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Estimates of probabilistic widths of the diagonal operator of finite-dimensional sets with the Gaussian measure

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Abstract

In this paper, we estimate the asymptotic orders of probabilistic and average widths of the compact embedding operators from the Sobolev space $W_2^r(\mathbb{T})$ into $L_q(\mathbb{T})$ $(1 \le q \le \infty)$ with the Gaussian measure.

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1 Introduction and main results

Problems of n-widths in the approximation theory have by now been studied in depth. A great deal of classical problems have been solved, and interesting new developments have appeared. For example, the problems of probabilistic, average and stochastic widths, which can reflect the behavior of function on the whole class and give information about the measure of the elements in the class that can be approximated to this or that degree, are the problems of this kind. For the results related to the probabilistic, average and stochastic widths, the reader may be referred to Sul'din [1, 2], Traub *et al.* [3], Maiorov [4–7], Mathé [8–12], Sun [13, 14], and Ritter [15]. The new developments in this direction can be found in Fang's papers [16–21]. Moreover, Carl and Pajor [22] proved an inequality with respect to the Gelfand numbers of an operator u from ℓ_1^N into a Hilbert space, from which one can immediately derive the inequality related to the Kolmogorov numbers by the known duality. In this article we continue the previous works and prove the estimates of probabilistic widths of the diagonal operators from $\mathbb{R}^m$ onto ℓ_q^m.

First, we recall some useful concepts. Let W be a bounded subset of a normed linear space X with the norm $\|\cdot\|_X$, and F_N be an N-dimensional subspace of X. The quantity

$$e(W, F_N, X) = \sup_{x \in W} e(x, F_N, X),$$

where

$$e(x, F_N, X) = \inf_{y \in F_N} \|x - y\|_X$$

is called the deviation of W from F_N. It shows how well the 'worst' elements of W can be approximated by F_N. The number

$$d_N(W,X) = \inf_{F_N} e(W,F_N,X) = \inf_{F_N} \sup_{x\in W} \inf_{y\in F_N} \|x-y\|_X,$$

where F_N runs through all possible linear subspaces of X of dimension at most N, is called the Kolmogorov's N-width of W in X. Assume that W contains a Borel field $\mathcal{B}$ consisting of open subsets of W and equipped with a probabilistic measure μ defined on $\mathcal{B}$. That is, μ is a σ-additive nonnegative function on $\mathcal{B}$, and $\mu(W) = 1$. Let $\delta \in [0,1]$ be an arbitrary number. The corresponding probabilistic Kolmogorov's (N,δ)-width of a set W with a measure μ in the space X is defined by

$$d_{N,\delta}(W,\mu,X) = \inf_{G_\delta} d_N(W\backslash G,X), \tag{1}$$

where G_δ runs through all possible subsets in $\mathcal{B}$ with measure $\mu(G_\delta) \le \delta$. The p-average Kolmogorov's N-width is defined by

$$d_N^{(a)}(W,\mu,X)_p = \inf_{F_N}\left(\int_W e(x,F_N,X)^p \, d\mu(x)\right)^{1/p}, \quad 0 < p < \infty, \tag{2}$$

where F_N in (2) runs over all linear subspaces of X of dimension at most N. Let ℓ_p^m be an m-dimensional normed space of vectors $x = (x_1,\dots,x_m) \in \mathbb{R}^m$, with a norm

$$\|x\|_{\ell_p^m} = \begin{cases} (\sum_{i=1}^m |x_i|^p)^{1/p}, & 1 \le p < \infty, \\ \max_{1\le i\le m} |x_i|, & p = \infty. \end{cases}$$

Consider in $\mathbb{R}^m$ the standard Gaussian measure $\nu = \nu_m$, which is defined as

$$\nu(G) = (2\pi)^{-m/2} \int_G \exp\left(-\frac{1}{2}\|x\|_2^2\right) dx,$$

where G is any Borel subset in $\mathbb{R}^m$. Obviously, $\nu(\mathbb{R}^m) = 1$.

Denote by $B_p^m(\rho) = \{x \in \ell_p^m : \|x\|_p \le \rho\}$ the ball of radius ρ in ℓ_p^m. Let $B_p^m = B_p^m(1)$.

Let $N = 0,1,\dots$, $\delta \in [0,1)$ be arbitrary and T_m be a linear invertible operator from $\mathbb{R}^m$ onto ℓ_q^m. We define the probabilistic (N,δ)-width of the operator acting in space $\mathbb{R}^m$ equipped with the Gaussian measure ν in ℓ_q^m-norm:

$$d_{N,\delta}\left(T_m : \mathbb{R}^m \to \ell_q^m, \nu\right) = \inf_G \inf_{\mathcal{L}_N} e\left(T_m\left(\mathbb{R}^m\backslash G\right), \mathcal{L}_N, \ell_q^m\right),$$

where $\nu(G) < \delta$, $\dim \mathcal{L}_N \le N$.

Maiorov in [6] proved the following result.

Theorem A [6] *For $m > 2N$, $\delta \in (0,1/2]$, $1 \le q \le 2$, then*

$$d_{N,\delta}\left(\mathbb{R}^m, \ell_q^m, \nu\right) \asymp m^{\frac{1}{q}-\frac{1}{2}}\sqrt{m + \ln(1/\delta)}.$$

In [22], Carl and Pajor proved the following result with respect to Gelfand numbers of an operator with values in a Hilbert space.

Theorem B [22] *Let T be an operator from ℓ_1^m into a Hilbert space H. Then*

$$d^N(T) \le C\left(\frac{\log(\frac{m}{N}+1)}{N}\right)^{1/2}\|T\|$$

for $1 \le N \le m$, $m = 1, 2, \dots$, *where* $C > 0$ *is a universal constant.*

Detailed facts about the usual widths, such as the Kolmogorov's N-widths and Nth Gelfand numbers (or Gelfand N-widths) of T were given in the books [23–26].

Remark 1

(a) Theorem A shows the asymptotic expression of the probabilistic widths of the identity embedding from $\mathbb{R}^m$ into ℓ_q^m, $1 \le q \le 2$.
(b) Theorem B gives the upper estimate of Gelfand numbers of operators from ℓ_1^m into a Hilbert space, and some of its striking applications in the geometry of Banach spaces and Rademacher processes can be found in [22]. By the dual relation, it is easy to obtain the similar upper estimate of the Kolmogorov's N-widths $d_N(T)$ of operators from ℓ_2^m into ℓ_∞^m, *i.e.*,

$$d_N(T) \le C\left(\frac{\log(\frac{m}{N}+1)}{N}\right)^{1/2}\|T\|.$$

(c) Motivated by Theorems A and B, in general cases, here we investigate the asymptotic estimate of probabilistic widths for diagonal operators from $\mathbb{R}^m$ onto ℓ_q^m, $1 \le q \le \infty$.

Now we are in a position to formulate our main results.

Theorem 1 *For* $m > N$, $\delta \in (0, 1/2]$, *then*

$$d_{N,\delta}\big(T_m : \mathbb{R}^m \to \ell_\infty^m, \nu\big) \le C\|T_m\|\sqrt{\big(1 + (1/N)\ln(1/\delta)\big)\ln(em/N)}.$$

Theorem 2 *For* $m > 2N$, $\delta \in (0, 1/2]$, *then*

$$d_{N,\delta}\big(T_m : \mathbb{R}^m \to \ell_1^m, \nu\big) \ge C'\|T_m\|\sqrt{m + \ln(1/\delta)}.$$

2 Proof of main results

In order to prove Theorems 1 and 2, we also need some auxiliary assertions.

Lemma 1 *Let* $\delta \in (0, \sqrt{2/e\pi}\,]$, *and let* T_m *be a bounded linear invertible operator from* $\mathbb{R}^m$ *onto* ℓ_∞^m. *Then, for any vector* $z \in \mathbb{R}^m$,

$$\nu\big(x : \big|(T_m x, z)\big| \ge 2\|T_m\|\sqrt{\ln(1/\delta)}\|z\|_2\big) \le \delta.$$

Proof First, assume that T_m is a diagonal operator of $\mathbb{R}^m$, *i.e.*, $T_m x = (\lambda_i x_i)_{i=1}^m$, for any $x \in \mathbb{R}^m$. Without loss of generality, assume that the sequence of the absolute of eigenvalues λ_i, $i = 1, \dots, m$, is arranged non-increasingly, *i.e.*, $|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_m| > 0$. It is known that

$$\|T_m\| = \max_{1\le i\le m} |\lambda_i| = |\lambda_1|.$$

Since ν is invariant with respect to orthogonal transformation of $\mathbb{R}^m$, it suffices to prove the lemma for the vector $z^* = (\|z\|_2, 0, \dots, 0)$. Let $\varepsilon \in (0, e^{-1}]$ be arbitrary. We have

$$\begin{aligned}
&\nu\bigl(x : \bigl|(T_m x, z^*)\bigr| \ge \|T_m\|\sqrt{\ln(1/\varepsilon)}\|z\|_2\bigr) \\
&\quad = \nu\bigl(x : \bigl|\lambda_1 x_1\bigr|\|z\|_2 \ge |\lambda_1|\sqrt{\ln(1/\varepsilon)}\|z\|_2\bigr) \\
&\quad = \nu\bigl(x : |x_1| \ge \sqrt{\ln(1/\varepsilon)}\bigr) \\
&\quad = \frac{2}{\sqrt{\pi}} \int_{\sqrt{(1/2)\ln(1/\varepsilon)}}^{\infty} \exp\bigl(-t^2\bigr)\,dt \le \left(\frac{2\varepsilon}{\pi}\right)^{1/2}.
\end{aligned} \tag{3}$$

Here we use the inequality

$$\int_u^{\infty} \exp\bigl(-t^2\bigr)\,dt < \frac{1}{2u}\exp\bigl(-u^2\bigr), \quad u \ge \frac{1}{\sqrt{2}}.$$

From (3) we obtain the assertion of the lemma by $\delta = \sqrt{2\varepsilon/\pi}$.

Next, assume that T_m is a symmetric transformation of $\mathbb{R}^m$, then there is an orthogonal matrix U of order m such that the matrix UT_mU^T is a diagonal matrix. Since the Gaussian measure is invariant for orthogonal transformation, the result holds for symmetric transformation T_m.

Finally, assume that T_m is a general invertible linear transformation from $\mathbb{R}^m$ onto $\mathbb{R}^m$, then there are two matrices U and S such that $T_m = US$, where U is an orthogonal matrix and S is a positive definite symmetric matrix. As the same reason above, the result holds for the transformation T_m.

Thus Lemma 1 is proved. □

The following inequality will be used (see [27]). For any integers N and m with $m > N \ge 1$, there exists a subspace H of $\mathbb{R}^m$ of dimension $\dim H \ge m - N$ such that for any $x \in H$,

$$\|x\|_2 \le c_0 \left(\frac{\ln(em/N)}{N}\right)^{1/2} \|x\|_1, \tag{4}$$

where c_0 is an absolute constant.

Let $G \subset \mathbb{R}^m$ be a set. We introduce in $\mathbb{R}^m$ another norm for the operator $T_m : \mathbb{R}^m \to \ell_\infty^m$:

$$\|x\|_G = \sup_{y\in\mathbb{R}^m\setminus G} \bigl|(T_m y, x)\bigr|.$$

Lemma 2 *For any $\delta \in (0, 1/2]$ and an arbitrary operator T_m from $\mathbb{R}^m$ onto ℓ_∞^m, there exists a subset $G = G_\delta$ of $\mathbb{R}^m$ with measure $\nu(G) \le \delta$ such that*

$$\sup_{z\in B_1^m\cap H} \|z\|_G \le c_1\|T_m\| \left(\frac{1}{N}\ln\frac{\exp(aN)}{\delta}\ln\frac{em}{N}\right)^{1/2},$$

where a and c_1 are absolute constants.

Proof Let $k = [8(N+1)/\ln(em/N)]$ and consider the k^{-1}-net

$$S = \{(s_1/k, \dots, s_m/k) : s_1, \dots, s_m \in \mathbb{Z}, |s_1| + \cdots + |s_m| \le k\}$$

for the B_1^m in ℓ_∞^m-norm. Using the inequality $\binom{m}{\ell} \le (\frac{em}{\ell})^\ell$, we estimate the cardinality of S:

$$\operatorname{card} S \le \sum_{\ell=1}^{k} 2^\ell \binom{m+\ell}{\ell} \le \sum_{\ell=1}^{k} 2^\ell \left(\frac{4em}{\ell}\right)^\ell \le \left(\frac{8em}{k}\right)^k \le e^{aN}, \tag{5}$$

where a is some absolute constant.

Consider the polyhedron $Q = B_1^m \cap k^{-1} B_\infty^m$. Let Q' be the set of extremal points of Q. The set Q' consists of vectors with k coordinates equal to $\pm k^{-1}$ and the remaining coordinates zero. This implies that $Q' \subset S$, and hence $\operatorname{card} Q' \le \exp(aN)$.

Let $\varepsilon = \delta/\exp(aN)$. In $\mathbb{R}^m$ we consider the set $G = \bigcup_{s\in S} G_s$, where

$$G_s = \{y \in \mathbb{R}^m : |(T_m y, s)| \ge 2\|T_m\|\sqrt{\ln(1/\varepsilon)}\|s\|_2\}.$$

Let $z \in B_1^m \cap H$ be any point, and $s \in S$ be a point closest to z in ℓ_∞^m-norm. Then $z = s + t$ for some $t \in Q$. From Lemma 1,

$$\|z\|_G \le \|s\|_G + \|t\|_G \le 2\|T_m\|\sqrt{\ln(1/\varepsilon)}\|s\|_2 + \|t\|_G. \tag{6}$$

Using the definition of Q, we have $\|t\|_2^2 \le \|t\|_1\|t\|_\infty \le k^{-1}$. From this and the definition of Q' and G,

$$\begin{aligned}\|t\|_G &\le \max_{t\in Q'} \|t\|_G \le \max_{t\in S\cap Q} \|t\|_G \\ &\le 2\|T_m\|\sqrt{\ln(1/\varepsilon)} \max_{t\in S\cap Q} \|t\|_2 \le 2\|T_m\|\sqrt{k^{-1}\ln(1/\varepsilon)}.\end{aligned}$$

Therefore from (6),

$$\begin{aligned}\|z\|_G &\le 2\|T_m\|\sqrt{\ln(1/\varepsilon)}\big(\|z\|_2 + \|t\|_2\big) + \|t\|_G \\ &\le 2\|T_m\|\sqrt{\ln(1/\varepsilon)}\big(\|z\|_2 + 2k^{-1/2}\big).\end{aligned} \tag{7}$$

Since $z \in H$, it follows from the inequalities (4) and (7) that

$$\|z\|_G \le c_1\|T_m\|\sqrt{k^{-1}\ln(1/\varepsilon)} \le c_2\|T_m\|\sqrt{(1/N)\ln(1/\varepsilon)\ln(em/N)}.$$

Using Lemma 1 and the inequality (5), we can estimate the measure of G:

$$\nu(G) \le \sum_{s\in S} \nu(G_s) \le \varepsilon \operatorname{card} S \le \varepsilon \exp(aN) = \delta.$$

Thus, Lemma 2 is proved. □

Proof of Theorem 1 Using the duality in $\mathbb{R}^m$ and Lemma 2, we have

$$\begin{aligned}\sup_{x\in\mathbb{R}^m\setminus G}\inf_{y\in H^\perp}\|T_m x-y\|_\infty &= \sup_{x\in\mathbb{R}^m\setminus G}\sup_{z\in H\cap B_1^m}\big|(T_m x,z)\big| \\ &= \sup_{z\in H\cap B_1^m}\|z\|_G \le c\|T_m\|\sqrt{(1/N)\ln(1/\varepsilon)\ln(em/N)} \\ &= c\|T_m\|\sqrt{(1/N)\ln\big(\exp(aN)/\delta\big)\ln(em/N)},\end{aligned}$$

where $H^\perp$ is the orthogonal complement of H and $\dim H^\perp \le N$. The proof of Theorem 1 is completed. □

Let us proceed to the proof of Theorem 2. For this, we first prove four lemmas. We introduce a definition. For arbitrary $\varepsilon > 0$, the ε-*cardinality* of a subset K of ℓ_1^m is defined to be

$$\mathcal{N}_\varepsilon(K) = \min\big\{\mathcal{N} : z_1,\dots,z_{\mathcal{N}} \in \mathbb{R}^m, e\big(K,\{z_1,\dots,z_{\mathcal{N}}\}\big) \le \varepsilon\big\},$$

where

$$e\big(K,\{z_1,\dots,z_{\mathcal{N}}\}\big) = \sup_{x\in K}\min_{i=1,\dots,\mathcal{N}}\|x-z_i\|_1$$

is the deviation of K from the set $\{z_1,\dots,z_{\mathcal{N}}\}$ in ℓ_1^m.

Let λ be a Lebesgue measure in $\mathbb{R}$, normalized by the condition $\lambda(B) = 1$, where $B = B_2^m$. We consider Kolmogorov's (N,δ)-width of the ball B with measure λ in the ℓ_1^m-norm:

$$d_{N,\delta} \equiv d_{N,\delta}\big(T_m : B \to \ell_1^m, \lambda\big) = \inf_G\inf_{\mathcal{L}} e\big(T_m(B\setminus G),\mathcal{L},\ell_1^m\big),$$

where the T_m is as above, the infima are over all possible subsets $G \subset B$ of measure $\lambda(G) \le \delta$ and all subspaces $\mathcal{L} \subset \mathbb{R}^m$ with $\dim\mathcal{L} \le N$.

Lemma 3 *Suppose that $D \subset B$ is an arbitrary subset with measure $\lambda(D) \le \delta$. Then, for any $\varepsilon > d_{N,\delta}$,*

$$\mathcal{N}_\varepsilon\big(T_m(B\setminus D)\big) \le \left(1+\frac{4\|T_m\|}{\varepsilon-d_{N,\delta}}\right)^m.$$

Proof Let $h > 0$ be any number, and let H be any subspace of $\mathbb{R}$ with $\dim H \le N$ such that

$$e\big(T_m(B\setminus D),H,\ell_1^m\big) - h \le d_{N,\delta}. \tag{8}$$

Let $\varepsilon' = \varepsilon - d_{N,\delta}$. We consider the set $Q = 2(T_m(B\setminus D)) \cap H$. Let $Q_{\varepsilon'} = \{z_1,\dots,z_{\mathcal{N}}\}$ be the maximal subset of Q such that $\|z_i - z_j\|_1 \ge \varepsilon'$ for all $i \ne j$. Clearly, by maximality $Q_{\varepsilon'}$ is a ε'-net of Q for $\|\cdot\|_1$. The balls $z_i + (\varepsilon'/2)B_1^m$ are disjoint and all contained in $Q + (\varepsilon'/2)B_1^m$. Therefore, taking volumes we can obtain

$$\sum_{i=1}^{\mathcal{N}}\operatorname{vol}\big(z_i + (\varepsilon'/2)B_1^m\big) \le \operatorname{vol}\big(Q + (\varepsilon'/2)B_1^m\big).$$

Hence, we have

$$\mathcal{N} \cdot (\varepsilon'/2) \operatorname{vol}(B_1^m) \le \operatorname{vol}(Q + (\varepsilon'/2)B_1^m). \tag{9}$$

By $Q = 2(T_m(B\backslash D)) \cap H \subset 2T_m(B) \subset 2T_m(B_1^m) \subset 2\|T_m\|(B_1^m)$ and (9), we have

$$\mathcal{N}(\varepsilon'/2)^m \operatorname{vol}(B_1^m) \le (2\|T_m\| + (\varepsilon'/2))^m \operatorname{vol}(B_1^m),$$

that is,

$$\mathcal{N} \le \left(1 + \frac{4\|T_m\|}{\varepsilon'}\right)^m. \tag{10}$$

Now, we need to establish $e(T_m(B\backslash D), \{z_1, \ldots, z_{\mathcal{N}}\}) \le \varepsilon$. Since $t \in Q \subset H$, we have from (8)

$$\begin{aligned}
\sup_{x\in T_m(B\backslash D)} \min_{t\in Q} \|x - t\|_1 &\le \sup_{x\in T_m(B\backslash D)} \min_{t\in Q} \min_{h_0\in H} (\|x - h_0\|_1 + \|h_0 - t\|_1) \\
&\le \sup_{x\in T_m(B\backslash D)} \min_{t\in Q} \|x - h_0\|_1 + \sup_{x\in T_m(B\backslash D)} \min_{t\in Q} \min_{h_0\in H} \|h_0 - t\|_1 \\
&\le d_{N,\delta} + h.
\end{aligned} \tag{11}$$

From the inequality (11), the definition of $Q_{\varepsilon'}$ and (8), it follows that

$$\begin{aligned}
e(T_m(B\backslash D), \{z_1, \ldots, z_{\mathcal{N}}\}) &= \sup_{x\in T_m(B\backslash D)} \min_{i=1,\ldots,\mathcal{N}} \|x - z_i\|_1 \\
&\le \sup_{x\in T_m(B\backslash D)} \min_{i=1,\ldots,\mathcal{N}} \min_{t\in Q} (\|x - t\|_1 + \|t - z_i\|_1) \\
&\le \sup_{x\in T_m(B\backslash D)} \min_{t\in Q} \Big(\|x - t\|_1 + \min_{i=1,\ldots,\mathcal{N}} \|t - z_i\|_1\Big) \\
&= \sup_{x\in T_m(B\backslash D)} \min_{t\in Q} \|x - t\|_1 + \varepsilon' \\
&\le d_{N,\delta} + h + \varepsilon' \\
&= \varepsilon + h.
\end{aligned}$$

Consequently, letting $h \to 0$, we get that $e(T_m(B\backslash D), \{z_1, \ldots, z_{\mathcal{N}}\}) \le \varepsilon$, which together with (10) completes the proof of Lemma 3. □

From the relation (see [28])

$$\operatorname{vol}(B_p^m) = [2\Gamma(1/p + 1)]^m / \Gamma(m/p + 1), \quad 1 \le p \le \infty,$$

the balls B_p^m satisfy the inequalities

$$(c_p' m)^{-m/p} < \operatorname{vol}(B_p^m) < (c_p'' m)^{-m/p}, \tag{12}$$

where Γ is the Euler Γ-function, and c_p', c_p'' depend only on p.

To estimate $\mathcal{N}_\varepsilon(T_m(B\backslash D))$ from below, we now need another auxiliary result.

Lemma 4 *If T_m is a diagonal operator from $\mathbb{R}^m$ onto ℓ_1^m, then*

$$\left|\det(T_m)\right| = \left|\prod_{i=1}^{m} \lambda_i(T_m)\right| \asymp \left(\frac{\|T_m\|}{\sqrt{m}}\right)^m,$$

where $\lambda_i(T_m)$, $i = 1, \ldots, m$, are non-zero eigenvalues of the operator T_m rearranged as usual so that $|\lambda_i(T_m)|$ is non-increasing and each eigenvalue is repeated according to its multiplicity.

Proof It is known that

$$\|T_m\| = \left(\sum_{i=1}^{m} \left(\lambda_i(T_m)\right)^2\right)^{1/2}.$$

Accordingly,

$$\sqrt{m}\left|\lambda_m(T_m)\right| \le \|T_m\| \le \sqrt{m}\left|\lambda_1(T_m)\right|.$$

Obviously,

$$\left|\lambda_m(T_m)\right|^m \le \left|\prod_{i=1}^{m} \lambda_i(T_m)\right| \le \left|\lambda_m(T_1)\right|^m,$$

from which the result of Lemma 4 follows immediately. □

Lemma 5 *If $\delta \in [0,1]$ and $\lambda(D) \le \delta$, then*

$$\mathcal{N}_\varepsilon\left(T_m(B\backslash D)\right) \ge \frac{1}{3}(1-\delta)\left(c_0\|T_m\|/\varepsilon\right)^m.$$

Proof We first establish the inequality

$$\mathcal{N}_\varepsilon\left(T_m(B\backslash D)\right) \ge \frac{1}{3}\frac{\lambda(T_m(B\backslash D))}{\lambda(B_1^m(2\varepsilon))}. \tag{13}$$

Indeed, suppose that (13) does not hold. Then, for $\mathcal{N} = \mathcal{N}_\varepsilon(T_m(B\backslash D))$ and some set of points $z_1^*, \ldots, z_{\mathcal{N}}^*$, by

$$\begin{aligned}
\varepsilon &\ge \sup_{x\in T_m(B\backslash D)} \min_{i=1,\ldots,\mathcal{N}} \left\|x - z_i^*\right\|_1 \\
&\ge \frac{1}{\lambda(T_m(B\backslash D))}\int_{T_m(B\backslash D)} \min_i \left\|x - z_i^*\right\|_1 \lambda(dx) \\
&\ge \frac{1}{\lambda(T_m(B\backslash D))}\int_{T_m(B\backslash D)\backslash \bigcup_{i=1}^{\mathcal{N}}(z_i^*+B_1^m(2\varepsilon))} \min_i \left\|x - z_i^*\right\|_1 \lambda(dx) \\
&\ge \frac{1}{\lambda(T_m(B\backslash D))}(2\varepsilon)\lambda\left(T_m(B\backslash D)\Big\backslash \bigcup_{i=1}^{\mathcal{N}}\left(z_i^* + B_1^m(2\varepsilon)\right)\right) \\
&= 2\varepsilon\frac{\lambda(T_m(B\backslash D)) - \mathcal{N}\cdot\lambda(B_1^m(2\varepsilon))}{\lambda(T_m(B\backslash D))}
\end{aligned}$$

$$\begin{aligned}
&= 2\varepsilon\left(1 - \mathcal{N}\cdot\frac{\lambda(B_1^m(2\varepsilon))}{\lambda(T_m(B\backslash D))}\right)\\
&\geq 2\varepsilon\cdot\frac{2}{3} = \frac{4}{3}\varepsilon,
\end{aligned}$$

we have obtained a contradiction.

In the sequel, we may as well assume that T_m is a diagonal operator from $\mathbb{R}^m$ onto ℓ_1^m. Using the inequality (13), (12) and Lemma 4, we have

$$\begin{aligned}
\mathcal{N}_\varepsilon\big(T_m(B\backslash D)\big) &\geq \frac{1}{3}\frac{\lambda(T_m(B\backslash D))}{\lambda(B_1^m(2\varepsilon))}\\
&= \frac{1}{3}\frac{|\det(T_m)|\lambda(B\backslash D)}{\lambda(B_1^m(2\varepsilon))}\\
&\geq \frac{1}{3}c\left(\frac{\|T_m\|}{\sqrt{m}}\right)^m(1-\delta)\frac{\mathrm{vol}(B)}{\mathrm{vol}(B_1^m(2\varepsilon))}\\
&\geq \frac{1}{3}\left(\frac{c_1\|T_m\|}{\sqrt{m}}\right)^m(1-\delta)\left(\frac{c_2\sqrt{m}}{2\varepsilon}\right)^m\\
&= \frac{1}{3}(1-\delta)\left(\frac{c_0\|T_m\|}{\varepsilon}\right)^m.
\end{aligned}$$

Next, assume that T_m is a symmetric transformation of $\mathbb{R}^m$, then there is an orthogonal matrix U of order m such that the matrix UT_mU^T is a diagonal matrix. Since the Lebesgue measure is invariant for orthogonal transformation, the result holds for symmetric transformation T_m.

Finally, in the general case, T_m is a general invertible linear transformation from $\mathbb{R}^m$ onto ℓ_1^m, then there are two matrices U and S such that $T_m = US$, where U is an orthogonal matrix and S is a positive definite symmetric matrix. As the same reason above, the result holds for the transformation T_m.

Thus, we complete the proof of Lemma 5. □

Lemma 6 *If* $\delta \in [0, 1/2]$, *then*

$$d_{N,\delta}\big(T_m : B \to \ell_1^m, \lambda\big) \geq c_1\|T_m\|.$$

Proof From Lemma 3 and Lemma 5, we get

$$c\left(1 + \frac{4\|T_m\|}{\varepsilon - d_{N,\delta}}\right)^m \geq N_\varepsilon\big(T_m(B\backslash D)\big) \geq \frac{1}{3}(1-\delta)\left(\frac{c_0\|T_m\|}{\varepsilon}\right)^m \geq \frac{1}{6}\left(\frac{c_0\|T_m\|}{\varepsilon}\right)^m. \tag{14}$$

Let $\varepsilon = 5d_{N,\delta}$. Taking the logarithm of the inequality (14), we get the inequality

$$d_{N,\delta}\big(T_m : B \to \ell_1^m, \lambda\big) \geq c_1\|T_m\|$$

for some constants c_1 and c_2 and N with $m \geq c_2N$. Lemma 6 is proved. □

Proof of Theorem 2 According to Lemma 6, for any $\delta \in [0, 1/2]$ and any subspace $\mathcal{L} \subset \mathbb{R}^m$ with $\dim\mathcal{L} \leq N$, there is a set $K \subset B$ with Lebesgue measure $\lambda(K) > \delta$ such that

$$e\big(T_mx, \mathcal{L}, \ell_1^m\big) \geq c_1\|T_m\| \tag{15}$$

for any element $x \in K$. On the unit sphere $S^{m-1} = \{x \in \mathbb{R}^m : \|x\|_2 = 1\}$, we consider the subset $K' = \{x/\|x\|_2 : x \in K\}$.

Let $\lambda_{S^{m-1}}$ be a Lebesgue measure on the sphere S^{m-1}. We prove that

$$\lambda_{S^{m-1}}(K') > \delta \lambda_{S^{m-1}}(S^{m-1}). \tag{16}$$

Indeed, assume that

$$\lambda_{S^{m-1}}(K') \le \delta \lambda_{S^{m-1}}(S^{m-1}).$$

We introduce in $\mathbb{R}^m$ a polar system of coordinates (r,s), where $r \ge 0$ and $s \in S^{m-1}$, and consider in B the cone

$$C = \{(r,s) : 0 \le r \le 1, s \in K'\}.$$

Then

$$\begin{aligned}\lambda(K) \le \lambda(C) &= \frac{1}{\operatorname{vol}(B)} \int_0^1 r^{m-1}\, dr \int_{K'} \lambda_{S^{m-1}}(ds) \\ &\le \frac{\delta}{\operatorname{vol}(B)} \int_0^1 r^{m-1}\, dr \int_{S^{m-1}} \lambda_{S^{m-1}}(ds) = \delta.\end{aligned}$$

We have obtained a contradiction.

Consider the set $K_t = \{(r,s) : r \ge t, s \in K'\}$, $t \ge 0$. Using the inequality (16), we estimate the Gaussian measure of K_t:

$$\begin{aligned}\nu(K_t) &= (2\pi)^{-m/2} \int_{K_t} \exp\left(-\frac{1}{2}\|u\|_2^2\right)(du) \\ &= (2\pi)^{-m/2} \int_t^\infty r^{m-1} \exp\left(-\frac{r^2}{2}\right) dr \int_{K'} \lambda_{S^{m-1}}(ds) \\ &\ge (2\pi)^{-m/2}\delta \int_t^\infty r^{m-1} \exp\left(-\frac{r^2}{2}\right) dr \int_{S^{m-1}} \lambda_{S^{m-1}}(ds) \\ &= \delta \nu\big(x : \|x\|_2 \ge t\big).\end{aligned} \tag{17}$$

A direct computation shows that for $t \ge \sqrt{m}$,

$$\nu\big(\|x\|_2 \ge t\big) \ge \exp\big(-t^2\big) \quad \text{and} \quad \nu\big(\|x\|_2 \ge \sqrt{m}\big) \ge c_0 > 0,$$

where c_0 is some absolute constant. It follows from this and (17) that for $t_0 = \max\{\sqrt{m}, \sqrt{\ln(1/\delta)}\}$ and for any $\delta \in (0, c_0]$,

$$\nu(K_{t_0}) \ge \delta \cdot \nu\big(\|x\|_2 > \max\{\sqrt{m}, \sqrt{\ln(1/\delta)}\}\big) > \delta^2. \tag{18}$$

For any element $y = rs \in K_{t_0}$, we have from (15)

$$\begin{aligned}e\big(T_m y, \mathcal{L}, \ell_1^m\big) &= re\big(T_m s, \mathcal{L}, \ell_1^m\big) \ge c_1 r \|T_m\| \ge c_2 \|T_m\| \max\{\sqrt{m}, \sqrt{\ln(1/\delta)}\} \\ &\ge \frac{c_2}{2} \|T_m\| \sqrt{m + \ln(1/\delta)}.\end{aligned} \tag{19}$$

Since $\mathscr{L}$ is an arbitrary subspace with $\dim \mathscr{L} \le N$, it follows from (18) and (19) that

$$d_{N,\delta^2}\bigl(T_m : \mathbb{R}^m \to \ell_1^m, \nu\bigr) \ge \frac{c_2}{2} \|T_m\| \sqrt{m + \ln(1/\delta)}.$$

Theorem 2 is a direct consequence of this. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors completed the paper together. They also read and approved the final manuscript.

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