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Razumikhin-type theorem and mean square asymptotic behavior of the backward Euler method for neutral stochastic pantograph equations

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Abstract

The Razumikhin-type stability theorems of neutral stochastic functional differential equations (NFDEs) were investigated by several authors, but there was almost no Razumikhin-type theorems on the general asymptotic stability of NFDEs with infinite delay. This paper investigates the Razumikhin-type pth moment asymptotic stability of neutral stochastic pantograph equations (NSPEs). Sufficient conditions of the pth moment asymptotic stability for NSPEs are obtained. The NSPE is a special class of NFDEs with infinite delay. We should emphasize that the Razumikhin-type theorem of this paper is established without taking difficulties from infinite delay into account. We also develop the backward Euler method for NSPEs. We prove that the backward Euler method can preserve the asymptotic behavior of the mean square stability of exact solutions under suitable conditions. Numerical examples are demonstrated for illustration.

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Keywords: neutral stochastic pantograph equations; Razumikhin-type theorem; pth moment asymptotic stability; backward Euler method; mean square stability

1 Introduction

Neutral stochastic functional differential equations (NSFDEs) can be used to model various phenomena and processes in the field of the chemical-engineering and aero elasticity. Stability analysis of NSFDEs has attracted increasing attention; for instance, see [1–5]. Among the methods contributed to the study of stability for NSFDEs, the Razumikhin technique is a powerful and effective method. In this paper, we consider the following neutral stochastic pantograph equation (NSPE):

$$d\big[x(t) - N\big(x(qt)\big)\big] = f\big(t, x(t), x(qt)\big)\,dt + g\big(t, x(t), x(qt)\big)\,dB(t), \qquad (1)$$

where $0 < q < 1$. NSPE (1) is an important extension of stochastic pantograph equations (see [6, 7]), and NSPE (1) stems from neutral pantograph equations (NPEs). NPEs play important roles in mathematical and industrial problems (see [8]). Many authors have studied NPEs numerically and analytically. We refer the reader to [8–14]. One class of

NPEs reads

$$\left[x(t) - N\bigl(x(qt)\bigr)\right]' = f\bigl(t, x(t), x(qt)\bigr). \tag{2}$$

Taking the environmental disturbances into account, we are led to NSPE (1). The almost sure asymptotic stability of Eq. (1) has been studied (please see [15]). In this paper, we aim to study the pth moment asymptotic stability of Eq. (1) by using the Razumikhin technique.

The Razumikhin technique has been successfully applied to study the stability of various NSFDEs (see, *e.g.*, [1–3, 16–20]). Recently, Wu *et al.* [17] studied the ψ^γ stability of the following NSFDE with unbounded delay:

$$d\bigl[x(t) - N(x_t)\bigr] = f(t, x_t)\, dt + g(t, x_t)\, dB(t) \tag{3}$$

by using the Razumikhin technique, where $x_t = \{x(t + \theta) : -\infty < \theta \le 0\}$, and some techniques are adopted to overcome the difficulties from infinite delay. Note that Eq. (1) is also a class of NSFDEs with time delay $\tau(t) = (1 - q)t \uparrow \infty$ as $t \to \infty$. For given $t > 0$, it is easy to see that the solution $x(t)$ of Eq. (1) depends on states of $[qt, t]$, but the solution $x(t)$ of Eq. (3) depends on states of $(-\infty, t]$. Obviously, the ψ^γ Razumikhin-type stability theorem of Eq. (3) obtained in [17] can be applied to Eq. (1). However, the general asymptotic stability of Eq. (3), one of important stability criteria, has not been considered. In many cases, the general asymptotic stability of the equilibrium of systems is much more economically and practically feasible in contrast to the exponential stability and general decay ψ^γ stability. Moreover, to the best of the authors' knowledge, so far no result has been concerned with Razumikhin-type theorems on the general asymptotic stability of Eq. (1).

In this paper, the Razumikhin-type pth moment asymptotic stability of Eq. (1) is established without taking difficulties from infinite delay into account. On the other hand, as most NSFDEs could not be solved explicitly, numerical solutions have become essential. Efficient numerical methods for various NSFDEs can be found in [21–23]. In this paper, we develop the backward Euler method for Eq. (1). We show that the backward Euler method can preserve the asymptotic behavior of the mean square stability of exact solutions to Eq. (1).

The paper is organized as follows. In Section 2, we introduce some necessary notations and definitions. In Section 3, we establish the Razumikhin-type theorem on the pth moment asymptotic stability for Eq. (1). In Section 4, we study the mean square asymptotic stability of the backward Euler method for Eq. (1). Numerical experiments are presented in the finial section.

2 Preliminaries

Throughout this paper, unless otherwise specified, we use the following notations. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge 0}, P)$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t \ge 0}$ satisfying the usual conditions (*i.e.*, it is right continuous and $\mathcal{F}_0$ contains all P-null sets). $B(t)$ is an r-dimensional Brownian motion defined on the probability space. $|\cdot|$ denotes the Euclidean norm in R^n. The inner product of x, y in R^n is denoted by $\langle x, y \rangle$ or $x^\top y$. If A is a vector or matrix, its transpose is denoted by $A^\top$. If A is a matrix, its norm $\|A\|$ is defined by $\|A\| = \sup\{|Ax| : |x| = 1\}$. Let $t \ge 0$ and $C([qt, t]; R^n)$ denote the family of all continuous functions ξ from $[qt, t]$ to R^n with the norm $\|\xi\| = \sup_{qt \le \theta \le t} |\xi(\theta)|$. For $p > 0$, de-

note by $L^p_{\mathcal{F}_t}([qt,t];R^n)$ the family of all $\mathcal{F}_t$-measurable, $C([qt,t];R^n)$-value random variables $\xi = \{\xi(\theta) : qt \leq \theta \leq t\}$ with $E\|\xi\|^p < \infty$.

Consider an n-dimensional neutral stochastic pantograph equation

$$d\big[x(t) - N\big(x(qt)\big)\big] = f\big(t, x(t), x(qt)\big)\,dt + g\big(t, x(t), x(qt)\big)\,dB(t), \quad t \geq t_0, \qquad (4)$$

with initial data $\{x(t) : qt_0 \leq t \leq t_0\} = \xi \in L^p_{\mathcal{F}_{t_0}}([qt_0,t_0];R^n)$. Here $0 < q < 1$, $f : [t_0,\infty) \times R^n \times R^n \to R^n$, $g : [t_0,\infty) \times R^n \times R^n \to R^{n \times r}$ and $N : R^n \to R^n$ are all continuous functions.

As usual, thought this paper, we assume that Eq. (4) has a unique global solution $x(t;\xi)$ on $t \geq qt_0$ with $E(\sup_{qt_0 \leq t < \infty} |x(t;\xi)|^p) < \infty$. Moreover, we assume that $f(t,0,0) = 0$, $g(t,0,0) = 0$, $N(0) = 0$ for the stability purpose, so that Eq. (4) admits a trivial solution $x(t;0) = 0$.

The following assumption and definition will be used in the following sections.

Assumption (H) Assume that there is a constant $\kappa \in (0,1)$ such that

$$\big|N(x) - N(y)\big| \leq \kappa|x - y|, \quad \forall x, y \in R^n.$$

Definition 2.1 The trivial solution of Eq. (4) is said to be:

(1) pth moment stable if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$E\big|x(t;\xi)\big|^p \leq \varepsilon, \quad \forall t \geq t_0,$$

whenever $E\|\xi\|^p < \delta$;

(2) pth moment asymptotically stable if it is pth moment stable and there exists a $\delta > 0$ such that $E\|\xi\|^p < \delta$ implies

$$\lim_{t \to \infty} E\big|x(t;\xi)\big|^p = 0;$$

(3) globally pth moment asymptotically stable if it is pth moment stable and, moreover, for all initial data $\xi \in L^p_{\mathcal{F}_{t_0}}([qt_0,t_0];R^n)$,

$$\lim_{t \to \infty} E\big|x(t;\xi)\big|^p = 0.$$

To study the stability of Eq. (4), we need to introduce some other notations. Let $C^{1,2}([qt_0,\infty) \times R^n; R_+)$ denote the family of all nonnegative functions $V(t,x)$ on $[qt_0,\infty)$, which are once differentiable in t and twice differentiable in x. Define an operator LV, associated with Eq. (4), acting on $V(t,x) \in C^{1,2}([qt_0,\infty) \times R^n; R_+)$ by

$$LV(t,x,y) = V_t\big(t, x - N(y)\big) + V_x\big(t, x - N(y)\big)f(t,x,y)$$

$$+ \frac{1}{2}\,\mathrm{trace}\big[g^\top(t,x,y)V_{xx}\big(t, x - N(y)\big)g(t,x,y)\big]$$

for $t \geq t_0$ and $x, y \in R^n$, where

$$V_t = \frac{\partial V(t,x)}{\partial t}, \qquad V_x = \left(\frac{\partial V(t,x)}{\partial x_1}, \frac{\partial V(t,x)}{\partial x_2}, \ldots, \frac{\partial V(t,x)}{\partial x_n}\right), \qquad V_{xx} = \left(\frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}\right)_{n \times n}.$$

The following inequality (see Lemma 4.1 of chapter 6 in [24]) will be used in this paper:
Let $p \geq 1$, $a, b \in R^n$. Then for any $\varepsilon > 0$,

$$|a + b|^p \leq (1 + \varepsilon)^{p-1}\left(|a|^p + \varepsilon^{1-p}|b|^p\right). \tag{5}$$

3 Razumikhin-type theorem

In this section, we investigate the pth moment asymptotic stability of Eq. (4) by using the Razumikhin-type technique.

Theorem 3.1 *Let $p \geq 1$, $c_2 \geq c_1 > 0$, $\lambda > 0$ and Assumption* (H) *hold. Assume that there exists a function $V(t,x) \in C^{1,2}([qt_0, \infty) \times R^n; R_+)$ such that*

$$c_1|x|^p \leq V(t,x) \leq c_2|x|^p, \quad \forall(t,x) \in [qt_0, \infty) \times R^n \tag{6}$$

and, moreover,

$$ELV\big(t, \varphi(t), \varphi(qt)\big) \leq -\lambda EV\big(t, \varphi(t) - N(\varphi(qt))\big), \quad t \geq t_0, \tag{7}$$

for all $\varphi \in L_{\mathcal{F}_t}^p([qt, t]; R^n)$ satisfying

$$EV\big(qt, \varphi(qt)\big) < \bar{p}EV\big(t, \varphi(t) - N(\varphi(qt))\big),$$

where $\bar{p} > \frac{c_2}{c_1}(1 - \kappa)^{-p}$. Then the trivial solution of Eq. (4) is globally pth moment asymptotically stable.

Proof For any given initial data $\xi \in L_{\mathcal{F}_{t_0}}^p([qt_0, t_0]; R^n)$, let $x(t) = x(t; \xi)$ be the solution of the system (4).

(1) For any given $\varepsilon > 0$, we will prove that there exists a $\delta > 0$ such that $E\|\xi\|^p < \delta$ implies $E|x(t)|^p \leq \varepsilon$ for $t \geq t_0$. Let $\delta = \frac{c_1(1-\kappa)^p}{c_2(1+\kappa)^p}\varepsilon$. Using the inequality (5) with $\varepsilon = \kappa$ and Assumption (H), we have

$$E\big|x(t_0) - N\big(x(qt_0)\big)\big|^p \leq (1+\kappa)^{p-1}\big(E\big|x(t_0)\big|^p + \kappa^{1-p}E\big|N\big(x(qt_0)\big)\big|^p\big)$$
$$\leq (1+\kappa)^{p-1}\big(E\big|x(t_0)\big|^p + \kappa E\big|x(qt_0)\big|^p\big)$$
$$< (1+\kappa)^{p-1}(\delta + \kappa\delta)$$
$$= (1+\kappa)^p\delta.$$

Since $E|x(t) - N(x(qt))|^p$ is continuous, there exists a $T > t_0$ such that

$$E\big|x(t) - N\big(x(qt)\big)\big|^p \leq (1+\kappa)^p\delta \quad \text{for } t_0 \leq t \leq T.$$

Then, by (6),

$$EV\big(t, x(t) - N\big(x(qt)\big)\big) \leq c_2(1+\kappa)^p\delta \quad \text{for } t_0 \leq t \leq T.$$

We assert that

$$EV\big(t, x(t) - N\big(x(qt)\big)\big) \leq c_2(1+\kappa)^p\delta \quad \text{for } t \geq t_0. \tag{8}$$

We will prove (8) by contradiction. Suppose that there exists a $t^* > T$ such that

$$EV\left(t^*, x\left(t^*\right) - N\left(x\left(qt^*\right)\right)\right) > c_2(1 + \kappa)^p \delta.$$

Set $\bar{t} = \max\{t : EV(s, x(s) - N(x(qs))) \leq c_2(1 + \kappa)^p \delta, s \in [t_0, t]\}$. Then there exists a sufficiently small $h > 0$ such that

$$
\begin{aligned}
& qt \leq \bar{t} \quad \text{for } t \in (\bar{t}, \bar{t} + h], \\
& EV\left(t, x(t) - N\left(x(qt)\right)\right) > c_2(1 + \kappa)^p \delta \quad \text{for } t \in (\bar{t}, \bar{t} + h].
\end{aligned}
\tag{9}
$$

Using the inequality (5) with $\varepsilon = \frac{\kappa}{1-\kappa}$ and Assumption (H), we obtain that

$$
\begin{aligned}
E\big|x(t)\big|^p &\leq \frac{E|x(t) - N(x(qt))|^p}{(1 - \kappa)^{p-1}} + \frac{E|N(x(qt))|^p}{\kappa^{p-1}} \\
&\leq \frac{E|x(t) - N(x(qt))|^p}{(1 - \kappa)^{p-1}} + \kappa E\big|x(qt)\big|^p.
\end{aligned}
\tag{10}
$$

Using (6) and (10), we have

$$
\begin{aligned}
E\big|x(t)\big|^p &\leq \frac{\sup_{t_0 \leq s \leq \bar{t}} E|x(s) - N(x(qs))|^p}{(1 - \kappa)^{p-1}} + \kappa \sup_{qt_0 \leq s \leq \bar{t}} E\big|x(s)\big|^p \\
&\leq \frac{\sup_{t_0 \leq s \leq \bar{t}} EV(s, x(s) - N(x(qs)))}{c_1(1 - \kappa)^{p-1}} + \kappa \sup_{qt_0 \leq s \leq \bar{t}} E\big|x(s)\big|^p \\
&\leq \frac{c_2(1 + \kappa)^p \delta}{c_1(1 - \kappa)^{p-1}} + \kappa \sup_{qt_0 \leq s \leq \bar{t}} E\big|x(s)\big|^p
\end{aligned}
$$

for $t \in [t_0, \bar{t}]$, which also holds for $t \in [qt_0, t_0]$. Then we have

$$
\sup_{qt_0 \leq t \leq \bar{t}} E\big|x(t)\big|^p \leq \frac{c_2(1 + \kappa)^p \delta}{c_1(1 - \kappa)^{p-1}} + \kappa \sup_{qt_0 \leq t \leq \bar{t}} E\big|x(t)\big|^p.
$$

That is,

$$
\sup_{qt_0 \leq t \leq \bar{t}} E\big|x(t)\big|^p \leq \frac{c_2(1 + \kappa)^p}{c_1(1 - \kappa)^p} \delta.
\tag{11}
$$

Using (6) yields

$$
EV\left(t, x(t)\right) \leq \frac{c_2^2(1 + \kappa)^p \delta}{c_1(1 - \kappa)^p} \quad \text{for } t \in [qt_0, \bar{t}].
\tag{12}
$$

Applying $\bar{p} > \frac{c_2}{c_1}(1 - \kappa)^{-p}$, (9) and (12), we obtain that

$$
\begin{aligned}
\bar{p} EV\left(t, x(t) - N\left(x(qt)\right)\right) &\geq \bar{p} c_2(1 + \kappa)^p \delta > \frac{c_2^2(1 + \kappa)^p \delta}{c_1(1 - \kappa)^p} \\
&\geq EV\left(qt, x(qt)\right) \quad \text{for } t \in [\bar{t}, \bar{t} + h].
\end{aligned}
$$

By Itô's formula and (7),

$$EV\big(t, x(t) - N\big(x(qt)\big)\big) \le EV\big(\bar{t}, x(\bar{t}) - N\big(x(q\bar{t})\big)\big) \le c_2(1 + \kappa)^p \delta \quad \text{for } t \in (\bar{t}, \bar{t} + h],$$

which contradicts (9). Therefore (8) must hold. In a similar way to (11), we have

$$\sup_{qt_0 \le s \le t} E\big|x(s)\big|^p \le \frac{c_2(1 + \kappa)^p \delta}{c_1(1 - \kappa)^p} \quad \text{for } t \ge t_0.$$

That is,

$$E\big|x(t)\big|^p \le \frac{c_2(1 + \kappa)^p \delta}{c_1(1 - \kappa)^p} = \varepsilon \quad \text{for } t \ge t_0.$$

The pth moment stability is proved.

(2) By the same argument as shown in (1), we can obtain that for any initial data $\xi \in L^p_{\mathcal{F}_{t_0}}([qt_0, t_0]; R^n)$, there exists a constant H_1 such that

$$EV\big(t, x(t) - N\big(x(qt)\big)\big) \le H_1 \quad \text{for } t \ge t_0,$$

$$E\big|x(t)\big|^p \le \frac{H_1}{c_1(1 - \kappa)^p} \quad \text{for } t \ge t_0$$

and

$$EV\big(t, x(t)\big) \le \frac{c_2 H_1}{c_1(1 - \kappa)^p} \quad \text{for } t \ge t_0.$$

We first show that

$$\lim_{t \to \infty} EV\big(t, x(t) - N\big(x(qt)\big)\big) = 0. \tag{13}$$

In order to prove (13), it is enough to show that for any given γ with $0 < \gamma < H_1$, there exists a $T > 0$ such that $EV(t, x(t) - N(x(qt))) < \gamma$ for $t > T$. Let $d > 0$ satisfy $\bar{p}\frac{c_1}{c_2}(1 - \kappa)^p \gamma > \gamma + d/(1 - \kappa)$. Obviously, $\bar{p}\frac{c_1}{c_2}(1 - \kappa)^p s > s + d/(1 - \kappa)$ for $s \ge \gamma$. Set $T_0 = \frac{\bar{p}\frac{c_1}{c_2}(1-\kappa)^p - 1}{\lambda(1-q)}$, $\bar{t}_i = \frac{t_0 + T_0}{q^{2i}}$, $\bar{s}_i = \frac{t_0 + T_0}{q^{2i+1}}$, $I_i = [\bar{s}_i, \bar{t}_{i+1}]$. Let M be the smallest nonnegative integer such that $H_1 \le \gamma + Md$.

We will show that

$$EV\big(t, x(t) - N\big(x(qt)\big)\big) \le \gamma + (M - i)d \quad \text{for } t \ge \bar{t}_i, i = 0, 1, 2, \ldots, M,$$

$$E\big|x(t)\big|^p \le \frac{\gamma + (M - (i+1))d + (1 + \kappa + \kappa^2 + \cdots + \kappa^i)d}{c_1(1 - \kappa)^p}$$

$$\text{for } t \ge \bar{t}_i, i = 0, 1, 2, \ldots, M, \tag{14}$$

$$EV\big(t, x(t)\big) \le c_2 \frac{\gamma + (M - (i+1))d + (1 + \kappa + \kappa^2 + \cdots + \kappa^i)d}{c_1(1 - \kappa)^p}$$

$$\text{for } t \ge \bar{t}_i, i = 0, 1, 2, \ldots, M.$$

We will prove (14) by induction. Obviously, (14) holds for $i = 0$. Assume that (14) holds for $1 \leq i \leq n < M$. Then we will obtain that there exists a $t^* \in I_n$ such that

$$EV\big(t^*, x\big(t^*\big) - N\big(x\big(qt^*\big)\big)\big) \leq \gamma + \big(M - (n+1)\big)d. \tag{15}$$

Otherwise, for all $t \in I_n$, we have

$$qt \geq \bar{t}_n,$$
$$EV\big(t, x(t) - N\big(x(qt)\big)\big) > \gamma + \big(M - (n+1)\big)d, \tag{16}$$

which implies that

$$\bar{p}EV\big(t, x(t) - N\big(x(qt)\big)\big) \geq \bar{p}\big(\gamma + \big(M - (n+1)\big)d\big) > c_2 \frac{\gamma + (M - (n+1))d + \frac{d}{1-\kappa}}{c_1(1-\kappa)^p}$$

$$> c_2 \frac{\gamma + (M - (n+1))d + (1 + \kappa + \kappa^2 + \cdots + \kappa^n)d}{c_1(1-\kappa)^p}$$

$$\geq EV\big(qt, x(qt)\big) \quad \text{for } t \in I_n. \tag{17}$$

Using Itô's formula, (7) and (16), we have

$$EV\big(\bar{t}_{n+1}, x(\bar{t}_{n+1}) - N\big(x(q\bar{t}_{n+1})\big)\big)$$

$$\leq EV\big(\bar{s}_n, x(\bar{s}_n) - N\big(x(q\bar{s}_n)\big)\big) - \lambda \int_{\bar{s}_n}^{\bar{t}_{n+1}} EV\big(t, x(t) - N\big(x(qt)\big)\big)\, dt$$

$$\leq \gamma + (M - n)d - \lambda\big(\gamma + \big(M - (n+1)\big)d\big)(\bar{t}_{n+1} - \bar{s}_n)$$

$$\leq \gamma + (M - n)d - \Big(\bar{p}\frac{c_1}{c_2}(1-\kappa)^p - 1\Big)\big(\gamma + \big(M - (n+1)\big)d\big)$$

$$\leq \gamma + (M - n)d - \frac{d}{1 - \kappa} < \gamma + \big(M - (n+1)\big)d, \tag{18}$$

which contradicts (16). So, (15) must hold. We assert that for all $t \geq t^*$,

$$EV\big(t, x(t) - N\big(x(qt)\big)\big) \leq \gamma + \big(M - (n+1)\big)d. \tag{19}$$

Otherwise, there is a $t > t^*$ such that

$$EV\big(t, x(t) - N\big(x(qt)\big)\big) > \gamma + \big(M - (n+1)\big)d. \tag{20}$$

Set $\bar{t} = \max\{t : EV(s, x(s) - N(x(qs))) \leq \gamma + (M - (n+1))d, s \in [t^*, t]\}$. Then there exists a sufficiently small $h > 0$ such that

$$qt \geq \bar{t}_n \quad \text{for } t \in [\bar{t}, \bar{t} + h],$$
$$EV\big(t, x(t) - N\big(x(qt)\big)\big) > \gamma + \big(M - (n+1)\big)d \quad \text{for } t \in (\bar{t}, \bar{t} + h], \tag{21}$$

which implies that

$$
\bar{p}EV\big(t, x(t) - N\big(x(qt)\big)\big) \geq \bar{p}\big(\gamma + \big(M - (n+1)\big)d\big) > c_2 \frac{\gamma + (M - (n+1))d + \frac{d}{1-\kappa}}{c_1(1-\kappa)^p}
$$

$$
> c_2 \frac{\gamma + (M - (n+1))d + (1 + \kappa + \kappa^2 + \cdots + \kappa^n)d}{c_1(1-\kappa)^p}
$$

$$
\geq EV\big(qt, x(qt)\big) \quad \text{for } t \in [\bar{t}, \bar{t} + h]. \tag{22}
$$

Then, by Itô's formula and (7),

$$
EV\big(\bar{t} + h, x(\bar{t} + h) - N\big(x\big(q(\bar{t} + h)\big)\big)\big) - EV\big(\bar{t}, x(\bar{t}) - N\big(x(q\bar{t})\big)\big) \leq 0.
$$

Hence $EV(\bar{t} + h, x(\bar{t} + h) - N(x(q(\bar{t} + h)))) \leq \gamma + (M - (n+1))d$, which contradicts (21). We therefore have

$$
EV\big(t, x(t) - N\big(x(qt)\big)\big) \leq \gamma + \big(M - (n+1)\big)d \quad \text{for } t \geq \bar{t}_{n+1} \geq t^*. \tag{23}
$$

On the other hand, using (6), (10) and (23), we can obtain that

$$
E\big|x(t)\big|^p \leq \frac{E|x(t) - N(x(qt))|^p}{(1-\kappa)^{p-1}} + \kappa E\big|x(qt)\big|^p
$$

$$
\leq \frac{EV(t, x(t) - N(x(qt)))}{c_1(1-\kappa)^{p-1}} + \kappa E\big|x(qt)\big|^p
$$

$$
\leq \frac{\gamma + (M - (n+1))d}{c_1(1-\kappa)^{p-1}} + \kappa \frac{\gamma + (M - (n+1))d + (1 + \kappa + \kappa^2 + \cdots + \kappa^n)d}{c_1(1-\kappa)^p}
$$

$$
= \frac{\gamma + (M - (n+2))d + (1 + \kappa + \kappa^2 + \cdots + \kappa^{n+1})d}{c_1(1-\kappa)^p} \quad \text{for } t \geq \bar{t}_{n+1}. \tag{24}
$$

Moreover, by (6),

$$
EV\big(t, x(t)\big) \leq c_2 \frac{\gamma + (M - (n+2))d + (1 + \kappa + \kappa^2 + \cdots + \kappa^{n+1})d}{c_1(1-\kappa)^p}
$$

for $t \geq \bar{t}_{n+1}$. That is, (14) holds for $i = n + 1$. By induction, (14) holds for $0 \leq i \leq M$. Thus

$$
EV\big(t, x(t) - N\big(x(qt)\big)\big) \leq \gamma \quad \text{for } t \geq T = \bar{t}_M.
$$

So, (13) must hold. Using (6) yields

$$
\lim_{t \to \infty} E\big|x(t) - N\big(x(qt)\big)\big|^p = 0.
$$

Since $E|x(t)|^p \leq \frac{H_1}{c_1(1-\kappa)^p}$, we conclude that $\limsup_{t \to \infty} E|x(t)|^p < \infty$. Using (10), we have

$$
\limsup_{t \to \infty} E\big|x(t)\big|^p \leq \frac{1}{(1-\kappa)^{p-1}} \lim_{t \to \infty} E\big|x(t) - N\big(x(qt)\big)\big|^p + \kappa \limsup_{t \to \infty} E\big|x(qt)\big|^p
$$

$$
= 0 + \kappa \limsup_{t \to \infty} E\big|x(t)\big|^p,
$$

which implies that

$$\limsup_{t \to \infty} E|x(t)|^p = 0.$$

The proof is complete. □

Remark The result obtained in Theorem 3.1 is different from the ones in [2, 16, 17, 19, 20]. The results obtained in [2, 16, 19, 20] only deal with NSFDEs with finite delay. It is easy to see that NSPE (4) is a class of NSFDEs with infinite delay. So, these results obtained in [2, 16, 19, 20] cannot be applied to NSPE (4). On the other hand, the result obtained in [17] studied the ψ^γ stability of NSFDEs with infinite delay. But Theorem 3.1 studied the pth moment asymptotic stability of NSPE (4). Since Theorem 3.1 is established without taking difficulties from infinite delay into account, the conditions of Theorem 3.1 are much weaker than the one obtained in [17]. So, our work is not trivial.

If we take $V(t, x) = |x|^p$, the condition (7) may be written as

$$E\left\{\frac{p}{2}|\varphi(t) - N(\varphi(qt))|^{p-4}\left(|\varphi(t) - N(\varphi(qt))|^2\left[2(\varphi(t) - N(\varphi(qt)))^\top f(t, \varphi(t), \varphi(qt))\right.\right.\right.$$

$$\left.\left.+ |g(t, \varphi(t), \varphi(qt))|^2\right] + (p-2)|(\varphi(t) - N(\varphi(qt)))^\top g(t, \varphi(t), \varphi(qt))|^2\right)\right\}$$

$$\leq -\lambda E|\varphi(t) - N(\varphi(qt))|^p, \quad t \geq t_0, \tag{25}$$

for all $\varphi \in L^p_{\mathcal{F}_t}([qt, t]; R^n)$ satisfying

$$E|\varphi(qt)|^p < \bar{p}E|\varphi(t) - N(\varphi(qt))|^p,$$

where $\bar{p} > (1 - \kappa)^{-p}$.

Corollary 3.1 *Let Assumption* (H) *hold. Assume that there are two positive constants* λ_1 *and* λ_2 *such that*

$$E\left\{\frac{p}{2}|\varphi(t) - N(\varphi(qt))|^{p-4}\left(|\varphi(t) - N(\varphi(qt))|^2\left[2(\varphi(t) - N(\varphi(qt)))^\top f(t, \varphi(t), \varphi(qt))\right.\right.\right.$$

$$\left.\left.+ |g(t, \varphi(t), \varphi(qt))|^2\right] + (p-2)|(\varphi(t) - N(\varphi(qt)))^\top g(t, \varphi(t), \varphi(qt))|^2\right)\right\}$$

$$\leq -\lambda_1 E|\varphi(t)|^p + \lambda_2 E|\varphi(qt)|^p \tag{26}$$

for all $t \geq t_0$ *and* $\varphi \in L^p_{\mathcal{F}_t}([qt, t]; R^n)$. *If*

$$0 < \kappa < \frac{1}{2} \quad and \quad \lambda_1 > \frac{\lambda_2}{(1 - 2\kappa)^p}, \tag{27}$$

then the trivial solution of Eq. (4) *is globally* pth *moment asymptotically stable.*

Proof As in Theorem 3.1, let $x(t) = x(t; \xi)$ be the solution of the system (4) for any given initial data $\xi \in L^p_{\mathcal{F}_{t_0}}([qt_0, t_0]; R^n)$. By the condition (27), we can choose $\bar{p}$ such that

$$\frac{1}{\kappa^p} > \bar{p} > \frac{1}{(1-\kappa)^p} \quad \text{and} \quad \lambda_1 > \frac{\lambda_2 \bar{p}}{(1 - \kappa(\bar{p})^{\frac{1}{p}})^p}. \tag{28}$$

Let $t \geq 0$ and $x(t)$ satisfy

$$E|x(qt)|^p < \bar{p} E|x(t) - N(x(qt))|^p.$$

Note that for any $\varepsilon \in (0, 1)$,

$$E|x(t) - N(x(qt))|^p \leq \frac{1}{(1-\varepsilon)^{p-1}} E|x(t)|^p + \frac{1}{\varepsilon^{p-1}} E|N(x(qt))|^p$$

$$\leq \frac{1}{(1-\varepsilon)^{p-1}} E|x(t)|^p + \frac{\kappa^p \bar{p}}{\varepsilon^{p-1}} E|x(t) - N(x(qt))|^p.$$

Hence

$$-E|x(t)|^p \leq -(1-\varepsilon)^{p-1}\left(1 - \frac{\kappa^p \bar{p}}{\varepsilon^{p-1}}\right) E|x(t) - N(x(qt))|^p. \tag{29}$$

It then follows from (26) and (29) that

$$E\left\{\frac{p}{2}|x(t) - N(x(qt))|^{p-4}\left(|x(t) - N(x(qt))|^2\left[2(x(t) - N(x(qt)))^\top f(t, x(t), x(qt))\right.\right.\right.$$

$$\left.\left.\left. + |g(t, x(t), x(qt))|^2\right] + (p-2)|(x(t) - N(x(qt)))^\top g(t, x(t), x(qt))|^2\right)\right\}$$

$$\leq -\lambda_1 E|x(t)|^p + \lambda_2 E|x(qt)|^p$$

$$\leq -\left[\lambda_1(1-\varepsilon)^{p-1}\left(1 - \frac{\kappa^p \bar{p}}{\varepsilon^{p-1}}\right) - \lambda_2 \bar{p}\right] E|x(t) - N(x(qt))|^2$$

$$= -\left[\lambda_1\left(1 - \kappa(\bar{p})^{\frac{1}{p}}\right)^p - \lambda_2 \bar{p}\right] E|x(t) - N(x(qt))|^2, \tag{30}$$

where $\varepsilon = \kappa(\bar{p})^{\frac{1}{p}}$. Using (28) yields

$$\lambda_1\left(1 - \kappa(\bar{p})^{\frac{1}{p}}\right)^p - \lambda_2 \bar{p} > 0.$$

That is, the condition (25) is satisfied. Hence the conclusion follows from Theorem 3.1. The proof is complete. □

Corollary 3.2 *Let Assumption* (H) *hold. Assume that there are four positive constants* λ_1, λ_2, λ_3 *and* λ_4 *such that*

$$2(x - N(y))^\top f(t, x, y) \leq -\lambda_1 |x|^2 + \lambda_2 |y|^2,$$

$$|g(t, x, y)|^2 \leq \lambda_3 |x|^2 + \lambda_4 |y|^2 \tag{31}$$

for all $t \geq t_0$ and $x, y \in R^n$. If

$$0 < \kappa < \frac{1}{2} \quad and \quad \lambda_1 - \lambda_3 > \frac{\lambda_2 + \lambda_4}{(1 - 2\kappa)^2}, \tag{32}$$

then the trivial solution of Eq. (4) *is mean square stable.*

Proof Using (31), we have

$$E\big(2\big(\varphi(t) - N\big(\varphi(qt)\big)\big)^\top f\big(t, \varphi(t), \varphi(qt)\big) + \big|g\big(t, \varphi(t), \varphi(qt)\big)\big|^2\big)$$
$$\leq -(\lambda_1 - \lambda_3)E\big|\varphi(t)\big|^2 + (\lambda_2 + \lambda_4)E\big|\varphi(qt)\big|^2$$

for all $\varphi \in L^p_{\mathcal{F}_t}([qt, t]; R^n)$. Then the desired conclusion is obtained from Corollary 3.1 as taking $p = 2$. The proof is complete. □

4 Mean square stability of the backward Euler method

In this section, we develop the backward Euler method for Eq. (4). We will show that the backward Euler method can preserve the mean square stability of exact solutions of Eq. (4).

To define the backward Euler method for Eq. (4), we introduce a mesh $\bar{H} = \{m; t_{-m}, \ldots, t_{-1}, t_0, t_1, \ldots, t_m, \ldots\}$ as follows. We assume that the initial time $t_0 > 0$. Set $t_m = q^{-1}t_0$. We define $m - 1$ grid points $t_1 < t_2 < \cdots < t_{m-1}$ in (t_0, t_m) by

$$t_i = t_0 + i\Delta_0 \quad \text{for } i = 1, 2, \ldots, m - 1,$$

where the initial step size $\Delta_0 = \frac{t_m - t_0}{m}$, and define the other grid points by

$$t_{km+i} = q^{-k}t_i \quad \text{for } k = -1, 0, 1, \ldots, i = 0, 1, 2, \ldots, m - 1.$$

Let $h_n = t_{n+1} - t_n$. It is easy to see that grid points t_n satisfy $qt_n = t_{n-m}$ for $n \geq 0$ and the step size h_n satisfies

$$qh_n = h_{n-m} \quad \text{for } n \geq 0 \text{ and } \lim_{n \to \infty} h_n = \infty.$$

For the given mesh $\bar{H}$, the backward Euler method for Eq. (4) is defined as follows:

$$Y_{n+1} - N(Y_{n+1-m})$$
$$= Y_n - N(Y_{n-m}) + h_n f(t_{n+1}, Y_{n+1}, Y_{n+1-m}) + g(t_n, Y_n, Y_{n-m})\Delta B_n \tag{33}$$

with the initial values

$$Y_n = \xi(t_n), \quad n = -m, -m + 1, \ldots, 0. \tag{34}$$

Here, Y_n ($n > 0$) is an approximation value of $x(t_n)$ and $\mathcal{F}_{t_n}$-measurable. $\Delta B_n = B(t_{n+1}) - B(t_n)$ is the Brownian motion increment.

Definition 4.1 The numerical solution Y_n ($n = 1, 2, \ldots$) of Eq. (4) is said to be mean square stable if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$E|Y_n|^2 \leq \varepsilon, \quad n = 1, 2, \ldots,$$

whenever the initial values satisfy $E(\max_{n=-m,\ldots,0} |\xi(t_n)|^2) < \delta$ and, moreover,

$$\lim_{n \to \infty} E|Y_n|^2 = 0.$$

Theorem 4.1 *Assume the backward Euler method* (33) *is well defined. Let Assumption* (H), *conditions* (31) *and* (32) *hold. Then the backward Euler method approximate solution* (33) *is mean square stable.*

Proof Set $\overline{Y}_n = Y_n - N(Y_{n-m})$. For $n \geq 0$, by (33),

$$\left|\overline{Y}_{n+1} - h_n f(t_{n+1}, Y_{n+1}, Y_{n+1-m})\right|^2 = \left|\overline{Y}_n + g(t_n, Y_n, Y_{n-m})\Delta B_n\right|^2.$$

Then we can obtain that

$$|\overline{Y}_{n+1}|^2 \leq |\overline{Y}_n|^2 + 2h_n \langle \overline{Y}_{n+1}, f(t_{n+1}, Y_{n+1}, Y_{n+1-m}) \rangle + \left|g(t_n, Y_n, Y_{n-m})\Delta B_n\right|^2$$
$$+ 2\langle \overline{Y}_n, g(t_n, Y_n, Y_{n-m})\Delta B_n \rangle,$$

which subsequently leads to

$$E|\overline{Y}_{n+1}|^2 \leq E|\overline{Y}_n|^2 + 2h_n E\langle \overline{Y}_{n+1}, f(t_{n+1}, Y_{n+1}, Y_{n+1-m}) \rangle$$
$$+ E\left|g(t_n, Y_n, Y_{n-m})\right|^2 h_n.$$

By the condition (31),

$$E|\overline{Y}_{n+1}|^2 \leq E|\overline{Y}_n|^2 - \lambda_1 h_n E|Y_{n+1}|^2 + \lambda_2 h_n E|Y_{n+1-m}|^2$$
$$+ \left(\lambda_3 E|Y_n|^2 + \lambda_4 E|Y_{n-m}|^2\right) h_n. \tag{35}$$

Using the equality $|a + b|^2 \leq 2|a|^2 + 2|b|^2$ yields

$$|\overline{Y}_{n+1}|^2 \geq \frac{1}{2}|Y_{n+1}|^2 - \left|N(Y_{n+1-m})\right|^2$$

and

$$|\overline{Y}_n|^2 \leq 2|Y_n|^2 + 2\left|N(Y_{n-m})\right|^2.$$

Inserting these inequalities into (35) and using Assumption (H), we have

$$\left(\frac{1}{2} + \lambda_1 h_n\right) E|Y_{n+1}|^2 \leq (2 + \lambda_3 h_n) E|Y_n|^2 + \left(\kappa^2 + \lambda_2 h_n\right) E|Y_{n+1-m}|^2$$
$$+ \left(2\kappa^2 + \lambda_4 h_n\right) E|Y_{n-m}|^2. \tag{36}$$

Let $A_n = 1/2 + \lambda_1 h_n$, $B_n = 2 + \lambda_3 h_n$, $C_n = \kappa^2 + \lambda_2 h_n$ and $D_n = 2\kappa^2 + \lambda_4 h_n$. Using these notations, (36) implies that

$$E|Y_{n+1}|^2 \leq \frac{B_n + C_n + D_n}{A_n} \max\left\{E|Y_n|^2, E|Y_{n+1-m}|^2, E|Y_{n-m}|^2\right\}. \tag{37}$$

By the condition (32), we have

$$0 < A = \lim_{n \to \infty} \frac{B_n + C_n + D_n}{A_n} = \frac{\lambda_2 + \lambda_3 + \lambda_4}{\lambda_1} < 1.$$

Let $\varepsilon \in (0, 1 - A)$. There exists a positive integer N such that

$$\left| \frac{B_n + C_n + D_n}{A_n} - A \right| < \varepsilon, \quad n \geq N,$$

that is,

$$\frac{B_n + C_n + D_n}{A_n} < A + \varepsilon < 1, \quad n \geq N.$$

Set $C = \max\{\max_{0 \leq n \leq N}\{\frac{B_n + C_n + D_n}{A_n}\}, 1\}$. Then, by (37),

$$E|Y_{n+1}|^2 \leq C^{N+1} \max_{n=-m,\ldots,0} E|Y_n|^2, \quad n \geq 0,$$

$$E|Y_{n+1}|^2 \leq (A + \varepsilon)^{\left[\frac{n-N}{m+1}\right]} \max_{-m \leq i \leq 0}\left\{E|Y_{N+i}|^2\right\}, \quad n \geq N, \tag{38}$$

where $[\cdot]$ denotes the 'greatest integer in' function. (38) implies

$$\lim_{n \to \infty} E|Y_n|^2 = 0.$$

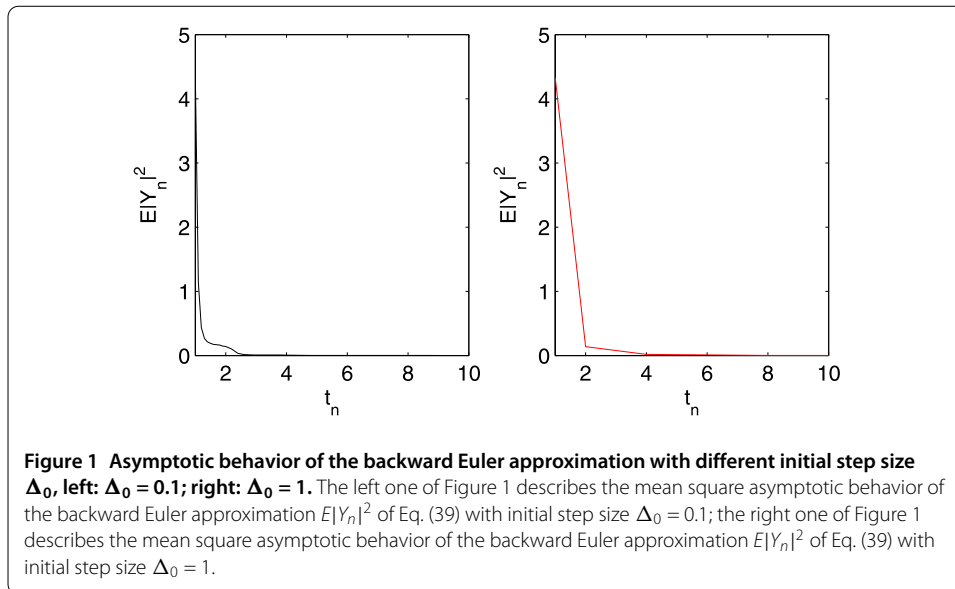The proof is complete. □

5 Numerical experiments

In this section, we present numerical experiments to illustrate theoretical results obtained in the previous sections.

Consider the following scalar problem:

$$\begin{cases} d[x(t) - \frac{1}{4}\sin(x(0.5t))] = (-12x(t) + 2\sin(x(0.5t)))\,dt + x(0.5t)\,dB(t), & t \geq 1, \\ x(t) = 1 + 2\cos(t), & 0 \leq t \leq 1. \end{cases} \tag{39}$$

For Eq. (39), we can obtain that $\kappa = 1/4$, $\lambda_1 = 14$, $\lambda_2 = 3/2$, $\lambda_3 = 0$ and $\lambda_4 = 1$ corresponding to Corollary 3.2. By Corollary 3.2, the solution $x(t)$ of Eq. (39) is mean square stable. Theorem 4.1 shows that the backward Euler method approximation Y_n of Eq. (39) can preserve the mean square stability of the exact solution $x(t)$.

A set of 20 blocks each containing 100 outcomes (ω_{ij}, $1 \leq i \leq 20$, $1 \leq j \leq 100$) is simulated, and $Y_n(\omega_{ij})$ denotes the nth step value of the backward Euler method approximation

Figure 1 Asymptotic behavior of the backward Euler approximation with different initial step size Δ_0, left: $\Delta_0 = 0.1$; right: $\Delta_0 = 1$. The left one of Figure 1 describes the mean square asymptotic behavior of the backward Euler approximation $E|Y_n|^2$ of Eq. (39) with initial step size $\Delta_0 = 0.1$; the right one of Figure 1 describes the mean square asymptotic behavior of the backward Euler approximation $E|Y_n|^2$ of Eq. (39) with initial step size $\Delta_0 = 1$.

of the jth trajectory in the ith block. We compute $E|Y_n|^2$ of the backward Euler method approximation by

$$E|Y_n|^2 \cong \frac{1}{2,000} \sum_{i=1}^{20} \sum_{j=1}^{100} |Y_n(\omega_{ij})|.$$

Two tests are developed with different initial step sizes $\Delta_0 = 0.1, 1$. Results are shown in Figure 1. It is easy to see that these results are compatible with the conclusion of Theorem 4.1.

Conclusion

In this paper, we have firstly investigated the pth moment asymptotic stability of NSPEs. Although the NSPE is a special class of NFDEs with infinite delay, we show that the Razumikhin-type theorem on the pth moment asymptotic stability of NSPEs can be established without taking into account difficulties from infinite delay. On the other hand, we also develop the backward Euler method for NSPEs. We show that the backward Euler method can preserve the mean square stability of exact solutions under suitable conditions. Numerical examples have been provided to demonstrate the validity of our results.

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