{\xi(t)}\right)^p dt\right]^{1/p} \\
&\quad \times \left[\int_{\pi/n}^{\pi}\left(\frac{\xi(t)(1-\cos^n(t/2)\cos(nt/2))}{t^2 t^{-\delta}\sin^{\beta}(t/2)}\right)^q dt\right]^{1/q} \\
&\le \frac{1}{n}\left[\int_{\pi/n}^{\pi}\left(\frac{t^{-\delta}|\phi_x(t)|}{\xi(t)}\right)^p dt\right]^{1/p}\left[\int_{\pi/n}^{\pi}\left(\frac{\xi(t)}{t^{2-\delta}\sin^{\beta}(t/2)}\right)^q dt\right]^{1/q} \\
&= O\big(n^{\delta-1}\big)\left[\int_{\pi/n}^{\pi}\left(\frac{\xi(t)}{t^{2-\delta+\beta}}\right)^q dt\right]^{1/q} = O\big(n^{\delta-1}\big)\left[\int_{1/\pi}^{n/\pi}\left(\frac{\xi(1/y)}{y^{\delta-\beta-2}}\right)^q \frac{dy}{y^2}\right]^{1/q} \\
&= O\left\{n^{\delta-1}\left(\frac{n}{\pi}\right)\xi\left(\frac{\pi}{n}\right)\right\}\left\{\int_{\epsilon_1}^{n/\pi} y^{(\beta-\delta+1)q-2}\,dy\right\}^{1/q} \quad \text{for some } 1/\pi < \epsilon_1 < n/\pi, \\
&= O\left(n^{\delta}\xi\left(\frac{1}{n}\right)\right)\left(\frac{n^{(\beta-\delta+1)q-1}-(\epsilon_1)^{(\beta-\delta+1)q-1}}{(\beta-\delta+1)q-1}\right)^{1/q} \\
&= O\left(n^{\delta}\xi\left(\frac{1}{n}\right)n^{\beta-\delta+1-1/q}\right) = O\left(n^{\beta+1/p}\xi\left(\frac{1}{n}\right)\right),
\end{aligned} \tag{5.3}$$

in view of increasing nature of $y\xi(1/y)$, $p^{-1}+q^{-1}=1$, $1 \le p \le \infty$, where ϵ_1 lie in $[\pi^{-1}, n\pi^{-1}]$.

Collecting (5.1)-(5.3), we get

$$\left|(CE)_n^1(f;x) - f(x)\right| = O\left(n^{\beta+1/p}\xi\left(\frac{1}{n}\right)\right).$$

Now, using the L_p-norm of a function, we get

$$\begin{aligned}
\left\|(CE)_n^1(f;x) - f(x)\right\|_p &= \left\{\int_0^{2\pi}\left|(CE)_n^1(f;x) - f(x)\right|^p dx\right\}^{1/p} \\
&= O\left(\int_0^{2\pi}\left(n^{\beta+1/p}\xi\left(\frac{1}{n}\right)\right)^p dx\right)^{1/p} \\
&= O\left(n^{\beta+1/p}\xi\left(\frac{1}{n}\right)\left(\int_0^{2\pi} dx\right)^{1/p}\right) = O\left(n^{\beta+1/p}\xi\left(\frac{1}{n}\right)\right).
\end{aligned}$$

This completes the proof of Theorem 3.1.

6 Corollaries and example

The following corollaries can be derived from Theorem 3.1.

Corollary 1 *If $\beta = 0$, then the generalized weighted $W(L^p, \xi(t))$ $(p \ge 1)$-class reduces to the class $\mathrm{Lip}(\xi(t), p)$, and the degree of approximation of a function $f(x) \in \mathrm{Lip}(\xi(t), p)$ is given by*

$$\left|(CE)_n^1(f;x) - f(x)\right| = O\big(n^{1/p}\xi(1/n)\big).$$

Proof The result follows by setting $\beta = 0$ in (3.1), we have

$$\|(CE)_n^1(f;x) - f(x)\|_p = \left\{\int_0^{2\pi} |(CE)_n^1(f;x) - f(x)|^p\,dx\right\}^{1/p} = O(n^{1/p}\xi(1/n)), \quad p \geq 1.$$

Thus, we get

$$|(CE)_n^1(f;x) - f(x)| \leq \left\{\int_0^{2\pi} |(CE)_n^1(f;x) - f(x)|^p\,dx\right\}^{1/p} = O(n^{1/p}\xi(1/n)), \quad p \geq 1.$$

This completes the proof of Corollary 1. □

Corollary 2 *If* $\beta = 0$, $\xi(t) = t^\alpha$, $0 < \alpha \leq 1$, *then the weighted* $W(L^p, \xi(t))$ $(p \geq 1)$*-class reduces to the class* $\operatorname{Lip}(\alpha, p)$, $(1/p) < \alpha < 1$ *and the degree of approximation of a* 2π*-periodic function* f *belonging to the class* $\operatorname{Lip}(\alpha, p)$ *is given by*

$$|(CE)_n^1(f;x) - f(x)| = O\left(\frac{1}{n^{\alpha - 1/p}}\right).$$

Proof Putting $\beta = 0$, $\xi(t) = t^\alpha$, $0 < \alpha \leq 1$ in Theorem 3.1, we have

$$\|(CE)_n^1(f;x) - f(x)\|_p = \left\{\int_0^{2\pi} |(CE)_n^1(f;x) - f(x)|^p\,dx\right\}^{1/p}$$

or

$$O\left(n^{\beta+1/p}\xi\left(\frac{1}{n}\right)\right) = \left\{\int_0^{2\pi} |(CE)_n^1(f;x) - f(x)|^p\,dx\right\}^{1/p},$$

or

$$O(1) = \left\{\int_0^{2\pi} |(CE)_n^1(f;x) - f(x)|^p\,dx\right\}^{1/p} O\left(\frac{1}{n^{\beta+1/p}\xi(\frac{1}{n})}\right),$$

since otherwise the right-hand side of the above equation will not be $O(1)$.

Hence

$$|(CE)_n^1(f;x) - f(x)| = O\left(\left(\frac{1}{n}\right)^\alpha n^{1/p}\right) = O\left(\frac{1}{n^{\alpha - 1/p}}\right).$$

This completes the proof of Corollary 2. □

Corollary 3 *If* $\beta = 0$, $\xi(t) = t^\alpha$ *for* $0 < \alpha < 1$ *and* $p \to \infty$ *in* (3.1), *then* $f \in \operatorname{Lip}\alpha$. *In this case, the degree of approximation of the function* $f \in \operatorname{Lip}\alpha$ $(0 < \alpha < 1)$ *class is given by*

$$|(CE)_n^1(f;x) - f(x)| = O(n^{-\alpha}).$$

Proof For $p \to \infty$ in Corollary 2, we get

$$\|(CE)_n^1(f;x) - f(x)\|_\infty = \sup_{0 \leq x \leq 2\pi} |(CE)_n^1(f;x) - f(x)| = O(n^{-\alpha}).$$

Thus, we have

$$\begin{aligned}\left|(CE)_n^1(f;x)-f(x)\right| &\le \left\|(CE)_n^1(f;x)-f(x)\right\|_\infty \\ &= \sup_{0\le x\le 2\pi}\left|(CE)_n^1(f;x)-f(x)\right| = O\left(n^{-\alpha}\right).\end{aligned}$$

This completes the proof of Corollary 3. □

Examples (i) Consider the infinite series

$$1-4\sum_{n=1}^{\infty}(-3)^{n-1}. \tag{6.1}$$

The nth partial sum of (6.1) is given by

$$s_n = 1-4\sum_{k=1}^{n}(-3)^{k-1} = (-3)^n.$$

Since $\lim_{n\to\infty} s_n$ does not exist, therefore the series (6.1) is not convergent and so

$$E_n^1 = 2^{-n}\sum_{k=0}^{n}\binom{n}{k}s_k = 2^{-n}\sum_{k=0}^{n}\binom{n}{k}(-3)^k = 2^{-n}\left[1+(-3)\right]^n = (-1)^n.$$

Also, $\lim_{n\to\infty} E_n^1$ does not exist. Therefore the series (6.1) is not $(E,1)$ summable.

Now,

$$\begin{aligned}\sigma_n^1 &= \frac{1}{n+1}\sum_{k=0}^{n}s_k = \frac{1}{n+1}\sum_{k=0}^{n}(-3)^k = \frac{1}{n+1}\,\frac{1\{1-(-3)^{n+1}\}}{1+3} \\ &= \frac{1}{4(n+1)}\left\{1-(-1)^n 3^{n+1}\right\}.\end{aligned}$$

Here $\lim_{n\to\infty}\sigma_n^1$ does not exist, the series (6.1) is not $(C,1)$ summable.

Finally,

$$(CE)_n^1 = \frac{1}{n+1}\sum_{\nu=0}^{n}E_\nu^1 = \frac{1}{n+1}\sum_{\nu=0}^{n}(-1)^\nu,$$

$$(CE)_n^1 \to 0 \quad \text{as } n\to\infty,$$

the series (6.1) is $(C,1)(E,1)$ summable.

Therefore the series (6.1) is neither $(C,1)$ summable nor $(E,1)$ summable. But it is $(C,1)(E,1)$ summable to 0. Therefore the product summability $(C,1)(E,1)$ is more powerful than $(C,1)$ and $(E,1)$. Consequently, $(C,1)(E,1)$ gives better approximation than the individual methods $(C,1)$ and $(E,1)$.

(ii) If we take the sequence $\{a^{2n}\}$for $a<0$, then there are two cases:

(a) If $-1<a<0$, then the sequence is already convergent and so it is $(C,1)$ summable.

(b) If $a<-1$, then the sequence $\{a^{2n}\}=\{a^2\}^n$ is not $(C,1)$ summable but $(C,1)(E,1)$ summable.

(iii) It is a well-known result that a Hausdorff matrix is a Nörlund matrix if and only if it is a Cesàro matrix. Cesàro means and Euler means both are Hausdorff means, but they are not comparable. Two matrices are called comparable if either they are equivalent (that is, they sum the same set of sequences), or one method is stronger than the other (that is, it has the larger convergence domain). Hardy [16], in his book on Divergent Series, showed that $(C,1)$ and $(E,1)$ methods are not comparable.

Note that the sequence $\{(-1)^{k-1}\sqrt{k}\}$ is $(C,1)$ summable but not bounded, whereas the sequence $x = \{x_k\}$ given by $x_1 = 1$, $x_2 = 0$ and

$$x_k = \begin{cases} 1, & \text{if } 2^{i-1} < k \leq 3(2^{i-2}) \ (i = 2,3,\ldots); \\ 0, & \text{otherwise}, \end{cases}$$

is bounded but not $(C,1)$ summable.

7 Conclusion

Several results concerning the degree of approximation of periodic functions belonging to the generalized weighted $W(L^p,\xi(t))$ $(p \geq 1)$-class by product $(C,1)(E,1)$ means of its Fourier series have been reviewed. Further, a proper set of conditions have been discussed to rectify the errors pointed out in Remarks 1 and 2. The theorem of this paper is an attempt to formulate the problem of approximation of the function $f \in W(L^p,\xi(t))$ $(p \geq 1)$ through trigonometric polynomials generated by the product summability $(C,1)(E,1)$ transform of the Fourier series of f in a simpler manner. The product summability $(C,1)(E,1)$ used in this paper plays an important role in signal theory as a double digital filter and the theory of machines in mechanical engineering.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

LNM computed lemmas and established the main theorem in this direction with appropriate Examples 1 and 2 as well as interesting corollaries. LNM and VNM conceived of the study and participated in its design and coordination. VNM, VS and LNM contributed equally and significantly to this work. All the authors drafted the manuscript, read and approved the final version of manuscript.

Author details

[1]Department of Applied Mathematics and Humanities, Sardar Vallabhbhai National Institute of Technology, Ichchhanath Mahadev Road, Surat, Gujarat 395007, India. [2]L. 1627 Awadh Puri Colony Beniganj, Phase - III, Opp. - I.T.I., Ayodhya Main Road, Faizabad, Uttar Pradesh 224 001, India.

Acknowledgements

The authors would like to thank the anonymous referees for several useful interesting comments and suggestions about the paper. Special thanks are to Professor Hari Mohan Srivastava for kind cooperation, smooth behavior during communication and for his efforts to send the reports of the manuscript timely. This research work is supported by CPDA, SVNIT, Surat (Gujarat), India.

Received: 18 January 2013 Accepted: 28 May 2013 Published: 25 June 2013

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doi:10.1186/1029-242X-2013-300
Cite this article as: Mishra et al.: **On trigonometric approximation of** $W(L^p, \xi(t))$ $(p \geq 1)$ **function by product** $(C, 1)(E, 1)$ **means of its Fourier series.** *Journal of Inequalities and Applications* 2013 **2013**:300.