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# New complexity analysis of interior-point methods for the Cartesian $P_*(\kappa)$-SCLCP

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**Abstract**

In this paper, we give a unified analysis for both large- and small-update interior-point methods for the Cartesian $P_*(\kappa)$-linear complementarity problem over symmetric cones based on a finite barrier. The proposed finite barrier is used both for determining the search directions and for measuring the distance between the given iterate and the $\mu$-center for the algorithm. The symmetry of the resulting search directions is forced by using the Nesterov-Todd scaling scheme. By means of Euclidean Jordan algebras, together with the feature of the finite kernel function, we derive the iteration bounds that match the currently best known iteration bounds for large- and small-update methods. Furthermore, our algorithm and its polynomial iteration complexity analysis provide a unified treatment for a class of primal-dual interior-point methods and their complexity analysis.

**MSC:** 90C33; 90C51

**Keywords:** interior-point methods; linear complementarity problem; Cartesian $P_*(\kappa)$-property; Euclidean Jordan algebra; polynomial complexity

## 1 Introduction

Let $(\mathcal{V}, \circ)$ be an $n$-dimensional Euclidean Jordan algebra with rank $r$ equipped with the standard inner product $\langle x, s\rangle = \operatorname{tr}(x \circ s)$. Let $\mathcal{K}$ be the corresponding symmetric cone. For a linear transformation $\mathcal{A} : \mathcal{V} \to \mathcal{V}$ and a $q \in \mathcal{V}$, the linear complementarity problem over symmetric cones, denoted by SCLCP, is to find $x, s \in \mathcal{V}$ such that

$$x \in \mathcal{K}, \quad s = \mathcal{A}(x) + q \in \mathcal{K} \quad \text{and} \quad x \circ s = 0. \tag{1}$$

Note that $\langle x, s\rangle = 0 \Leftrightarrow x \circ s = 0$ (Lemma 2.2 in [1]).

The SCLCP is a wide class of problems that contains linear complementarity problem (LCP), second-order cone linear complementarity problem (SOCLCP) and semidefinite linear complementarity problem (SDLCP) as special cases. For an overview of these and related results, we refer to the survey paper [2] and references within.

There are many solution approaches for SCLCP. Among them, the interior-point methods (IPMs) gain much more attention. Faybusovich [3] made the first attempt to generalize IPMs to symmetric optimization (SO) and SCLCP using the 'machinery' of Euclidean Jordan algebras. Potra [4] proposed an infeasible corrector-predictor IPM for the monotone SCLCP. Yoshise [5] proposed the homogeneous model for the monotone nonlinear complementarity problems (NCP) over symmetric cones (SCNCP) and analyzed IPM to solve it.

Let $\mathcal{V}$ be a Cartesian product of a finite number of simple Euclidean Jordan algebras $(\mathcal{V}_j, \circ)$ with dimensions $n_j$ and ranks $r_j$ for $j = 1, \ldots, N$, that is, $\mathcal{V} = \mathcal{V}_1 \times \cdots \times \mathcal{V}_N$ with its cone of squares $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_N$, where $\mathcal{K}_j$ are the corresponding cones of squares of $\mathcal{V}_j$ for $j = 1, \ldots, N$. The dimension of $\mathcal{V}$ is $n = \sum_{j=1}^{N} n_j$ and the rank is $r = \sum_{j=1}^{N} r_j$. Recall that a Euclidean Jordan algebra is said to be simple if it cannot be represented as the orthogonal direct sum of two Euclidean Jordan algebras.

We call SCLCP the Cartesian $P_*(\kappa)$-SCLCP if the linear transformation $\mathcal{A}$ has the Cartesian $P_*(\kappa)$-property for some nonnegative constant $\kappa$, *i.e.*,

$$(1 + 4\kappa) \sum_{j \in \mathcal{I}_+(x)} \langle x^{(j)}, [\mathcal{A}(x)]^{(j)} \rangle + \sum_{j \in \mathcal{I}_-(x)} \langle x^{(j)}, [\mathcal{A}(x)]^{(j)} \rangle \geq 0, \quad \forall x \in \mathcal{V}, \tag{2}$$

where $\mathcal{I}_+(x) = \{j : \langle x^{(j)}, [\mathcal{A}(x)]^{(j)} \rangle > 0\}$ and $\mathcal{I}_-(x) = \{j : \langle x^{(j)}, [\mathcal{A}(x)]^{(j)} \rangle < 0\}$ are two index sets. It is closely related to the Cartesian $P_0$- and $P$-properties which were first introduced by Chen and Qi [6] over the space of symmetric matrices, and later extended by Pan and Chen [7] and Luo and Xiu [8] to the space of second-order cones and the general Euclidean Jordan algebras, respectively.

The Cartesian $P_*(\kappa)$-SCLCP is indeed the generalization of $P_*(\kappa)$-LCP, which was first introduced by Kojima *et al.* [9]. They established the existence of the central path and designed and analyzed IPMs for solving $P_*(\kappa)$-LCP. The theoretical importance of this class of LCPs lays in the fact that this is the largest class for which polynomial convergence of IPMs can be proved without additional conditions (such as boundedness of the level sets).

Luo and Xiu [8] were the first to establish a theoretical framework of path-following interior-point algorithms for the Cartesian $P_*(\kappa)$-SCLCP and to prove the global convergence and the iteration complexities of the proposed algorithms. Wang and Bai [10] analyzed a class of IPMs for the Cartesian $P_*(\kappa)$-SCLCP based on a parametric kernel function different from the logarithmic kernel function. Lesaja *et al.* [11] gave a unified analysis of kernel-based IPMs for the Cartesian $P_*(\kappa)$-SCLCP and derived the currently best known iteration bounds for large- and small-update methods for some special eligible kernel functions. Wang and Lesaja [12] generalize Roos's full-Newton step feasible IPM for LO [13] and Gu *et al.* extension to SO [14], to the Cartesian $P_*(\kappa)$-SCLCP. Liu *et al.* [15] proposed smoothing Newton methods for the Cartesian $P_0$- and $P$-SCLCPs. Huang and Lu [16] presented a globally convergent smoothing method with a linear rate of convergence for the Cartesian $P_*(\kappa)$-SCLCP.

Bai *et al.* [17] introduced a finite kernel function as follows:

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{e^{\sigma(1-t)} - 1}{\sigma}, \quad \sigma \geq 1, \tag{3}$$

which is not a kernel function in the usual sense (see, *e.g.*, [18, 19]). It has a finite value at the boundary of the feasible region, *i.e.*,

$$\lim_{t \downarrow 0} \psi(t) = \psi(0) = -\frac{1}{2} + \frac{e^{\sigma} - 1}{\sigma} < \infty. \tag{4}$$

However, the iteration bound of a large-update method based on this kernel function is shown to be $O(\sqrt{n} \log n \log \frac{n}{\varepsilon})$. Recently, Ghami *et al.* [20] studied the generalization of

the finite kernel function $\psi(t)$ as follows:

$$\psi_{p,\sigma}(t) = \frac{t^{p+1}-1}{p+1} + \frac{e^{\sigma(1-t)}-1}{\sigma}, \quad p \in [0,1], \sigma \geq 1. \tag{5}$$

This parametric kernel function also has finite value at the boundary of the feasible region and its growth terms are between linear and quadratic. They proposed a class of primal-dual interior-point algorithms for LO and the extension to SDO [21] based on the parametric kernel function $\psi_{p,\sigma}(t)$, respectively. Meanwhile, the results for LO in [17, 20] were extended to $P_*(\kappa)$-LCP by Wang and Bai in [22], again matching the best known iteration bounds for LO with the addition $1 + 2\kappa$. An interesting question here is whether we can directly extend the interior-point algorithms for LO in [17] to the Cartesian $P_*(\kappa)$-SCLCP. As we will see later, LO cannot be trivially generalized to the Cartesian $P_*(\kappa)$-SCLCP context. The analysis of the algorithm proposed in this paper is more complicated than in the LO case mainly due to the fact that the search directions are no longer orthogonal.

In this paper, we consider a generalization of kernel-based IPMs discussed in the paper [17] to the Cartesian $P_*(\kappa)$-SCLCP. The paper also extends the results of the paper [22] where we consider the same type of IPMs for $P_*(\kappa)$-LCP, however, only over the nonnegative orthant. Our goal is to provide a unified analysis for both large- and small-update IPMs for the Cartesian $P_*(\kappa)$-SCLCP based on the finite barrier. Although the proposed algorithm is an exact extension of the algorithms for LO and $P_*(\kappa)$-LCP, the Cartesian $P_*(\kappa)$-property makes the analysis of the method far more complicated. Furthermore, we loose the orthogonality of the scaled search directions in the Cartesian $P_*(\kappa)$-SCLCP case. This also yields many difficulties in the analysis of the algorithm for the Cartesian $P_*(\kappa)$-SCLCP. However, we manage to prove the same good characteristics as in the LO case. The obtained complexity results match the best known iteration bounds known for large- and small-update methods, namely $O((1+2\kappa)\sqrt{r}\log r \log\frac{r}{\varepsilon})$ and $O((1+2\kappa)\sqrt{r}\log\frac{r}{\varepsilon})$, respectively. The order of the iteration bounds almost coincides with the bounds derived for LO in [17], except that the iteration bounds in the Cartesian $P_*(\kappa)$-SCLCP case are multiplied by the factor $(1 + 2\kappa)$.

The paper is organized as follows. In Section 2, we briefly describe some concepts, properties, and results from Euclidean Jordan algebras. In Section 3, we provide and develop some useful properties of the finite kernel function $\psi(t)$ and the corresponding barrier function $\Psi(v)$. In Section 4, we mainly study primal-dual IPMs for the Cartesian $P_*(\kappa)$-SCLCP based on the finite kernel function. The analysis and complexity bounds of the algorithm are presented in Sections 5 and 6, respectively. Finally, some conclusions and remarks follow in Section 7.

Notations used throughout the paper are as follows. $\mathbf{R}^n$, $\mathbf{R}^n_+$, and $\mathbf{R}^n_{++}$ denote the set of all vectors (with $n$ components), the set of nonnegative vectors, and the set of positive vectors, respectively. The largest eigenvalue of $x$ and the smallest eigenvalue of $x$ are defined by $\lambda_{\max}(x)$ and $\lambda_{\min}(x)$. The Löwner partial ordering '$\succeq_{\mathcal{K}}$' of $\mathcal{V}$ defined by a symmetric cone $\mathcal{K}$ is defined by $x \succeq_{\mathcal{K}} s$ if $x - s \in \mathcal{K}$. The interior of $\mathcal{K}$ is denoted as $\operatorname{int}\mathcal{K}$, and we write $x \succ_{\mathcal{K}} s$ if $x - s \in \operatorname{int}\mathcal{K}$. Finally, if $g(x) \geq 0$ is a real-valued function of a real nonnegative variable, the notation $g(x) = O(x)$ means that $g(x) \leq \bar{c}x$ for some positive constant $\bar{c}$, and $g(x) = \Theta(x)$ that $c_1 x \leq g(x) \leq c_2 x$ for two positive constants $c_1$ and $c_2$.

## 2 Preliminaries

In what follows, we assume that the reader is familiar with the basic concepts of Euclidean Jordan algebras and symmetric cones. The detailed information can be found in the monograph of Faraut and Korányi [23] and in [1, 14, 24–29] as it relates to optimization.

The bilinear form on $\mathcal{V}$ is defined as

$$x \diamond s := \left(x^{(1)} \circ s^{(1)}, \ldots, x^{(N)} \circ s^{(N)}\right)^T, \tag{6}$$

where $x = (x^{(1)}, \ldots, x^{(N)})^T$ and $s = (s^{(1)}, \ldots, s^{(N)})^T$ in $\mathcal{V}$ with $x^{(j)}, s^{(j)} \in \mathcal{V}_j$, $j = 1, \ldots, N$. If $e^{(j)} \in \mathcal{V}_j$ is the identity element in the Euclidean Jordan algebra $\mathcal{V}_j$, then the vector

$$e = \left(e^{(1)}, \ldots, e^{(N)}\right)^T \tag{7}$$

is the identity element in $\mathcal{V}$.

For each $x^{(j)} \in \mathcal{V}_j$ with $j = 1, \ldots, N$, the Lyapunov transformation and the quadratic representation of $\mathcal{V}_j$ are given by

$$L\left(x^{(j)}\right)y^{(j)} = x^{(j)} \circ y^{(j)} \quad \text{and} \quad P\left(x^{(j)}\right) := 2L\left(x^{(j)}\right)^2 - L\left(\left(x^{(j)}\right)^2\right), \tag{8}$$

where $L(x^{(j)})^2 = L(x^{(j)})L(x^{(j)})$. They can be adjusted to the Cartesian product structure $\mathcal{V}$ as follows:

$$L(x) = \operatorname{diag}\left(L\left(x^{(1)}\right), \ldots, L\left(x^{(N)}\right)\right) \quad \text{and} \quad P(x) = \operatorname{diag}\left(P\left(x^{(1)}\right), \ldots, P\left(x^{(N)}\right)\right). \tag{9}$$

The spectral decomposition of $x^{(j)} \in \mathcal{V}_j$ with respect to the Jordan frame $\{c_1^{(j)}, \ldots, c_{r_j}^{(j)}\}$ is given by

$$x^{(j)} = \sum_{i=1}^{r_j} \lambda_i\left(x^{(j)}\right)c_i^{(j)}, \quad j = 1, \ldots, N, \tag{10}$$

where $\lambda_1(x^{(j)}), \ldots, \lambda_{r_j}(x^{(j)})$ are the corresponding eigenvalues. The spectral decomposition of $x \in \mathcal{V}$ can be defined straightforwardly by using the spectral decomposition of components $x^{(j)} \in \mathcal{V}_j$ as follows:

$$x = \left(\sum_{i=1}^{r_1} \lambda_i\left(x^{(1)}\right)c_i^{(1)}, \ldots, \sum_{i=1}^{r_N} \lambda_i\left(x^{(N)}\right)c_i^{(N)}\right)^T. \tag{11}$$

It enables us to extend the definition of any real-valued, continuous univariate function to elements of a Euclidean Jordan algebra, using the eigenvalues. In particular this holds for the finite kernel function.

Let $x \in \mathcal{V}$ with the spectral decomposition as defined (11). The vector-valued function $\psi(x)$ is defined by

$$\psi(x) = \left(\psi\left(x^{(1)}\right), \ldots, \psi\left(x^{(N)}\right)\right)^T, \tag{12}$$

where

$$\psi\left(x^{(j)}\right) = \psi\left(\lambda_1\left(x^{(j)}\right)\right)c_1^{(j)} + \cdots + \psi\left(\lambda_{r_j}\left(x^{(j)}\right)\right)c_{r_j}^{(j)}, \quad j = 1, \ldots, N. \tag{13}$$

Furthermore, if $\psi(t)$ is differentiable, the derivative $\psi'(t)$ exists, and we also have a vector-valued function $\psi'(x)$, namely

$$\psi'(x) = \left(\psi'\left(x^{(1)}\right), \ldots, \psi'\left(x^{(N)}\right)\right)^T, \tag{14}$$

where

$$\psi'\left(x^{(j)}\right) = \psi'\left(\lambda_1\left(x^{(j)}\right)\right)c_1^{(j)} + \cdots + \psi'\left(\lambda_{r_j}\left(x^{(j)}\right)\right)c_{r_j}^{(j)}, \quad j = 1, \ldots, N. \tag{15}$$

It should be noted that $\psi'(x)$, which does not mean that the derivative of the vector-valued function $\psi(x)$ defined by (12) is just a vector-valued function induced by the derivative $\psi'(t)$ of the function $\psi(t)$.

The Peirce decomposition of $x^{(j)} \in \mathcal{V}_j$ with respect to the Jordan frame $\{c_1^{(j)}, \ldots, c_{r_j}^{(j)}\}$ is given by

$$x^{(j)} = \sum_{i=1}^{r_j} x_i^{(j)} c_i^{(j)} + \sum_{i<m_j} x_{im_j}^{(j)}, \quad j = 1, \ldots, N, \tag{16}$$

with $x_i^{(j)} \in \mathbf{R}$, $i = 1, \ldots, r_j$ and $x_{im_j}^{(j)} \in \mathcal{V}_{im_j}^{(j)}$, $1 \le i < m_j \le r_j$. The $\mathcal{V}_{im_j}^{(j)}$ for $1 \le i < m_j \le r_j$ are the Peirce subspaces of $\mathcal{V}_j$ induced by the Jordan frame $c_1^{(j)}, \ldots, c_{r_j}^{(j)}$. The Peirce decomposition of $x \in \mathcal{V}$ can be defined straightforwardly by using the Peirce decomposition of components $x^{(j)} \in \mathcal{V}_j$ as follows:

$$x = \left(\sum_{i=1}^{r_1} x_i^{(1)} c_i^{(1)} + \sum_{i<m_1} x_{im_1}^{(1)}, \ldots, \sum_{i=1}^{r_N} x_i^{(N)} c_i^{(N)} + \sum_{i<m_N} x_{im_N}^{(N)}\right)^T. \tag{17}$$

The canonical inner product is defined as

$$\langle x, s\rangle = \sum_{j=1}^{N} \left\langle x^{(j)}, s^{(j)}\right\rangle = \sum_{j=1}^{N} \operatorname{tr}\left(x^{(j)} \circ s^{(j)}\right). \tag{18}$$

We recall the following definitions:

$$\begin{aligned} &\operatorname{tr}\left(x^{(j)}\right) = \sum_{i=1}^{r_j} \lambda_i\left(x^{(j)}\right), \qquad \det\left(x^{(j)}\right) = \prod_{i=1}^{r_j} \lambda_i\left(x^{(j)}\right) \quad \text{and} \\ &\left\|x^{(j)}\right\|_F = \sqrt{\sum_{i=1}^{r_j} \lambda_i^2\left(x^{(j)}\right)}, \quad j = 1, \ldots, N. \end{aligned} \tag{19}$$

Then, in $\mathcal{V}$ we have

$$\operatorname{tr}(x) = \sum_{j=1}^{N} \operatorname{tr}\left(x^{(j)}\right), \qquad \det(x) = \prod_{j=1}^{N} \det\left(x^{(j)}\right) \quad \text{and} \quad \|x\|_F = \sqrt{\sum_{j=1}^{N} \left\|x^{(j)}\right\|_F^2}. \tag{20}$$

Furthermore, we define

$$\lambda_{\max}(v) = \max\{\lambda_i(x^{(j)}) : j = 1,\dots,N, 1 \le i \le r_j\} \tag{21}$$

and

$$\lambda_{\min}(v) = \min\{\lambda_i(x^{(j)}) : j = 1,\dots,N, 1 \le i \le r_j\}. \tag{22}$$

It follows from (21), (22), and (20) that

$$|\lambda_{\max}(x)| \le \|x\|_F \quad \text{and} \quad |\lambda_{\min}(x)| \le \|x\|_F. \tag{23}$$

Furthermore, we have the following important result.

**Lemma 2.1** (Lemma 14 in [28]) *Let $x, s \in \mathcal{V}$. Then*

$$\lambda_{\min}(x) - \|s\|_F \le \lambda_{\min}(x+s) \le \lambda_{\max}(x+s) \le \lambda_{\max}(x) + \|s\|_F.$$

Before ending this section, we need to consider the separable spectral functions induced by the univariate functions. Let $f : D \to \mathbf{R}$ be a univariate function on the open set $D \subseteq \mathbf{R}$ that is differentiable or even continuously differentiable if necessary. Let $x^{(j)} = \sum_{i=1}^{r_j} \lambda_i(x^{(j)})c_i^{(j)}$ be the spectral decomposition of $x^{(j)} \in \mathcal{V}_j$ with respect to the Jordan frame $c_1^{(j)},\dots,c_{r_j}^{(j)}$ for each $j$, $j = 1,\dots,N$. Then we define the real -valued separable spectral function $F(x^{(j)}) : \mathcal{V}_j \to \mathbf{R}$ and the vector-valued separable spectral function $G : \mathcal{V}_j \to \mathcal{V}_j$ by

$$F(x^{(j)}) := \sum_{i=1}^{r_j} f(\lambda_i(x^{(j)})) \quad \text{and} \quad G(x) := \sum_{i=1}^{r_j} f(\lambda_i(x^{(j)}))c_i^{(j)}, \quad j = 1,\dots,N, \tag{24}$$

respectively. The first derivative $D_xF(x^{(j)})$ of the function $F(x^{(j)})$ and the first derivative $D_xG(x^{(j)})$ of the function $G(x^{(j)})$ are given by

$$D_xF(x^{(j)}) = \sum_{i=1}^{r_j} f'(\lambda_i^{(j)})c_i^{(j)} \tag{25}$$

and

$$D_xG(x^{(j)}) = \sum_{i=1}^{r_j} f'(\lambda_i^{(j)})\mathcal{V}_{ii}^{(j)} + \sum_{\substack{i<m_j \\ \lambda_i^{(j)}=\lambda_{m_j}^{(j)}}} f'(\lambda_i^{(j)})\mathcal{V}_{im_j}^{(j)} + \sum_{\substack{i<m_j \\ \lambda_i^{(j)}\neq\lambda_{m_j}^{(j)}}} \frac{f(\lambda_i^{(j)}) - f(\lambda_{m_j}^{(j)})}{\lambda_i^{(j)} - \lambda_{m_j}^{(j)}}\mathcal{V}_{im_j}^{(j)}, \tag{26}$$

respectively, where $\lambda_i^{(j)} = \lambda_i(x^{(j)})$, $\lambda_{m_j}^{(j)} = \lambda_{m_j}(x^{(j)})$, and $\mathcal{V}_{im_j}^{(j)}$, $1 \le i \le m_j \le r$, are orthogonal projection operators that appear in the Peirce decomposition of $\mathcal{V}_j$ with respect to the Jordan frame $c_1^{(j)},\dots,c_{r_j}^{(j)}$.

The above results, as well as a more general treatment of spectral functions, their derivatives and various properties can be found in [24, 27].

Now, the separable spectral functions can be adjusted to the Cartesian product structure $\mathcal{V}$ as follows:

$$F(x) = \sum_{j=1}^{N} F\big(x^{(j)}\big) \quad \text{and} \quad G(x) = \big(G\big(x^{(1)}\big), \ldots, G\big(x^{(N)}\big)\big)^T. \tag{27}$$

It follows directly from (25) and (26) that

$$\begin{aligned} &D_x F(x) = \big(D_x F\big(x^{(1)}\big), \ldots, D_x F\big(x^{(N)}\big)\big)^T \quad \text{and} \\ &D_x G(x) = \big(D_x G\big(x^{(1)}\big), \ldots, D_x G\big(x^{(N)}\big)\big)^T. \end{aligned} \tag{28}$$

## 3 Properties of the finite kernel (barrier) function

In this section, we provide and develop some useful properties of the finite kernel function and the corresponding barrier function that are needed in the analysis of the algorithm. For ease of reference, we give the first three derivatives of $\psi(t)$ with respect to $t$ as follows:

$$\psi'(t) = t - e^{\sigma(1-t)}, \qquad \psi''(t) = 1 + \sigma e^{\sigma(1-t)} \quad \text{and} \quad \psi'''(t) = -\sigma^2 e^{\sigma(1-t)}. \tag{29}$$

We can conclude that

$$\begin{aligned} &\psi(1) = \psi'(1) = 0, \qquad \psi''(t) > 0, \quad t > 0, \\ &\psi'''(t) < 0, \quad t > 0 \quad \text{and} \quad \lim_{t \to \infty} \psi(t) = +\infty. \end{aligned} \tag{30}$$

It follows from (30) that $\psi(t)$ is strictly convex and $\psi''(t)$ is monotonically decreasing in $t \in (0, +\infty)$.

The property described below in Lemma 3.1 is exponential convexity, which has been proven to be very useful in the analysis of kernel-based primal-dual IPMs (see, *e.g.*, [18, 19]).

**Lemma 3.1** (Lemma 2.4 in [17]) *If* $t_1 \geq \frac{1}{\sigma}$ *and* $t_2 \geq \frac{1}{\sigma}$, *then*

$$\psi\big(\sqrt{t_1 t_2}\big) \leq \frac{1}{2}\big(\psi(t_1) + \psi(t_2)\big).$$

Note that $\psi(t)$ is exponentially convex, whenever $t \geq \frac{1}{\sigma}$. The following lemma makes clear that when $v$ belongs to the level set $\{v : \Psi(v) \leq L\}$, for some given $L \geq 8$, the exponential convexity is guaranteed and it is proved that the value of $\sigma$ is large enough.

**Lemma 3.2** (Lemma 2.5 in [17]) *Let* $L \geq 8$ *and* $\Psi(v) \leq L$. *If* $\sigma \geq 1 + 2\log(1 + L)$, *then* $\lambda_{\min}(v) \geq \frac{3}{2\sigma}$.

Corresponding to the finite kernel function $\psi(t)$ defined by (3), we define the barrier function on $\operatorname{int} \mathcal{K}$ as follows:

$$\Psi(v) := \Psi(x, s; \mu) := \operatorname{tr}\big(\psi(v)\big). \tag{31}$$

It follows immediately from (12) and (20) that

$$\Psi(v) = \operatorname{tr}\big(\psi(v)\big) = \sum_{j=1}^{N} \operatorname{tr}\big(\psi\big(v^{(j)}\big)\big) = \sum_{j=1}^{N}\sum_{i=1}^{r_j} \psi\big(\lambda_i\big(v^{(j)}\big)\big). \tag{32}$$

According to the properties of the finite kernel function $\psi(t)$, we can conclude that $\Psi(v)$ is nonnegative and strictly convex with respect to $v \succ_{\mathcal{K}} 0$ and vanishes at its global minimal point $v = e$, *i.e.*,

$$\Psi(v) = 0 \quad \Leftrightarrow \quad \psi(v) = 0 \quad \Leftrightarrow \quad \psi'(v) = 0 \quad \Leftrightarrow \quad v = e.$$

Furthermore, we have, by (28),

$$D_v\Psi(v) = \left(\sum_{i=1}^{r_1} \psi'\big(\lambda_i\big(v^{(1)}\big)\big)c_i^{(1)}, \ldots, \sum_{i=1}^{r_N} \psi'\big(\lambda_i\big(v^{(N)}\big)\big)c_i^{(N)}\right)^T = \psi'(v). \tag{33}$$

This means that the derivative of the barrier function $\Psi(v)$ in essence coincides with the vector-valued function $\psi'(v)$ defined by (14) and (15).

As the consequence of Lemma 3.1, we have the following theorem, which is indeed a slight modification of Theorem 4.3.2 in [29]. Thus, we omit its proof.

**Theorem 3.3** *Let $x, s \in \operatorname{int}\mathcal{K}$. If $\lambda_{\min}(x) \geq \frac{1}{\sigma}$ and $\lambda_{\min}(s) \geq \frac{1}{\sigma}$, then*

$$\Psi\Big(\big(P(x)^{1/2}s\big)^{1/2}\Big) \leq \frac{1}{2}\big(\Psi(x) + \Psi(s)\big).$$

**Lemma 3.4** *If $t \geq 1$, then*

$$\psi(t) \leq \frac{1+\sigma}{2}(t-1)^2.$$

*Proof* From Taylor's theorem and the fact that $\psi''(1) = 1 + \sigma$, the inequality is straightforward. □

**Lemma 3.5** *If $t \geq 1$, then*

$$t\psi'(t) \geq \psi(t).$$

*Proof* Defining $f(t) := t\psi'(t) - \psi(t)$, we have $f(1) = 0$ and

$$f'(t) = t\psi''(t) \geq 0.$$

This implies the desired result. □

The following lemma can be directly obtained from Lemma 2.5 in [22], which provides the lower and upper bounds of the inverse function of the finite kernel function $\psi(t)$ for $t \geq 1$.

**Lemma 3.6** *Let $\varrho : [0,\infty) \to [1,\infty)$ be the inverse function of the finite kernel function $\psi(t)$ for $t \geq 1$. If $\sigma \geq 1$, then*

$$\sqrt{1+2s} \leq \varrho(s) \leq \left(2s + \frac{2+\sigma}{\sigma}\right)^{\frac{1}{2}}. \tag{34}$$

*If $\sigma \geq 2$, then*

$$\varrho(s) \leq 1 + \sqrt{s}\left(2s + \frac{2+\sigma}{\sigma}\right)^{\frac{1}{4}}. \tag{35}$$

For the analysis of the algorithm, we define the norm-based proximity measure $\delta(v)$ as follows:

$$\delta(v) := \frac{1}{2}\left\|\psi'(v)\right\|_F. \tag{36}$$

It follows from (14) and (20) that

$$\delta(v) = \frac{1}{2}\sqrt{\sum_{j=1}^{N}\sum_{i=1}^{r_j}\psi'\left(\lambda_i\left(v^{(j)}\right)\right)^2}. \tag{37}$$

We can conclude that $\delta(v) \geq 0$ and $\delta(v) = 0$ if and only if $\Psi(v) = 0$.

Clearly, $\delta(v)$ and $\Psi(v)$ depend only on the eigenvalues $\lambda_i(v^{(j)})$ of the symmetric cone $v^{(j)}$ for each $j$, $j = 1,\ldots,N$. The following theorem gives a lower bound on $\delta(v)$ in terms of $\Psi(v)$, which is precisely the same as its LO counterpart $\delta(v)$ (*cf.* Theorem 4.8 in [18]).

**Theorem 3.7** *If $v \in \operatorname{int}\mathcal{K}$, then*

$$\delta(v) \geq \frac{1}{2}\psi'\left(\varrho\left(\Psi(v)\right)\right).$$

**Corollary 3.8** *If $v \in \operatorname{int}\mathcal{K}$ and $\Psi(v) \geq 1$, then*

$$\delta(v) \geq \frac{1}{6}\sqrt{\Psi(v)}.$$

*Proof* From (34) and the fact that $\Psi(v) \geq 1$ and $\sigma \geq 1$, we have

$$\varrho\left(\Psi(v)\right) \leq \left(2\Psi(v) + \frac{2+\sigma}{\sigma}\right)^{\frac{1}{2}} \leq \left(5\Psi(v)\right)^{\frac{1}{2}} \leq 3\left(\Psi(v)\right)^{\frac{1}{2}}.$$

Thus, we have, by Theorem 3.7 and Lemma 3.5,

$$\delta(v) \geq \frac{1}{2}\psi'\left(\varrho\left(\Psi(v)\right)\right) \geq \frac{\psi(\varrho(\Psi(v)))}{2\varrho(\Psi(v))} = \frac{\Psi(v)}{2\varrho(\Psi(v))} \geq \frac{\Psi(v)}{6(\Psi(v))^{\frac{1}{2}}} = \frac{1}{6}\sqrt{\Psi(v)}.$$

This completes the proof of the corollary. □

In what follows, we consider the derivatives of the function $\Psi(x(t))$ with respect to $t$, where $x(t) = x_0 + tu \in \operatorname{int}\mathcal{K}$ with $t \in \mathbf{R}$ and $u \in \mathcal{V}$. For more details, we refer to [29].

It follows from (11) and (17) that the spectral decomposition of $x(t)$ with respect to the Jordan frame $\{c_1^{(1)},\dots,c_{r_1}^{(1)},\dots,c_1^{(N)},\dots,c_{r_N}^{(N)}\}$ can be defined by

$$x(t)=\left(\sum_{i=1}^{r_1}\lambda_i\big(x(t)^{(1)}\big)c_i^{(1)},\dots,\sum_{i=1}^{r_N}\lambda_i\big(x(t)^{(N)}\big)c_i^{(N)}\right)^T, \tag{38}$$

and the Pierce decomposition of $u$ can be defined by

$$u=\left(\sum_{i=1}^{r_1}u_i^{(1)}c_i^{(1)}+\sum_{i<m_1}u_{im_1}^{(1)},\dots,\sum_{i=1}^{r_N}u_i^{(N)}c_i^{(N)}+\sum_{i<m_N}u_{im_N}^{(N)}\right)^T. \tag{39}$$

From (28), after some elementary reductions, we can derive the first two derivatives of the general function $\Psi(x(t))$ with respect to $t$ as follows:

$$D_t\Psi\big(x(t)\big)=\sum_{j=1}^{N}\operatorname{tr}\left(\sum_{i=1}^{r_j}\psi'\big(\lambda_i\big(x(t)^{(j)}\big)\big)c_i^{(j)}\circ u^{(j)}\right) \tag{40}$$

and

$$\begin{aligned}D_t^2\Psi\big(x(t)\big)=\sum_{j=1}^{N}\Bigg(&\sum_{i=1}^{r_j}\psi''\big(\lambda_i^{(j)}\big)\big(u_i^{(j)}\big)^2+\sum_{\substack{i<m_j\\ \lambda_i^{(j)}=\lambda_{m_j}^{(j)}}}\psi''\big(\lambda_i^{(j)}\big)\operatorname{tr}\big(\big(u_{im_j}^{(j)}\big)^2\big)\\&+\sum_{\substack{i<m_j\\ \lambda_i^{(j)}\neq\lambda_{m_j}^{(j)}}}\frac{\psi'(\lambda_i^{(j)})-\psi'(\lambda_{m_j}^{(j)})}{\lambda_i^{(j)}-\lambda_{m_j}^{(j)}}\operatorname{tr}\big(\big(u_{im_j}^{(j)}\big)^2\big)\Bigg),\end{aligned} \tag{41}$$

where $\lambda_i^{(j)}=\lambda_i(x(t)^{(j)})$ and $\lambda_{m_j}^{(j)}=\lambda_{m_j}(x(t)^{(j)})$.

Note that $\psi''(t)$ is monotonically decreasing in $t\in(0,+\infty)$. Under the assumption that $i<m_j$ implies $\lambda_i(x(t))\geq\lambda_{m_j}(x(t))$, we can conclude that

$$D_t^2\Psi\big(x(t)\big)\leq\sum_{j=1}^{N}\left(\sum_{i=1}^{r_j}\psi''\big(\lambda_i^{(j)}\big)\big(u_i^{(j)}\big)^2+\sum_{i<m_j}\psi''\big(\lambda_{m_j}^{(j)}\big)\operatorname{tr}\big(\big(u_{im_j}^{(j)}\big)^2\big)\right), \tag{42}$$

which bounds the second-order derivative of $\Psi(x(t))$ with respect to $t$.

## 4 Interior-point algorithm for the Cartesian $P_*(\kappa)$-SCLCP

In this section, we first introduce the central path for the Cartesian $P_*(\kappa)$-SCLCP. Next, we mainly derive the new search directions induced by the finite kernel function $\psi(t)$. Finally, we present the generic polynomial interior-point algorithm for the Cartesian $P_*(\kappa)$-SCLCP.

### 4.1 The central path for the Cartesian $P_*(\kappa)$-SCLCP

Throughout the paper, we assume that the Cartesian $P_*(\kappa)$-SCLCP satisfies the interior-point condition (IPC), *i.e.*, there exists $(x^0\succ_{\mathcal{K}}0, s^0\succ_{\mathcal{K}}0)$ such that $s^0=\mathcal{A}(x^0)+q$. For this

and other properties of the Cartesian $P_*(\kappa)$-SCLCP, we refer to [8]. Under the IPC holds, by relaxing the complementarity slackness $x \diamond s = 0$ with $x \diamond s = \mu e$, we obtain the following system:

$$\begin{pmatrix} \mathcal{A}(x) - s \\ x \diamond s \end{pmatrix} = \begin{pmatrix} -q \\ \mu e \end{pmatrix}, \quad x, s \succeq_{\mathcal{K}} 0, \tag{43}$$

where $\mu > 0$ is a parameter. The parameterized system (43) has a unique solution for each $\mu > 0$. This solution is denoted as $(x(\mu), s(\mu))$ and we call $(x(\mu), s(\mu))$ the $\mu$-center of the Cartesian $P_*(\kappa)$-SCLCP. The set of $\mu$-centers (with $\mu$ running through all positive real numbers) gives a homotopy path, which is called *the central path* of the Cartesian $P_*(\kappa)$-SCLCP. If $\mu \to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition $x \diamond s = 0$, the limit yields a solution for the Cartesian $P_*(\kappa)$-SCLCP (see, *e.g.*, [8]).

### 4.2 The new search directions for the Cartesian $P_*(\kappa)$-SCLCP

To obtain the search directions for the Cartesian $P_*(\kappa)$-SCLCP, the usual approach is to use Newton's method and to linearize the system (43). In what follows, we briefly outline the details.

For any strictly feasible $x \succ_{\mathcal{K}} 0$ and $s \succ_{\mathcal{K}} 0$, we want to find displacements $\Delta x$ and $\Delta s$ such that

$$\begin{pmatrix} \mathcal{A}(x + \Delta x) - (s + \Delta s) \\ (x + \Delta x) \diamond (s + \Delta s) \end{pmatrix} = \begin{pmatrix} -q \\ \mu e \end{pmatrix}. \tag{44}$$

Neglecting the term $\Delta x \diamond \Delta s$ on the left-hand side expression of the second equation, we obtain the following Newton system for the search directions $\Delta x$ and $\Delta s$:

$$\begin{pmatrix} \mathcal{A}(\Delta x) - \Delta s \\ s \diamond \Delta x + x \diamond \Delta s \end{pmatrix} = \begin{pmatrix} 0 \\ \mu e - x \diamond s \end{pmatrix}. \tag{45}$$

Due to the fact that $x$ and $s$ do not operator commute in general, *i.e.*, $L(x)L(s) \neq L(s)L(x)$, this system does not always have a unique solution. It is well known that this difficulty can be solved by applying a scaling scheme. This goes as follows.

**Lemma 4.1** (Lemma 28 in [28]) *Let $u \in \operatorname{int} \mathcal{K}$. Then*

$$x \diamond s = \mu e \quad \Leftrightarrow \quad P(u)x \diamond P\big(u^{-1}\big)s = \mu e.$$

Now we replace the second equation of the system (44) by

$$P(u)(x + \Delta x) \diamond P\big(u^{-1}\big)(s + \Delta s) = \mu e. \tag{46}$$

Applying Newton's method again, and neglecting the term $P(u)\Delta x \diamond P(u^{-1})\Delta s$, we get

$$\begin{pmatrix} \mathcal{A}(\Delta x) - \Delta s \\ P(u^{-1})(s) \diamond P(u)\Delta x + P(u)(x) \diamond P(u^{-1})\Delta s \end{pmatrix} = \begin{pmatrix} 0 \\ \mu e - P(u)(x) \diamond P(u^{-1})(s) \end{pmatrix}. \tag{47}$$

By choosing $u$ appropriately, this system can be used to define the commutative class of search directions (see, *e.g.*, [28]). In the literature the following three choices are well known: $u = s^{1/2}$, $u = x^{1/2}$, and $u = w^{-1/2}$, where $w$ is the NT-scaling point of $x$ and $s$. The first two choices lead to the so-called $xs$-direction and $sx$-direction, respectively. In this paper, we consider the third choice that is called NT-scaling scheme and the resulting direction is called NT search direction. This scaling scheme was first proposed by Nesterov and Todd for self-scaled cones [30, 31] and then adapted by Faybusovich [1, 26] for symmetric cones.

**Lemma 4.2** (Lemma 3.2 in [26]) *Let $x, s \in \operatorname{int} \mathcal{K}$. Then there exists a unique scaling point $w \in \operatorname{int} \mathcal{K}$ such that*

$$x = P(w)s.$$

*Moreover,*

$$w = P(x)^{\frac{1}{2}}\left(P\left(x^{\frac{1}{2}}\right)s\right)^{-\frac{1}{2}}\left[= P\left(s^{-\frac{1}{2}}\right)\left(P\left(s^{\frac{1}{2}}\right)x\right)^{\frac{1}{2}}\right].$$

As a consequence of the above lemma, there exists $\tilde{v} \in \operatorname{int} \mathcal{K}$ such that

$$\tilde{v} = P(w)^{-\frac{1}{2}}x = P(w)^{\frac{1}{2}}s. \tag{48}$$

Note that $P(w)^{\frac{1}{2}}$ and its inverse $P(w)^{-\frac{1}{2}}$ are automorphisms of $\mathcal{K}$ (see, *e.g.*, [14, 29]). This leads to the definition of the following *variance vector*:

$$v := \frac{1}{\sqrt{\mu}}P(w)^{-\frac{1}{2}}x\left[= \frac{1}{\sqrt{\mu}}P(w)^{\frac{1}{2}}s\right]. \tag{49}$$

Furthermore, we define

$$\overline{\mathcal{A}} := P(w)^{\frac{1}{2}}\mathcal{A}P(w)^{\frac{1}{2}}, \qquad d_x := \frac{P(w)^{-\frac{1}{2}}\Delta x}{\sqrt{\mu}} \quad \text{and} \quad d_s := \frac{P(w)^{\frac{1}{2}}\Delta s}{\sqrt{\mu}}. \tag{50}$$

The transformation $\overline{\mathcal{A}}$ also has the Cartesian $P_*(\kappa)$-property (*cf.* Proposition 3.4 in [8]).

Using (49) and (50), after some elementary reductions, we obtain the scaled Newton system as follows:

$$\begin{pmatrix} \overline{\mathcal{A}}(d_x) - d_s \\ d_x + d_s \end{pmatrix} = \begin{pmatrix} 0 \\ v^{-1} - v \end{pmatrix}. \tag{51}$$

Since the linear transformation $\overline{\mathcal{A}}$ has the Cartesian $P_*(\kappa)$-property, the system (51) has a unique solution [8].

So far we have described the scheme that defines the classical NT-direction for the Cartesian $P_*(\kappa)$-SCLCP. The approach in this paper differs only in one detail. Given the finite kernel function $\psi(t)$ defined by (3) and the associated vector-valued function $\psi'(v)$ defined by (14) and (15), we replace the right-hand side of the second equation in (51) by

$-\psi'(v)$, *i.e.*, minus the derivative of the barrier function $\Psi(v)$. Thus we consider the following system:

$$\begin{pmatrix} \overline{\mathcal{A}}(d_x) - d_s \\ d_x + d_s \end{pmatrix} = \begin{pmatrix} 0 \\ -\psi'(v) \end{pmatrix}. \tag{52}$$

Since the system (52) has the same matrix of coefficients as the system (51), also the system (52) has a unique solution.[a]

The new search directions $d_x$ and $d_s$ are computed by solving (52), thus $\Delta x$ and $\Delta s$ are obtained from (50). If $(x,s) \neq (x(\mu), s(\mu))$, then $(\Delta x, \Delta s)$ is nonzero. By taking a default step size $\alpha$ along the search directions, we get the new iteration point $(x_+, s_+)$ according to

$$x_+ := x + \alpha \Delta x \quad \text{and} \quad s_+ := s + \alpha \Delta s. \tag{53}$$

Furthermore, we can easily verify that

$$\begin{aligned} x \diamond s = \mu e \quad &\Leftrightarrow \quad v = e \quad \Leftrightarrow \quad \psi'(v) = 0 \\ &\Leftrightarrow \quad \psi(v) = 0 \quad \Leftrightarrow \quad \Psi(v) = 0. \end{aligned} \tag{54}$$

Hence, the value of $\Psi(v)$ can be considered as a measure for the distance between the given iterate $(x,s)$ and the $\mu$-center $(x(\mu), s(\mu))$.

### 4.3 The generic interior-point algorithm for the Cartesian $P_*(\kappa)$-SCLCP

Define the $\tau$-neighborhood of the central path as follows:

$$\mathcal{N}(\tau) := \left\{(x,s) \in \operatorname{int} \mathcal{K} \times \operatorname{int} \mathcal{K} : s = \mathcal{A}(x) + q, \Psi(v) \leq \tau\right\}.$$

It is clear from the above description that the closeness of $(x,s)$ to $(x(\mu), s(\mu))$ is measured by the value of $\Psi(v)$, with $\tau > 0$ as a threshold value. If $\Psi(v) \leq \tau$, then we start a new *outer iteration* by performing a $\mu$-update, *i.e.*, $\mu_+ := (1-\theta)\mu$; otherwise, we enter an *inner iteration* by computing the search directions using (52) and (50) at the current iterates with respect to the current value of $\mu$ and apply (53) to get new iterates. If necessary, we repeat the procedure until we find iterates that are in the $\tau$-neighborhood of the central path. Then $\mu$ is again reduced by the factor $1-\theta$ with $0 < \theta < 1$, and we apply inner iteration(s) targeting at the new $\mu$-centers, and so on. This process is repeated until $\mu$ is small enough, say until $r\mu < \varepsilon$. At this stage, we have found a $\varepsilon$-approximate solution of the Cartesian $P_*(\kappa)$-SCLCP.

The generic interior-point algorithm for the Cartesian $P_*(\kappa)$-SCLCP is now presented as follows.

**Generic IPMs for the Cartesian $P_*(\kappa)$-SCLCP**

**Input:**

A threshold parameter $\tau \geq 1$;

an accuracy parameter $\varepsilon > 0$;

a fixed barrier update parameter $\theta$, $0 < \theta < 1$;

a strictly feasible $(x^0, s^0)$ and $\mu^0 := \langle x^0, s^0 \rangle / r$ such that $\Psi(x^0, s^0; \mu^0) \leq \tau$.

**begin**

$x := x^0$; $s := s^0$; $\mu := \mu^0$;

**while** $r\mu \geq \varepsilon$ **do**

**begin**

$\mu := (1 - \theta)\mu$;

**while** $\Psi(x, s; \mu) > \tau$ **do**

**begin**

solve system (52) and use (50) to obtain $(\Delta x, \Delta s)$;

choose a default step size $\alpha$;

update $x := x + \alpha \Delta x$, $s := s + \alpha \Delta s$;

**end**

**end**

**end**

## 5 Analysis of the algorithm

In this section, we first discuss the growth behavior of the barrier function during an outer iteration. Next, we choose the default step size and obtain an upper bound for the decrease of the barrier function during an inner iteration. Finally, we show that the default step size yields sufficient decrease of the barrier function value during each inner iteration.

### 5.1 Growth behavior of the barrier function during an outer iteration

It should be mentioned that during the course of the algorithm the largest values of $\Psi(v)$ occur just after the update of $\mu$. So, next we derive an estimate for the effect of a $\mu$-update on the value of $\Psi(v)$.

It follows from (32) that

$$\Psi(\beta v) = \sum_{j=1}^{N} \sum_{i=1}^{r_j} \psi\left(\beta \lambda_i\left(v^{(j)}\right)\right),$$

which means that $\Psi(\beta v)$ depends only on the eigenvalues $\lambda_i(v^{(j)})$ of the symmetric cone $v^{(j)}$ for each $j$, $j = 1, \ldots, N$. The growth behavior of the proximity $\Psi(v)$ is precisely the same as its LO counterpart $\Psi(\beta v)$ (*cf.* Theorem 3.2 in [18]).

**Theorem 5.1** *If $v \in \mathcal{K}_+$ and $\beta \geq 1$, then*

$$\Psi(\beta v) \leq r\psi\left(\beta \varrho\left(\frac{\Psi(v)}{r}\right)\right).$$

**Corollary 5.2** *Let $0 < \theta < 1$ and $v_+ = \frac{v}{\sqrt{1-\theta}}$. If $\Psi(v) \leq \tau$, then*

$$\Psi(v_+) \leq r\psi\left(\frac{\varrho(\frac{\tau}{r})}{\sqrt{1-\theta}}\right).$$

*Proof* With $\beta = \frac{1}{\sqrt{1-\theta}} > 1$ and $\Psi(v) \leq \tau$, the corollary follows immediately from Theorem 5.1. □

### 5.2 Choice of the default step size

From (53) and (50), after some elementary reductions, we have

$$x_+ = \sqrt{\mu} P(w)^{\frac{1}{2}}(v + \alpha d_x) \quad \text{and} \quad s_+ = \sqrt{\mu} P(w)^{-\frac{1}{2}}(v + \alpha d_s). \tag{55}$$

Thus,

$$v_+ := \frac{1}{\sqrt{\mu}} P(w_+)^{-\frac{1}{2}} x_+ = \frac{1}{\sqrt{\mu}} P(w_+)^{\frac{1}{2}} s_+,$$

or equivalently,

$$v_+ = P(w_+)^{-\frac{1}{2}} P(w)^{\frac{1}{2}}(v + \alpha d_x) = P(w_+)^{\frac{1}{2}} P(w)^{-\frac{1}{2}}(v + \alpha d_s),$$

where, according to Lemma 4.2,

$$w_+ := P(x_+)^{\frac{1}{2}} \left( \left( P(x_+)^{\frac{1}{2}} s_+ \right)^{-\frac{1}{2}} \right).$$

To calculate the decrease of the barrier function $\Psi(v)$ during an inner iteration, it is standard to consider the decrease as a function of $\alpha$ defined by

$$f(\alpha) := \Psi(v_+) - \Psi(v).$$

Our aim is to find an upper bound for $f(\alpha)$ by using the exponential convexity of $\psi(t)$, and according to Lemma 3.1. In order to do this, we assume for the moment that

$$\lambda_{\min}\left(v^{(j)} + \alpha d_x^{(j)}\right) \geq \frac{1}{\sigma} \quad \text{and} \quad \lambda_{\min}\left(v^{(j)} + \alpha d_s^{(j)}\right) \geq \frac{1}{\sigma}, \quad j = 1, \ldots, N. \tag{56}$$

However, working with $f(\alpha)$ may not be easy because in general $f(\alpha)$ is not convex. Thus, we are searching for the convex function $f_1(\alpha)$ that is an upper bound of $f(\alpha)$ and whose derivatives are easier to calculate than those of $f(\alpha)$. The key element in this process is replacing $v_+$ with a similar element that will allow the use of exponential-convexity of the barrier function. By Proposition 5.9.3 in [29], we have

$$v_+ \sim \left( P(v + \alpha d_x)^{\frac{1}{2}}(v + \alpha d_s) \right)^{\frac{1}{2}}$$

and therefore

$$\Psi(v_+) = \Psi\left( \left( P(v + \alpha d_x)^{\frac{1}{2}}(v + \alpha d_s) \right)^{\frac{1}{2}} \right).$$

Theorem 3.3 implies that

$$\Psi(v_+) \leq \frac{1}{2}\left( \Psi(v + \alpha d_x) + \Psi(v + \alpha d_s) \right).$$

Hence, we have

$$f(\alpha) \le f_1(\alpha) := \frac{1}{2}\big(\Psi(v+\alpha d_x) + \Psi(v+\alpha d_s)\big) - \Psi(v),$$

which means that $f_1(\alpha)$ gives an upper bound for the decrease of the barrier function $\Psi(v)$. Furthermore, we can conclude that $f(0) = f_1(0) = 0$.

From (40), we have

$$\begin{aligned} f_1'(\alpha) &= \frac{1}{2}\sum_{j=1}^{N}\Big(\operatorname{tr}\big(\psi'\big(v^{(j)}+\alpha d_x^{(j)}\big)\circ d_x^{(j)}\big) + \operatorname{tr}\big(\psi'\big(v^{(j)}+\alpha d_s^{(j)}\big)\circ d_s^{(j)}\big)\Big) \\ &= \frac{1}{2}\big(\operatorname{tr}\big(\psi'(v+\alpha d_x)\diamond d_x\big) + \operatorname{tr}\big(\psi'(v+\alpha d_s)\diamond d_s\big)\big). \end{aligned}$$

This gives, by (52) and (36),

$$f_1'(0) = \frac{1}{2}\operatorname{tr}\big(\psi'(v)\diamond(d_x+d_s)\big) = -\frac{1}{2}\operatorname{tr}\big(\psi'(v)\diamond\psi'(v)\big) = -\frac{1}{2}\big\|\psi'(v)\big\|_F^2 = -2\delta(v)^2 < 0.$$

Hence, we can conclude that $f_1'(\alpha)$ is monotonically decreasing in a neighborhood of $\alpha = 0$.

Furthermore, we have, by (41) and (42),

$$\begin{aligned} f_1''(\alpha) \le{}& \frac{1}{2}\sum_{j=1}^{N}\left(\sum_{i=1}^{r_j}\psi''\big(\lambda_i\big(\eta^{(j)}\big)\big)\big(d_{xi}^{(j)}\big)^2 + \sum_{i<m_j}\psi''\big(\lambda_{m_j}\big(\eta^{(j)}\big)\big)\operatorname{tr}\big(\big(d_{xim_j}^{(j)}\big)^2\big)\right) \\ &+ \frac{1}{2}\sum_{j=1}^{N}\left(\sum_{i=1}^{r_j}\psi''\big(\lambda_i\big(\gamma^{(j)}\big)\big)\big(d_{si}^{(j)}\big)^2 + \sum_{i<m_j}\psi''\big(\lambda_{m_j}\big(\gamma^{(j)}\big)\big)\operatorname{tr}\big(\big(d_{sim_j}^{(j)}\big)^2\big)\right). \end{aligned} \tag{57}$$

Contrary to the LO case, the vectors $d_x$ and $d_s$ are not necessarily orthogonal any more. However, the Cartesian $P_*(\kappa)$-property of SCLCP still allows us to find a good lower bound of the inner product $\langle d_x, d_s\rangle$.

In order to facilitate discussion, we denote

$$\delta := \delta(v), \qquad \delta_+ := \sum_{v\in J_+}\big\langle d_x^{(v)}, d_s^{(v)}\big\rangle \quad \text{and} \quad \delta_- := -\sum_{v\in J_-}\big\langle d_x^{(v)}, d_s^{(v)}\big\rangle. \tag{58}$$

**Lemma 5.3** *One has*

$$\langle d_x, d_s\rangle \ge -4\kappa\delta^2.$$

*Proof* Since the linear transformation $\mathcal{A}$ has the Cartesian $P_*(\kappa)$-property, we have

$$(1+4\kappa)\sum_{j\in\mathcal{I}_+(\Delta x)}\big\langle \Delta x^{(j)}, \big[\mathcal{A}(\Delta x)\big]^{(j)}\big\rangle + \sum_{j\in\mathcal{I}_-(\Delta x)}\big\langle \Delta x^{(j)}, \big[\mathcal{A}(\Delta x)\big]^{(j)}\big\rangle \ge 0, \tag{59}$$

where $\mathcal{I}_+(\Delta x) = \{j : \langle \Delta x^{(j)}, [\mathcal{A}(\Delta x)]^{(j)}\rangle > 0\}$ and $\mathcal{I}_-(\Delta x) = \{j : \langle \Delta x^{(j)}, [\mathcal{A}(\Delta x)]^{(j)}\rangle < 0\}$ are two index sets. It follows from (50) and $\mathcal{A}(\Delta x) = \Delta s$ that $\langle \Delta x, \Delta s\rangle = \mu\langle d_x, d_s\rangle$. This enables us to rewrite (59) as

$$(1+4\kappa)\sum_{j\in\mathcal{I}_+(\Delta x)}\big\langle d_x^{(j)}, d_s^{(j)}\big\rangle + \sum_{j\in\mathcal{I}_-(\Delta x)}\big\langle d_x^{(j)}, d_s^{(j)}\big\rangle \ge 0. \tag{60}$$

Hence, it follows that

$$\langle d_x, d_s \rangle \geq -4\kappa \sum_{j \in \mathcal{I}_+(\Delta x)} \langle d_x^{(j)}, d_s^{(j)} \rangle. \tag{61}$$

Using the arithmetic-geometric mean inequality $\langle a, b \rangle \leq \frac{1}{4}\langle a+b, a+b \rangle$, we have

$$\sum_{j \in \mathcal{I}_+(\Delta x)} \langle d_x^{(j)}, d_s^{(j)} \rangle \leq \frac{1}{4} \sum_{j \in \mathcal{I}_+(\Delta x)} \| d_x^{(j)} + d_s^{(j)} \|_F^2 \leq \frac{1}{4} \sum_{j=1}^{N} \| d_x^{(j)} + d_s^{(j)} \|_F^2 = \frac{1}{4} \| d_x + d_s \|_F^2 = \delta^2.$$

Substitution of this inequality into (61) yields

$$\langle d_x, d_s \rangle \geq -4\kappa\delta^2.$$

This completes the proof of the lemma. □

The key steps in the analysis of the algorithm are based on the effort to find upper bounds on $\|d_x\|$ and $\|d_s\|$ in terms of the proximity measure $\delta$. The following lemma yields their upper bounds.

**Lemma 5.4** *One has*

$$\|d_x\|_F \leq 2\sqrt{1+2\kappa}\,\delta \quad \textit{and} \quad \|d_s\|_F \leq 2\sqrt{1+2\kappa}\,\delta.$$

*Proof* From Lemma 5.3, we have

$$\|d_x\|_F^2 + \|d_s\|_F^2 = \|d_x + d_s\|^2 - 2\langle d_x, d_s \rangle \leq 4\delta^2 + 8\kappa\delta^2 = 4(1+2\kappa)\delta^2. \tag{62}$$

This implies the inequalities in the statement of the lemma. □

**Lemma 5.5** *One has*

$$f_1''(\alpha) \leq 2(1+2\kappa)\delta^2 \psi''\big(\lambda_{\min}(v) - 2\alpha\sqrt{1+2\kappa}\,\delta\big).$$

*Proof* From Lemma 2.1 and Lemma 5.4, we have

$$\lambda_i\big((v+\alpha d_x)^{(j)}\big) \geq \lambda_{\min}\big((v+\alpha d_x)^{(j)}\big) \geq \lambda_{\min}\big(v^{(j)}\big) - \big\|\alpha d_x^{(j)}\big\|_F \geq \lambda_{\min}\big(v^{(j)}\big) - 2\alpha\sqrt{1+2\kappa}\,\delta,$$
$$\lambda_i\big((v+\alpha d_s)^{(j)}\big) \geq \lambda_{\min}\big((v+\alpha d_s)^{(j)}\big) \geq \lambda_{\min}\big(v^{(j)}\big) - \big\|\alpha d_s^{(j)}\big\|_F \geq \lambda_{\min}\big(v^{(j)}\big) - 2\alpha\sqrt{1+2\kappa}\,\delta.$$

Let

$$d_x^{(j)} = \sum_{i=1}^{r_j} d_{xi}^{(j)} c_i^{(j)} + \sum_{i<m_j} d_{xim_j}^{(j)}, \quad j = 1, \ldots, N,$$

be the Peirce decomposition of $d_x^{(j)}$ with respect to the Jordan frame $\{c_1^{(j)}, \ldots, c_{r_j}^{(j)}\}$, and let

$$d_s^{(j)} = \sum_{i=1}^{r_j} d_{si}^{(j)} b_i^{(j)} + \sum_{i<m_j} d_{sim_j}^{(j)}, \quad j = 1, \ldots, N,$$

be the Peirce decomposition of $d_s^{(j)}$ with respect to the Jordan frame $\{b_1^{(j)}, \dots, b_{r_j}^{(j)}\}$. We have

$$\|d_x\|_F^2 = \sum_{j=1}^{N} \|d_x^{(j)}\|_F^2 = \sum_{j=1}^{N} \langle d_x^{(j)}, d_x^{(j)} \rangle = \sum_{j=1}^{N} \left( \sum_{i=1}^{r_j} (d_{xi}^{(j)})^2 + \sum_{i<m_j} \operatorname{tr}\big((d_{xim_j}^{(j)})^2\big) \right)$$

and

$$\|d_s\|_F^2 = \sum_{j=1}^{N} \|d_s^{(j)}\|_F^2 = \sum_{j=1}^{N} \langle d_s^{(j)}, d_s^{(j)} \rangle = \sum_{j=1}^{N} \left( \sum_{i=1}^{r_j} (d_{si}^{(j)})^2 + \sum_{i<m_j} \operatorname{tr}\big((d_{sim_j}^{(j)})^2\big) \right).$$

Since $\psi''(t)$ is monotonically decreasing in $t \in (0, +\infty)$, we have, by (57),

$$\begin{aligned} f_1''(\alpha) &\le \frac{1}{2}\psi''\big(\lambda_{\min}(v) - 2\alpha\sqrt{1+2\kappa}\delta\big) \sum_{j=1}^{N} \left( \sum_{i=1}^{r_j} (d_{xi}^{(j)})^2 + \sum_{i<m_j} \operatorname{tr}\big((d_{xim_j}^{(j)})^2\big) \right) \\ &\quad + \frac{1}{2}\psi''\big(\lambda_{\min}(v) - 2\alpha\sqrt{1+2\kappa}\delta\big) \sum_{j=1}^{N} \left( \sum_{i=1}^{r_j} (d_{si}^{(j)})^2 + \sum_{i<m_j} \operatorname{tr}\big((d_{sim_j}^{(j)})^2\big) \right) \\ &\le \frac{1}{2}\psi''\big(\lambda_{\min}(v) - 2\alpha\sqrt{1+2\kappa}\delta\big)\big(\|d_x\|_F^2 + \|d_s\|_F^2\big) \\ &\le 2(1+2\kappa)\delta^2\psi''(v_{\min} - 2\alpha\sqrt{1+2\kappa}\delta). \end{aligned}$$

The last inequality holds due to the fact that (62). This completes the proof of the lemma. □

From this point on, the analysis of the algorithm follows almost completely the similar analyses in [17, 22] with straightforward modifications that take into account the Cartesian $P_*(\kappa)$-property. Therefore, the intermediate results are omitted and only main results are mentioned without the proofs.

In particular, the step size $\alpha$ satisfies the following condition:

$$\alpha \ge \frac{1}{(1+2\kappa)\psi''(\rho(L\delta))}. \tag{63}$$

It follows from (63) and the definition of $\rho$ that

$$\alpha \ge \frac{1}{(1+2\kappa)(1+\sigma e^{\sigma(1-t)})}, \quad t \in \left[\frac{1}{\sigma}, 1\right] \text{ and } e^{\sigma(1-t)} - t = 4\delta. \tag{64}$$

Using the second equation of (64), we have

$$e^{\sigma(1-t)} = t + 4\delta \le 1 + 4\delta.$$

It follows from Corollary 3.8 and $\Psi(v) \ge 1$ that

$$\delta \ge \frac{1}{6}\sqrt{\Psi(v)} \ge \frac{1}{6}.$$

Hence, we have

$$\alpha \geq \frac{1}{\sigma(1+2\kappa)(1+e^{\sigma(1-t)})} \geq \frac{1}{2\sigma(1+2\kappa)(1+2\delta)} \geq \frac{1}{16\sigma\delta(1+2\kappa)}.$$

In the sequel, we use the notation

$$\tilde{\alpha} = \frac{1}{16\sigma\delta(1+2\kappa)}, \tag{65}$$

and we will use $\tilde{\alpha}$ as the default step size. It is obvious that $\alpha \geq \tilde{\alpha}$.

Now, to validate the above analysis, we need to show that $\tilde{\alpha}$ satisfies (56). In fact, from Lemmas 2.1, 3.2, 5.4 and (65), we have

$$\begin{aligned}\lambda_{\min}(v+\tilde{\alpha}d_x) &\geq \lambda_{\min}(v) - \tilde{\alpha}\|d_x\| \geq \frac{3}{2\sigma} - \frac{1}{16\sigma\delta(1+2\kappa)}2\sqrt{1+2\kappa}\delta \\ &\geq \frac{3}{2\sigma} - \frac{1}{8\sigma} \geq \frac{11}{8\sigma} \geq \frac{1}{\sigma}\end{aligned}$$

and

$$\begin{aligned}\lambda_{\min}(v+\tilde{\alpha}d_s) &\geq \lambda_{\min}(v) - \tilde{\alpha}\|d_s\| \geq \frac{3}{2\sigma} - \frac{1}{16\sigma\delta(1+2\kappa)}2\sqrt{1+2\kappa}\delta \\ &\geq \frac{3}{2\sigma} - \frac{1}{8\sigma} \geq \frac{11}{8\sigma} \geq \frac{1}{\sigma}.\end{aligned}$$

### 5.3 Decrease of the value of $\Psi(v)$ during an inner iteration

In what follows, we will show that the barrier function $\Psi(v)$ in each inner iteration with the default step size $\tilde{\alpha}$, as defined by (65), is decreasing. For this, we need the following technical result.

**Lemma 5.6** (Lemma 3.12 in [19]) *Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and let $h(t)$ attain its (global) minimum at $t^* > 0$. If $h''(t)$ is increasing for $t \in [0, t^*]$, then*

$$h(t) \leq \frac{th'(0)}{2}, \quad 0 \leq t \leq t^*.$$

As a consequence of Lemma 5.6 and the fact that $f(\alpha) \leq f_1(\alpha)$, which is a twice differentiable convex function with $f_1(0) = 0$, and $f_1'(0) = -2\delta^2 < 0$, we can easily prove the following lemma.

**Lemma 5.7** *If the step size $\alpha$ is such that $\alpha \leq \tilde{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2.$$

The following theorem states the results which show that the default step size (65) yields sufficient decrease of the barrier function value during each inner iteration.

**Theorem 5.8** *One has*

$$f(\tilde{\alpha}) \leq -\frac{(\Psi(v))^{\frac{1}{2}}}{96\sigma(1+2\kappa)}.$$

*Proof* From Lemma 5.7, Corollary 3.8 and (65), we have

$$f(\tilde{\alpha}) \leq -\tilde{\alpha}\delta^2 = -\frac{1}{16\sigma\delta(1+2\kappa)}\delta^2 = -\frac{\delta}{16\sigma(1+2\kappa)} \leq -\frac{(\Psi(v))^{\frac{1}{2}}}{96\sigma(1+2\kappa)}.$$

This completes the proof of the theorem. □

## 6 Complexity of the algorithm

In this section, we first derive an upper bound for the number of the iteration bounds by our algorithm. Then we obtain the iteration bounds that match the currently best known iteration bounds for large- and small-update methods, respectively.

### 6.1 Iteration bound for a large-update method

For the analysis of the iterations of the algorithm, we need to count how many inner iterations are required to return to the situation where $\Psi(v) \leq \tau$. We denote the value of $\Psi(v)$ after the $\mu$-update as $\Psi_0$, the subsequent values in the same outer iteration are denoted as $\Psi_k$, $k = 1, \ldots, K$, where $K$ denotes the total number of inner iterations in the outer iteration. According to the decrease of $f(\tilde{\alpha})$, we get

$$\Psi_{k+1} \leq \Psi_k - \beta(\Psi_k)^{1-\gamma}, \quad k = 0, 1, \ldots, K-1, \tag{66}$$

where $\beta = \frac{1}{96\sigma(1+2\kappa)}$ and $\gamma = \frac{1}{2}$.

**Lemma 6.1** (Lemma 14 in [19]) *Suppose that $t_0, t_1, \ldots, t_K$ is a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta t_k^{1-\gamma}, \quad k = 0, 1, \ldots, K-1,$$

*where $\beta > 0$ and $0 < \gamma \leq 1$. Then $K \leq \lceil \frac{t_0^{\gamma}}{\beta\gamma} \rceil$.*

Combining Lemma 6.1 and (66), we can easily verify the following main result.

**Theorem 6.2** *One has*

$$K \leq 192\sigma(1+2\kappa)(\Psi_0)^{\frac{1}{2}}.$$

By applying Corollary 5.2, (34), and the fact that $\psi(t) \leq \frac{t^2}{2}$ when $t \geq 1$, we have

$$\Psi_0 \leq r\psi\left(\frac{\varrho(\frac{\tau}{r})}{\sqrt{1-\theta}}\right) \leq r\psi\left(\frac{\sqrt{\frac{2\tau}{r} + \frac{2+\sigma}{\sigma}}}{\sqrt{1-\theta}}\right) \leq \frac{r}{2(1-\theta)}\left(\frac{2\tau}{r} + \frac{2+\sigma}{\sigma}\right).$$

From the above expression with $\theta = \Theta(1)$ and $\tau = O(r)$, and also applying Lemma 3.2, we can conclude that $\sigma = O(\log r)$.

The number of outer iterations is bounded above by $\frac{1}{\theta}\log\frac{r}{\varepsilon}$ (*cf.* Lemma Π.17 in [13]). By multiplying the number of outer iterations and the number of inner iterations, we get an upper bound for the total number of iterations, namely

$$\frac{192\sigma(1+2\kappa)}{\theta}\sqrt{\frac{r}{2(1-\theta)}\left(\frac{2\tau}{r}+\frac{2+\sigma}{\sigma}\right)}\log\frac{r}{\varepsilon}.$$

After some elementary reductions, we have the following theorem, which gives the currently best known iteration bound for the large-update method.

**Theorem 6.3** *For the large-update method, which is characterized by* $\theta = \Theta(1)$ *and* $\tau = O(r)$, *then the algorithm requires at most*

$$O\left((1+2\kappa)\sqrt{r}\log r\log\frac{r}{\varepsilon}\right)$$

*iterations. The output gives a* $\varepsilon$*-approximate solution of the Cartesian* $P_*(\kappa)$*-SCLCP.*

### 6.2 Iteration bound for a small-update method

It is not hard to show that if the above analysis is used for a small-update method, the iteration bound would not be as good as it can be for these types of methods. For the analysis of the iteration bound of a small-update method, we need to estimate the upper bound of $\Psi_0$ more accurately. It should be noted that the following analysis only holds for $\sigma \geq 2$.

By applying Corollary 5.2, (35), Lemma 3.4, and the fact that $1-\sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$, we have

$$\begin{aligned}
\Psi_0 &\leq r\psi\left(\frac{\varrho(\frac{\tau}{r})}{\sqrt{1-\theta}}\right)\\
&\leq r\psi\left(\frac{1+\sqrt{\frac{\tau}{r}}(\frac{2\tau}{r}+\frac{2+\sigma}{\sigma})^{\frac{1}{4}}}{\sqrt{1-\theta}}\right)\\
&\leq \frac{r(1+\sigma)}{2}\left(\frac{1+\sqrt{\frac{\tau}{r}}(\frac{2\tau}{r}+\frac{2+\sigma}{\sigma})^{\frac{1}{4}}}{\sqrt{1-\theta}}-1\right)^2\\
&\leq \frac{1+\sigma}{2(1-\theta)}\left(\theta\sqrt{r}+\sqrt{\tau}\left(\frac{2\tau}{r}+\frac{2+\sigma}{\sigma}\right)^{\frac{1}{4}}\right)^2.
\end{aligned}$$

From the above expression with $\theta = \Theta(\frac{1}{\sqrt{r}})$ and $\tau = O(1)$, and also applying Lemma 3.2, we can conclude that $\sigma = O(1)$. It follows from Theorem 6.2 that the total number of iterations is bounded above by

$$\frac{192\sigma\sqrt{1+\sigma}(1+2\kappa)}{\theta\sqrt{2(1-\theta)}}\left(\theta\sqrt{r}+\sqrt{\tau}\left(\frac{2\tau}{r}+\frac{2+\sigma}{\sigma}\right)^{\frac{1}{4}}\right)\log\frac{r}{\varepsilon}.$$

After some elementary reductions, we have the following theorem, which gives the currently best known iteration bound for a small-update method.

**Theorem 6.4** *For a small-update method, which is characterized by* $\theta = \Theta(\frac{1}{\sqrt{r}})$ *and* $\tau = O(1)$, *then the algorithm requires at most*

$$O\left((1+2\kappa)\sqrt{r}\log\frac{r}{\varepsilon}\right)$$

*iterations. The output gives a* $\varepsilon$*-approximate solution of the Cartesian* $P_*(\kappa)$*-SCLCP.*

## 7 Conclusions and remarks

In this paper, we have shown that primal-dual IPMs for LO [17] and $P_*(\kappa)$-LCP [22] based on the finite barrier can be extended to the context of the Cartesian $P_*(\kappa)$-SCLCP. The iteration bounds for large- and small-update methods are obtained, namely $O((1+2\kappa)\sqrt{r}\log r\log\frac{r}{\varepsilon})$ and $O((1+2\kappa)\sqrt{r}\log\frac{r}{\varepsilon})$, respectively. In both cases, we were able to match the best known iteration bounds for these types of methods. Moreover, this unifies the analysis for the $P_*(\kappa)$-LCP, the Cartesian $P_*(\kappa)$-SOCLCP, and the Cartesian $P_*(\kappa)$-SDLCP.

Some interesting topics for further research remain. One possible topic is to investigate whether it is possible to replace NT-scaling scheme by some other scaling schemes and still obtain polynomial-time iteration bounds. Another worthwhile direction for further research may be the development of infeasible kernel-based IPMs for SCLCP.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors carried out the proof. All authors conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

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**Endnote**
[a] It may be worth mentioning that if we use the kernel function of the classical logarithmic barrier function, *i.e.*, $\psi(t) = \frac{1}{2}(t^2-1) - \log t$, then $\psi'(t) = t - t^{-1}$, whence $-\psi'(v) = v^{-1} - v$, and hence system (52) then coincides with the classical system (51).

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