Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Hardy-type inequalities on a half-space in the Heisenberg group

Heng-Xing Liu* and Jing-Wen Luan

*Correspondence:
hxliu.math@whu.edu.cn
School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, People's Republic of China

Abstract
We prove some Hardy-type inequalities on half-spaces for Kohn's sub-Laplacian in the Heisenberg group. Furthermore, the constants we obtained are sharp.
MSC: Primary 26D10; 35H20

Keywords: Hardy inequality; Heisenberg group; sharp constant

1 Introduction

The Hardy inequality in $\mathbb{R}^N$ reads that for all $u \in C_0^\infty(\mathbb{R}^N)$ and $N \geq 3$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \, dx \tag{1.1}$$

and the constant $\frac{(N-2)^2}{4}$ in (1.1) is sharp. Recently, it has been proved by Nazarov ([1], Proposition 4.1, see also [2]) that the following Hardy inequality is valid for $f \in C_0^\infty(\mathbb{R}_+^N)$:

$$\int_{\mathbb{R}_+^N} \left|\nabla u(x)\right|^2 dx \geq \frac{N^2}{4} \int_{\mathbb{R}_+^N} \frac{u(x)^2}{|x|^2} \, dx, \tag{1.2}$$

where $\mathbb{R}_+^N = \{(x_1, \ldots, x_n) | x_1 > 0\}$, and the constant $\frac{N^2}{4}$ is sharp. This shows that the Hardy constant jumps from $\frac{(N-2)^2}{4}$ to $\frac{N^2}{4}$ when the singularity of the potential reaches the boundary. For more information about this inequality and its applications, we refer to [3–10] and the references therein.

The aim of this note is to prove an analogous Hardy-type inequality on a half-space for Kohn's sub-Laplacian in Heisenberg groups $\mathbb{H}^n$. It has been proved by D'Ambrosio ([11], Theorem 3.3) that for $u \in C_0^\infty(\mathbb{H}^n)$, the following holds:

$$\int_{\mathbb{H}^n} |\nabla_H u|^2 \, dx \, dt \geq (n-1)^2 \int_{\mathbb{H}^n} \frac{u^2}{|x|^2} \, dx \, dt, \tag{1.3}$$

where ∇_H is the horizontal gradient associated with Kohn's sub-Laplacian on $\mathbb{H}^n$ (for details, see Section 2). Furthermore, the constant $(n-1)^2$ in (1.3) is sharp (see [12], Theorem 3.13). In this note we shall show that when the singularity is on the boundary, the Hardy constant also jumps. In fact, we have the following.

Theorem 1.1 *For all* $u \in C_0^\infty(\mathbb{H}_+^n)$, *the following holds*:

$$\int_{\mathbb{H}_+^n} |\nabla_H u|^2 \,dx\,dt \geq n^2 \int_{\mathbb{H}_+^n} \frac{u^2}{|x|^2} \,dx\,dt, \tag{1.4}$$

where $\mathbb{H}_+^n = \{(x,t) \in \mathbb{H}^n : x_1 > 0\}$, *and the constant* n^2 *in* (1.4) *is sharp*.

In order to prove Theorem 1.1, we use a new technique which is different from that in [1, 2]. In fact, it seems that the method used in [1, 2] cannot be applied to Kohn's sub-Laplacian.

With the same technique, we obtain the following sharp Hardy inequality on $\mathbb{H}_{k_+}^n = \{(x,t) \in \mathbb{H}^n : x_1 > 0, \ldots, x_k > 0\}$.

Theorem 1.2 *Let* $1 \leq k \leq 2n$. *For all* $u \in C_0^\infty(\mathbb{H}_{k_+}^n)$, *the following holds*:

$$\int_{\mathbb{H}_{k_+}^n} |\nabla_H u|^2 \,dx\,dt \geq (n+k-1)^2 \int_{\mathbb{H}_{k_+}^n} \frac{u^2}{|x|^2} \,dx\,dt. \tag{1.5}$$

Furthermore, the constant $(n+k-1)^2$ *in* (1.5) *is sharp*.

2 Proofs

Let $\mathbb{H}^n = (\mathbb{R}^{2n} \times \mathbb{R}, \circ)$ be the $(2n+1)$-dimensional Heisenberg group whose group structure is given by

$$(x,t) \circ (x',t') = \left(x + x', t + t' + 2\sum_{j=1}^{n} (x'_{2j}x_{2j-1} - x'_{2j-1}x_{2j})\right).$$

The vector fields

$$X_{2j-1} = \frac{\partial}{\partial x_{2j-1}} + 2x_{2j}\frac{\partial}{\partial t}, \qquad X_{2j} = \frac{\partial}{\partial x_{2j}} - 2x_{2j-1}\frac{\partial}{\partial t}$$

$(j = 1, \ldots, n)$ are left invariant and generate the Lie algebra of $\mathbb{H}^n$. Kohn's sub-Laplace on $\mathbb{H}^n$ is

$$\Delta_H = \sum_{j=1}^{2n} X_j^2 = \sum_{j=1}^{2n} \frac{\partial^2}{\partial x_j^2} + 4|x|^2 \frac{\partial^2}{\partial t^2} + 4\sum_{j=1}^{n} \left(x_{2j}\frac{\partial}{\partial x_{2j-1}} - x_{2j-1}\frac{\partial}{\partial x_{2j}}\right)\frac{\partial}{\partial t}$$

and the horizontal gradient is the $(2n)$-dimensional vector given by

$$\nabla_H = (X_1, \ldots, X_{2n}) = \nabla_x + 2\Lambda x \frac{\partial}{\partial t},$$

where $\nabla_x = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{2n}})$, Λ is a skew symmetric and orthogonal matrix given by

$$\Lambda = \operatorname{diag}(J_1, \ldots, J_n), \quad J_1 = \cdots = J_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

By the definition of ∇_H, we have, for $\alpha \in \mathbb{R}$ and $|x| \neq 0$,

$$\begin{aligned}\Delta_H\big(x_1|x|^\alpha\big) &= \sum_{j=1}^{2n} \frac{\partial^2}{\partial x_j^2}\big(x_1|x|^\alpha\big) = \sum_{j=2}^{2n} \frac{\partial^2}{\partial x_j^2}\big(x_1|x|^\alpha\big) + \frac{\partial^2}{\partial x_1^2}\big(x_1|x|^\alpha\big) \\ &= x_1 \sum_{j=2}^{2n} \frac{\partial^2}{\partial x_j^2}|x|^\alpha + x_1 \frac{\partial^2}{\partial x_1^2}|x|^\alpha + 2\frac{\partial |x|^\alpha}{\partial x_1} \\ &= x_1 \sum_{j=1}^{2n} \frac{\partial^2}{\partial x_j^2}|x|^\alpha + 2\frac{\partial |x|^\alpha}{\partial x_1} \\ &= \alpha(2n+\alpha)x_1|x|^{\alpha-2}. \end{aligned} \tag{2.1}$$

Similarly,

$$\begin{aligned}\Delta_H\left(|x|^\alpha \prod_{i=1}^k x_i\right) &= \sum_{j=1}^{2n} \frac{\partial^2}{\partial x_j^2}\left(|x|^\alpha \prod_{i=1}^k x_i\right) \\ &= \sum_{j=k+1}^{2n} \frac{\partial^2}{\partial x_j^2}\left(|x|^\alpha \prod_{i=1}^k x_i\right) + \sum_{l=1}^{k} \frac{\partial^2}{\partial x_l^2}\left(|x|^\alpha \prod_{i=1}^k x_i\right) \\ &= \prod_{i=1}^k x_i \sum_{j=1}^{2n} \frac{\partial^2}{\partial x_j^2}|x|^\alpha + 2k|x|^{\alpha-2}\prod_{i=1}^k x_i \\ &= \alpha(2n+2k+\alpha-2)|x|^{\alpha-2}\prod_{i=1}^k x_i. \end{aligned} \tag{2.2}$$

Proof of Theorem 1.1 Using the substitution $u = x_1|x|^{-n}f$, we get

$$\begin{aligned}\int_{\mathbb{H}_+^n} |\nabla_H u|^2 &= \int_{\mathbb{H}_+^n} \left[\big|\nabla_H\big(x_1|x|^{-n}\big)\big|^2 f^2 + |\nabla_H f|^2 \frac{x_1^2}{|x|^{2n}} + \frac{\langle \nabla_H(x_1^2|x|^{-2n}), \nabla_H f^2\rangle}{2} \right] \\ &\geq \int_{\mathbb{H}_+^n} \left(\big|\nabla_H\big(x_1|x|^{-n}\big)\big|^2 f^2 + \frac{\langle \nabla_H(x_1^2|x|^{-2n}), \nabla_H f^2\rangle}{2} \right) \\ &= \int_{\mathbb{H}_+^n} f^2 \left(\big|\nabla_H\big(x_1|x|^{-n}\big)\big|^2 - \frac{1}{2}\Delta_H\big(x_1^2|x|^{-2n}\big) \right). \end{aligned}$$

Using the following identity, for $g \in C^2(\mathbb{H}^n)$,

$$\frac{1}{2}\Delta_H g^2 = \frac{1}{2}\sum_{j=1}^{2n} X_j^2 g^2 = g\sum_{j=1}^{2n} X_j^2 g + \sum_{j=1}^{m} |X_j g|^2 = g\Delta_H g + |\nabla_H g|^2, \tag{2.3}$$

we have, by (2.1),

$$\begin{aligned}\big|\nabla_H\big(x_1|x|^{-n}\big)\big|^2 - \frac{1}{2}\Delta_H\big(x_1^2|x|^{-2n}\big) &= -x_1|x|^{-n}\Delta_H\big(x_1|x|^{-n}\big) \\ &= -x_1|x|^{-n}\cdot(-n)\cdot n x_1|x|^{-n-2} \\ &= n^2 x_1^2|x|^{-2n-2}. \end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{H}_+^n} |\nabla_H u|^2\,dx\,dt &\ge \int_{\mathbb{H}_+^n} f^2\left(\left|\nabla_H\left(x_1|x|^{-n}\right)\right|^2 - \frac{1}{2}\Delta_H\left(x_1^2|x|^{-2n}\right)\right)dx\,dt \\ &= n^2\int_{\mathbb{H}_+^n} f^2 x_1^2|x|^{-2n-2}\,dx\,dt \\ &= n^2\int_{\mathbb{H}_+^n}\frac{u^2}{|x|^2}\,dx\,dt. \end{aligned}\tag{2.4}$$

Now we show the constant n^2 in (1.4) is sharp. Choosing

$$g(x,t) = \phi(x)\omega(t),$$

where $\phi \in C_0^\infty(\mathbb{R}_+^{2n})$ and $\omega \in C_0^\infty(\mathbb{R})$, we have

$$\nabla_H g(x,t) = \nabla_x g(x,t) + 2\Lambda x\frac{\partial}{\partial t}g(x,t) = \omega(t)\nabla_x\phi(x) + 2\phi(x)\omega'(t)\Lambda x.$$

Therefore,

$$\begin{aligned}\left|\nabla_H g(x,t)\right|^2 &= \left\langle\omega(t)\nabla_x\phi(x) + 2\phi(x)\omega'(t)\Lambda x, \omega(t)\nabla_x\phi(x) + 2\phi(x)\omega'(t)\Lambda x\right\rangle \\ &= \omega^2(t)|\nabla_x\phi|^2 + 4\phi^2|x|^2\left|\omega'(t)\right|^2 + 4\omega(t)\omega'(t)\phi\langle\Lambda x, \nabla_x\phi\rangle. \end{aligned}\tag{2.5}$$

To get the last equation, we use the fact $|\Lambda x|^2 = |x|^2$.

Since

$$\int_{-\infty}^{+\infty}\omega(t)\omega'(t)\,dt = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega^2(t)}{dt}\,dt = 0,$$

we have, by (2.5),

$$\begin{aligned}\frac{\int_{\mathbb{H}_+^n}|\nabla_H g(x,t)|^2\,dx\,dt}{\int_{\mathbb{H}_+^n}\frac{g^2}{|x|^2}\,dx\,dt} &= \frac{\int_{\mathbb{H}_+^n}\omega^2(t)|\nabla_x\phi|^2\,dx\,dt + 4\int_{\mathbb{H}_+^n}\phi^2|x|^2|\omega'(t)|^2\,dx\,dt}{\int_{\mathbb{R}_+^{2n}}\frac{\phi^2}{|x|^2}\,dx\cdot\int_{\mathbb{R}}\omega^2\,dt} \\ &= \frac{\int_{\mathbb{R}_+^{2n}}|\nabla_x\phi|^2\,dx}{\int_{\mathbb{R}_+^{2n}}\frac{\phi^2}{|x|^2}\,dx} + 4\frac{\int_{\mathbb{R}}|\omega'(t)|^2\,dt}{\int_{\mathbb{R}}\omega^2\,dt}\cdot\frac{\int_{\mathbb{R}_+^{2n}}\phi^2|x|^2\,dx}{\int_{\mathbb{R}_+^{2n}}\frac{\phi^2}{|x|^2}\,dx}. \end{aligned}$$

Notice that

$$\inf_{\omega\in C_0^\infty(\mathbb{R})\setminus\{0\}}\frac{\int_{\mathbb{R}}|\omega'(t)|^2\,dt}{\int_{\mathbb{R}}\omega^2\,dt} = 0,\tag{2.6}$$

we have

$$\inf_{u\in C_0^\infty(\mathbb{H}_+^n)\setminus\{0\}}\frac{\int_{\mathbb{H}_+^n}|\nabla_H u|^2\,dx\,dt}{\int_{\mathbb{H}_+^n}\frac{u^2}{|x|^2}\,dx\,dt} \le \inf_{\phi\in C_0^\infty(\mathbb{R}^{2n})\setminus\{0\}}\frac{\int_{\mathbb{R}_+^{2n}}|\nabla_x\phi|^2\,dx}{\int_{\mathbb{R}_+^{2n}}\frac{\phi^2}{|x|^2}\,dx} = n^2.$$

Here we use the sharp Hardy inequality (1.2). This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 The proof is similar to that of Theorem 1.1. Using the substitution $u = f|x|^{-n-k+1}\prod_{i=1}^{k} x_i$, we get

$$\begin{aligned}
\int_{\mathbb{H}^n_{k_+}} |\nabla_H u|^2 &= \int_{\mathbb{H}^n_{k_+}} \Bigg[\left| \nabla_H \left(|x|^{-n-k+1} \prod_{i=1}^{k} x_i \right) \right|^2 f^2 + |\nabla_H f|^2 \frac{\prod_{i=1}^{k} x_i^2}{|x|^{2n}} \\
&\quad + \frac{1}{2} \left\langle \nabla_H \left(|x|^{-2n-2k+2} \prod_{i=1}^{k} x_i^2 \right), \nabla_H f^2 \right\rangle \Bigg] \\
&\geq \int_{\mathbb{H}^n_{k_+}} \left(\left| \nabla_H \left(|x|^{-n-k+1} \prod_{i=1}^{k} x_i \right) \right|^2 f^2 + \frac{1}{2} \left\langle \nabla_H \left(|x|^{-2n-2k+2} \prod_{i=1}^{k} x_i^2 \right), \nabla_H f^2 \right\rangle \right) \\
&= \int_{\mathbb{H}^n_{k_+}} f^2 \left(\left| \nabla_H \left(|x|^{-n-k+1} \prod_{i=1}^{k} x_i \right) \right|^2 - \frac{1}{2} \Delta_H \left(|x|^{-2n-2k+2} \prod_{i=1}^{k} x_i^2 \right) \right).
\end{aligned}$$

Using the identities (2.3) and (2.2), we have

$$\begin{aligned}
&\left| \nabla_H \left(|x|^{-n-k+1} \prod_{i=1}^{k} x_i \right) \right|^2 - \frac{1}{2} \Delta_H \left(|x|^{-2n-2k+2} \prod_{i=1}^{k} x_i^2 \right) \\
&\quad = -|x|^{-n-k+1} \prod_{i=1}^{k} x_i \cdot \Delta_H \left(|x|^{-n-k+1} \prod_{i=1}^{k} x_i \right) \\
&\quad = (n+k-1)^2 |x|^{-2n-2k} \prod_{i=1}^{k} x_i^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{H}^n_{k_+}} |\nabla_H u|^2 &\geq \int_{\mathbb{H}^n_{k_+}} f^2 \left(\left| \nabla_H \left(|x|^{-n-k+1} \prod_{i=1}^{k} x_i \right) \right|^2 - \frac{1}{2} \Delta_H \left(|x|^{-2n-2k+2} \prod_{i=1}^{k} x_i^2 \right) \right) \\
&= (n+k-1)^2 \int_{\mathbb{H}^n_{k_+}} f^2 |x|^{-2n-2k} \prod_{i=1}^{k} x_i^2 \\
&= (n+k-1)^2 \int_{\mathbb{H}^n_{k_+}} \frac{u^2}{|x|^2}.
\end{aligned}$$

To see the constant $(n+k-1)^2$ in (1.5) is sharp, we consider the function

$$h(x,t) = \psi(x)\omega(t),$$

where $\psi \in C_0^\infty(\mathbb{R}^{2n}_{k_+})$ and $\omega \in C_0^\infty(\mathbb{R})$. Here we denote by $\mathbb{R}^{2n}_{k_+} = \{x \in \mathbb{R}^{2n} : x_1 > 0, \ldots, x_k > 0\}$. Then

$$\begin{aligned}
\left| \nabla_H h(x,t) \right|^2 &= \left\langle \omega(t)\nabla_x \psi(x) + 2\psi(x)\omega'(t)\Lambda x, \omega(t)\nabla_x \psi(x) + 2\psi(x)\omega'(t)\Lambda x \right\rangle \\
&= \omega^2(t)|\nabla_x \psi|^2 + 4\psi^2|x|^2 \left|\omega'(t)\right|^2 + 4\omega(t)\omega'(t)\psi \langle \Lambda x, \nabla_x \psi \rangle
\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{H}^n_{k_+}} \left|\nabla_H h(x,t)\right|^2 dx\,dt &= \int_{\mathbb{H}^n_{k_+}} \left(\omega^2(t)|\nabla_x\psi|^2 + 4\psi^2|x|^2\left|\omega'(t)\right|^2\right) dx\,dt \\ &\quad + 4\int_{\mathbb{R}^{2n}_{k_+}} \psi\langle\Lambda x, \nabla_x\psi\rangle\, dx \cdot \int_{\mathbb{R}} \omega(t)\omega'(t)\, dt \\ &= \int_{\mathbb{H}^n_{k_+}} \left(\omega^2(t)|\nabla_x\psi|^2 + 4\psi^2|x|^2\left|\omega'(t)\right|^2\right) dx\,dt \\ &\quad + 4\int_{\mathbb{R}^{2n}_{k_+}} \psi\langle\Lambda x, \nabla_x\psi\rangle\, dx \cdot \frac{1}{2}\int_{\mathbb{R}} d\omega^2(t) \\ &= \int_{\mathbb{H}^n_{k_+}} \left(\omega^2(t)|\nabla_x\psi|^2 + 4\psi^2|x|^2\left|\omega'(t)\right|^2\right) dx\,dt.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\int_{\mathbb{H}^n_{k_+}} |\nabla_H h(x,t)|^2\, dx\,dt}{\int_{\mathbb{H}^n_{k_+}} \frac{h^2}{|x|^2}\, dx\,dt} &= \frac{\int_{\mathbb{H}^n_{k_+}} \omega^2(t)|\nabla_x\psi|^2\, dx\,dt + 4\int_{\mathbb{H}^n_{k_+}} \psi^2|x|^2|\omega'(t)|^2\, dx\,dt}{\int_{\mathbb{R}^{2n}_{k_+}} \frac{\psi^2}{|x|^2}\, dx \cdot \int_{\mathbb{R}} \omega^2\, dt} \\ &= \frac{\int_{\mathbb{R}^{2n}_+} |\nabla_x\psi|^2\, dx}{\int_{\mathbb{R}^{2n}_{k_+}} \frac{\psi^2}{|x|^2}\, dx} + 4\frac{\int_{\mathbb{R}} |\omega'(t)|^2\, dt}{\int_{\mathbb{R}} \omega^2\, dt} \cdot \frac{\int_{\mathbb{R}^{2n}_{k_+}} \psi^2|x|^2\, dx}{\int_{\mathbb{R}^{2n}_{k_+}} \frac{\psi^2}{|x|^2}\, dx}.\end{aligned}$$

Thus, by (2.6),

$$\begin{aligned}\inf_{u\in C_0^\infty(\mathbb{H}^n_{k_+})\setminus\{0\}} \frac{\int_{\mathbb{H}^n_{k_+}} |\nabla_H u|^2\, dx\,dt}{\int_{\mathbb{H}^n_{k_+}} \frac{u^2}{|x|^2}\, dx\,dt} &\le \inf_{\psi\in C_0^\infty(\mathbb{R}^{2n})\setminus\{0\}} \frac{\int_{\mathbb{R}^{2n}_+} |\nabla_x\psi|^2\, dx}{\int_{\mathbb{R}^{2n}_{k_+}} \frac{\psi^2}{|x|^2}\, dx} \\ &= (n+k-1)^2.\end{aligned}$$

Here we use the sharp Hardy inequality ([9], Theorem 1.1)

$$\int_{\mathbb{R}^{2n}_+} |\nabla f|^2\, dx \ge (n+k-1)^2 \int_{\mathbb{R}^{2n}_+} \frac{f^2}{|x|^2}\, dx.$$

The proof of Theorem 1.2 is therefore completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgements
The first author is supported by the National Natural Science Foundation of China (No. 11171259) and the second author is supported by the National Natural Science Foundation of China (No. 11201346).

Received: 30 April 2012 Accepted: 23 May 2013 Published: 10 June 2013

References

1. Nazarov, AI: Hardy-Sobolev inequalities in a cone. J. Math. Sci. **132**(4), 419-427 (2006)
2. Filippas, S, Tertikas, A, Tidblom, J: On the structure of Hardy-Sobolev-Maz'ya inequalities. J. Eur. Math. Soc. **11**(6), 1165-1185 (2009)
3. Caldiroli, P, Musina, R: Stationary states for a two-dimensional singular Schrödinger equation. Boll. Unione Mat. Ital, B **4**(3), 609-633 (2001)
4. Caldiroli, P, Musina, R: Rellich inequalities with weights. Calc. Var. Partial Differ. Equ. **45**(1-2), 147-164 (2012). doi:10.1007/s00526-011-0454-3
5. Cazacu, C: On Hardy inequalities with singularities on the boundary. C. R. Math. Acad. Sci. Paris, Sér. I **349**, 273-277 (2011)
6. Cazacu, C: Hardy inequalities with boundary singularities. arXiv:1009.0931
7. Fall, MM: On the Hardy-Poincaré inequality with boundary singularities. arXiv:1008.4785
8. Fall, MM, Musina, R: Hardy-Poincaré inequalities with boundary singularities. Proc. R. Soc. Edinb., Sect. A, Math. **142**(4), 769-786 (2012)
9. Su, D, Yang, Q-H: On the best constants of Hardy inequality in $\mathbb{R}^{n-k} \times (\mathbb{R}_+)^k$ and related improvements. J. Math. Anal. Appl. **389**, 48-53 (2012)
10. Yang, Q: Hardy type inequalities related to Carnot-Carathéodory distance on the Heisenberg group. Proc. Am. Math. Soc. **141**(1), 351-362 (2013)
11. D'Ambrosio, L: Some Hardy inequalities on the Heisenberg group. Differ. Equ. **40**, 552-564 (2004)
12. D'Ambrosio, L: Hardy-type inequalities related to degenerate elliptic differential operators. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **IV**, 451-486 (2005)

doi:10.1186/1029-242X-2013-291
Cite this article as: Liu and Luan: **Hardy-type inequalities on a half-space in the Heisenberg group.** *Journal of Inequalities and Applications* 2013 **2013**:291.