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On a class of spiral-like functions with respect to a boundary point related to subordination

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Abstract

For $\mu \in \mathbb{C}$, φ a starlike univalent function, the class of functions f that are spiral-like with respect to a boundary point satisfying the subordination

$$\frac{2}{\mu}\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z} \prec \varphi(z), \quad z \in \mathbb{D},$$

is investigated. The integral representation, growth and distortion theorem are proved by relating these functions with Ma and Minda starlike functions. Some earlier results are shown to be a special case of the results obtained.

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1 Introduction and motivation

Let $\mathbb{D} = \{z : |z| < 1\}$ be an open unit disk of the complex plane $\mathbb{C}$ and let $\mathcal{A}$ be a class of analytic functions f normalized by $f(0) = 0$ and $f'(0) = 1$. Let w_0 be an interior or a boundary point of a set $\mathcal{D}$ in $\mathbb{C}$. The set $\mathcal{D}$ is starlike with respect to w_0 if the line segment joining w_0 to every other point in $\mathcal{D}$ lies in the interior of $\mathcal{D}$. If a function $f \in \mathcal{A}$ maps $\mathbb{D}$ onto a starlike domain with respect to origin, then f is a starlike function. The class of starlike functions with respect to origin is denoted by $\mathcal{S}^*$. Analytically,

$$\mathcal{S}^* := \left\{ f \in \mathcal{A} : \operatorname{Re}\frac{zf'(z)}{f(z)} > 0 \right\}.$$

Robertson [1] took a leap forward with the characterization of the class $\mathcal{S}^*$ and defined the class $\mathcal{S}_b^*$ of starlike functions with respect to a boundary point. Geometrically, it is the characterization of a function $f \in \mathcal{S}_b = \{f(z) = 1 + d_1 z + d_2 z^2 + \cdots \mid f \text{ univalent}\}$ such that $f(\mathbb{D})$ is starlike with respect to the boundary point $f(1) := \lim_{r\to 1^-} f(r) = 0$ and lies in a half-plane. The analytic description given by Robertson was

$$\mathcal{S}_b^* := \left\{ f \in \mathcal{S}_b : \operatorname{Re}\left(2\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z} \right) > 0 \right\}.$$

This was partially proved in [1]. It was only in 1984 that the characterization was validated by Lyzzaik [2]. Todorov [3] associated this class with a functional $f(z)/(1-z)$ and obtained a structured formula and coefficient estimates in the year 1986. Later, Silverman and Silvia [4] gave a full description of the class of univalent functions on $\mathbb{D}$, the image of which is star-shaped, with respect to a boundary point. Since then, this class of starlike functions with respect to a boundary point has gained notable interest among geometric function theorist and also other researchers. Among them, Abdullah *et al.* [5] studied the properties of functions in this class. The distortion results for starlike functions with respect to a boundary point were obtained in [6, 7]. The dynamical characterizations of functions starlike with respect to a boundary point can be found in [8]. In the year 2001, Lecko [9] gave another representation of starlike functions with respect to a boundary point. Also, Lecko and Lyzzaik obtained different characterizations of this class in [10].

Following the studies on the class of starlike functions, many authors extensively studied the class of spiral-like functions. For recent work on the class of spiral-like functions, see [11]. Later, there was interest towards the class of spiral-like functions with respect to a boundary point. See [12–15]. Aharonov *et al.* [16] gave a comprehensive definition for spiral-shaped domains with respect to a boundary point.

Definition 1.1 A simply connected domain $\Omega \subset \mathbb{C}$, $0 \in \partial\Omega$, is called a spiral-shaped domain with respect to a boundary point if there is a number $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > 0$ such that, for any point $\omega \in \Omega$, the curve $e^{-t\mu}\omega$, $t \geq 0$, is contained in Ω.

It was also showed in [16] (see also [17]) that each spiral-like function with respect to a boundary point is a complex power of starlike function with respect to a boundary point. In particular, if $\mu \in \mathbb{R}$ in Definition 1.1, then Ω is called a star-shaped domain with respect to a boundary point. The following was proved in the same.

Theorem 1.1 *Let f be an analytic function with $f(0) = 1, f(1) = 0$, and let it be a spiral-like function with respect to a boundary point. Then there exists a number $\mu \in \Omega := \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1, \lambda \neq 0\}$ such that*

$$\operatorname{Re}\left(\frac{2}{\mu}\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}\right) > 0. \tag{1.1}$$

Conversely, if f is a univalent function with $f(0) = 1$ and $f(1) = 0$ satisfies (1.1) *for some $\mu \in \Omega$, then f is a spiral-like function with respect to a boundary point.*

Elin [18] then considered the class of spiral-like functions of order β $(0 < \beta \leq 1)$ with respect to a boundary point and obtained interesting results including the distortion and covering theorems.

On the other hand, Ma and Minda [19] gave a unified presentation of the class starlike using the method of subordination. For two functions h and g in $\mathcal{A}$, the function h is subordinate to g, written

$$h(z) \prec g(z), \quad z \in \mathbb{D},$$

if there exists a function $w \in \mathcal{A}$, with $w(0) = 0$ and $|w(z)| < 1$, such that $h(z) = g(w(z))$. In particular, if the function g is univalent in $\mathbb{D}$, then $h(z) \prec g(z)$ is equivalent to $h(0) = g(0)$

and $h(\mathbb{D}) \subset g(\mathbb{D})$. A function $h \in \mathcal{A}$ is starlike if $zh'(z)/h(z)$ is subordinated to $(1+z)/(1-z)$. Ma and Minda [19] introduced the class

$$\mathcal{S}^*(\varphi) = \left\{ h \in \mathcal{A} : \frac{zh'(z)}{h(z)} \prec \varphi(z) \right\},$$

where φ is an analytic function with a positive real part in $\mathbb{D}$, $\varphi(\mathbb{D})$ is symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. A function $f \in \mathcal{S}^*(\varphi)$ is called Ma and Minda starlike (with respect to φ). The class $\mathcal{S}^*(\beta)$ consisting of starlike functions of order β, $0 \le \beta < 1$ and the class $\mathcal{S}^*(A,B)$ of Janowski starlike functions are special cases of $\mathcal{S}^*(\varphi)$ when $\varphi(z) := (1 + (1-2\beta)z)/(1-z)$ and $\varphi(z) := (1+Az)/(1+Bz)$ for $-1 \le B < A \le 1$, respectively.

In the same direction and motivated mainly by [18] and [19], we consider the following class.

Definition 1.2 Let $f \in \mathcal{S}_b$, $f(0) = 1$ and $\mu \in \Omega := \{\lambda \in \mathbb{C} : |\lambda - 1| \le 1, \lambda \ne 0\}$. Also, let φ be an analytic function with a positive real part $\mathbb{D}$, let $\varphi(\mathbb{D})$ be symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. The function $f \in \mathcal{S}_b^*(\mu, \varphi)$ if the subordination

$$\frac{2}{\mu}\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z} \prec \varphi(z), \quad z \in \mathbb{D}, \tag{1.2}$$

holds.

For $\varphi(z) = (1+Az)/(1+Bz)$ $(-1 \le B < A \le 1)$, denote the class $\mathcal{S}_b^*(\mu,\varphi)$ by $\mathcal{S}_b^*(\mu,A,B)$. For $0 \le \beta < 1$, $A = 1-2\beta$ and $B = -1$, denote $\mathcal{S}_b^*(\mu,A,B)$ by $\mathcal{S}_b^*(\mu,\beta)$.

The class $\mathcal{S}_b^*(\mu,\varphi)$ defined by subordination is investigated to obtain representation, estimates for f and f' and subordination conditions. We obtained some interesting result in a wider context and our approach is mainly based on [19].

2 Representation for the class $\mathcal{S}_b^*(\mu, \varphi)$

The following result provides an integral representation of functions belonging to the class $\mathcal{S}_b^*(\mu,\varphi)$.

Theorem 2.1 *The function $f \in \mathcal{S}_b^*(\mu,\varphi)$ if and only if there exists p satisfying $p \prec \varphi$ such that*

$$f(z) = (1-z)^{\mu} \exp\left(\frac{\mu}{2} \int_0^z \frac{p(\zeta) - 1}{\zeta}\, d\zeta \right).$$

Proof Let $f \in \mathcal{S}_b^*(\mu,\varphi)$. Then define $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{2}{\mu}\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}.$$

Then $f \in \mathcal{S}_b^*(\mu,\varphi)$ implies that $p \prec \varphi$. Rewriting the above equation as

$$\frac{2}{\mu}\frac{f'(z)}{f(z)} + \frac{2}{1-z} = \frac{p(z) - 1}{z}$$

and integrating from 0 to z, it follows that

$$\log\left(\frac{f(z)^{\frac{2}{\mu}}}{(1-z)^2}\right) = \int_0^z \frac{p(\zeta)-1}{\zeta}\,d\zeta.$$

An exponentiation gives

$$f(z)^{\frac{2}{\mu}} = (1-z)^2 \exp\left(\int_0^z \frac{p(\zeta)-1}{\zeta}\,d\zeta\right).$$

The desired result follows from this. The converse follows easily. □

3 Estimates for f and f' in the class $\mathcal{S}_b^*(\mu,\varphi)$

Theorem 3.1 *Let h_φ be an analytic function with $h_\varphi(0) = 0$, $h'_\varphi(0) = 1$ satisfying the equation $zh'_\varphi(z)/h_\varphi(z) = \varphi(z)$. If $f \in \mathcal{S}_b^*(\mu,\varphi)$, then*

$$\frac{-h_\varphi(-r)}{r}|1-z|^2 \le \left|f(z)^{\frac{2}{\mu}}\right| \le \frac{h_\varphi(r)}{r}|1-z|^2, \quad |z| = r. \tag{3.1}$$

Proof Define the function $h \in \mathcal{A}$ by

$$h(z) = \frac{z}{(1-z)^2} f(z)^{\frac{2}{\mu}}, \quad z \in \mathbb{D}. \tag{3.2}$$

Since f is univalent and $f(1) := \lim_{r\to 1^-} f(r) = 0$, it is clear that $f(z) \neq 0$ in $\mathbb{D}$. Therefore, the function h is well defined and analytic in $\mathbb{D}$. A computation shows that

$$\frac{zh'(z)}{h(z)} = \frac{2}{\mu}\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}. \tag{3.3}$$

Hence we have the relation $f \in \mathcal{S}_b^*(\mu,\varphi)$ if and only if $h \in \mathcal{S}^*(\varphi)$. Ma and Minda [19, Corollary 1′] have shown that for $h \in \mathcal{S}^*(\varphi)$,

$$-h_\varphi(-r) \le \left|h(z)\right| \le h_\varphi(r), \quad |z| = r.$$

Using this inequality for h in (3.2) gives

$$-h_\varphi(-r) \le \left|\frac{z}{(1-z)^2} f(z)^{\frac{2}{\mu}}\right| \le h_\varphi(r), \quad |z| = r$$

and hence the desired result follows. □

If $\mathcal{S}_b^*(\mu,A,B)$ and hence

$$h_\varphi(z) = \begin{cases} z(1+Bz)^{\frac{A-B}{B}}, & B \neq 0, \\ z\exp(Az), & B = 0, \end{cases}$$

then

$$|1-z|^2(1-Br)^{\frac{A-B}{B}} \le \left|f(z)^{\frac{2}{\mu}}\right| \le |1-z|^2(1+Br)^{\frac{A-B}{B}} \quad \text{for } B \neq 0,$$

$$|1-z|^2\exp(-Ar) \le \left|f(z)^{\frac{2}{\mu}}\right| \le |1-z|^2\exp(Ar) \quad \text{for } B = 0.$$

If $\mathcal{S}_b^*(\mu,\beta)$ and

$$h_\varphi(z) = \frac{z}{(1-z)^{2-2\beta}},$$

then

$$\frac{|1-z|^2}{(1+r)^{2-2\beta}} \le \bigl|f(z)^{\frac{2}{\mu}}\bigr| \le \frac{|1-z|^2}{(1-r)^{2-2\beta}}.$$

In particular, for $0 \neq \mu \in \mathbb{R}$, the inequality reduces to the following inequality [18]:

$$\frac{|1-z|^\mu}{(1+r)^{\mu(1-\beta)}} \le \bigl|f(z)\bigr| \le \frac{|1-z|^\mu}{(1-r)^{\mu(1-\beta)}}.$$

Theorem 3.2 *Let $\varphi(z) = zh_\varphi'(z)/h_\varphi(z)$ and $f \in \mathcal{S}_b^*(\mu,\varphi)$. Then, for $|z| = r$,*

$$\left|\arg\frac{f(z)^{\frac{1}{\mu}}}{(1-z)}\right| \le \frac{1}{2}\max_{|z|=r}\arg\frac{h_\varphi(z)}{z}.$$

For $0 \neq \mu \in \mathbb{R}$,

$$\left|\arg\frac{f(z)}{(1-z)^\mu}\right| \le \frac{|\mu|}{2}\max_{|z|=r}\arg\frac{h_\varphi(z)}{z}.$$

Proof For a function $h \in \mathcal{S}^*(\varphi)$, in the paper [19, Corollary 3′] it is shown that

$$\left|\arg\frac{h(z)}{z}\right| \le \max_{|z|=r}\arg\frac{h_\varphi(z)}{z}, \quad |z| = r.$$

The result then follows easily as the relation (3.3) holds. □

Corollary 3.1 *If $f \in \mathcal{S}_b^*(\mu,A,B)$, then for $|z| = r$,*

$$\left|\arg\frac{f(z)^{\frac{1}{\mu}}}{(1-z)}\right| \le \frac{A-B}{2B}\max_{|z|=r}\arg(1+Bz) \quad \textit{for } B \neq 0$$

and

$$\left|\arg\frac{f(z)^{\frac{1}{\mu}}}{(1-z)}\right| \le \frac{1}{2}\max_{|z|=r}\arg\exp(Az) \quad \textit{for } B = 0.$$

Corollary 3.2 *If $f \in \mathcal{S}_b^*(\mu,\beta)$, then for $|z| = r$*

$$\left|\arg\frac{f(z)^{\frac{1}{\mu}}}{(1-z)}\right| \le (1-\beta)\max_{|z|=r}\arg\frac{1}{(1-z)}.$$

Theorem 3.3 *Let $\varphi(z) = zh_\varphi'(z)/h_\varphi(z)$ and*

$$\min_{|z|=r}\bigl|\varphi(z)\bigr| = \varphi(-r) \quad \textit{and} \quad \max_{|z|=r}\bigl|\varphi(z)\bigr| = \varphi(r). \tag{3.4}$$

Also, let

$$H_{\varphi 1} = \frac{|\mu||1-z|^{\mu}}{2r}\left(\frac{h_{\varphi}(-r)}{-r}\right)^{\frac{\mu}{2}}\left(-\left|\frac{1+z}{1-z}\right| + \varphi(-r)\right)$$

and

$$H_{\varphi 2} = \frac{|\mu||1-z|^{\mu}}{2r}\left(\frac{h_{\varphi}(r)}{r}\right)^{\frac{\mu}{2}}\left(\left|\frac{1+z}{1-z}\right| + \varphi(r)\right).$$

For $\mu \in \mathbb{R}$, if $f \in \mathcal{S}_b^*(\mu,\varphi)$ then

$$H_{\varphi 1} \le \left|f'(z)\right| \le H_{\varphi 2}.$$

Proof By Definition 1.2, for $f \in \mathcal{S}_b^*(\mu,\varphi)$, we have

$$\frac{2}{\mu}\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z} \prec \varphi(z), \quad z \in \mathbb{D}.$$

When (3.4) holds, the above subordination indicates that

$$\varphi(-r) \le \left|\frac{2}{\mu}\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}\right| \le \varphi(r), \quad |z| = r.$$

This shows that

$$-\left|\frac{1+z}{1-z}\right| + \varphi(-r) \le \left|\frac{2}{\mu}\frac{zf'(z)}{f(z)}\right| \le \left|\frac{1+z}{1-z}\right| + \varphi(r)$$

or

$$\frac{|\mu|}{2r}\left(-\left|\frac{1+z}{1-z}\right| + \varphi(-r)\right) \le \left|\frac{f'(z)}{f(z)}\right| \le \frac{|\mu|}{2r}\left(\left|\frac{1+z}{1-z}\right| + \varphi(r)\right). \tag{3.5}$$

For $\mu \in \mathbb{R}$, Theorem 3.1 gives

$$|1-z|^{\mu}\left(\frac{h_{\varphi}(-r)}{-r}\right)^{\frac{\mu}{2}} \le \left|f(z)\right| \le |1-z|^{\mu}\left(\frac{h_{\varphi}(r)}{r}\right)^{\frac{\mu}{2}}. \tag{3.6}$$

Combining (3.5) and (3.6), the desired results follows. □

We have the following corollaries as (3.4) holds.

Corollary 3.3 *Let* $\varphi(z) = zh_{\varphi}'(z)/h_{\varphi}(z)$. *For* $B \neq 0$, *let*

$$H_{\varphi 1} = \frac{|\mu||1-z|^{\mu}}{2r}(1-Br)^{\frac{\mu(A-B)}{2B}}\left(-\left|\frac{1+z}{1-z}\right| + \frac{1-Ar}{1-Br}\right)$$

and

$$H_{\varphi 2} = \frac{|\mu||1-z|^{\mu}}{2r}(1+Br)^{\frac{\mu(A-B)}{2B}}\left(\left|\frac{1+z}{1-z}\right| + \frac{1+Ar}{1+Br}\right).$$

For $B = 0$, let

$$H_{\varphi 1} = \frac{|\mu||1-z|^{\mu}}{2r} \exp\left(\frac{-\mu A r}{2}\right)\left(-\left|\frac{1+z}{1-z}\right| - r\exp(-Ar)\right)$$

and

$$H_{\varphi 2} = \frac{|\mu||1-z|^{\mu}}{2r} \exp\left(\frac{\mu A r}{2}\right)\left(\left|\frac{1+z}{1-z}\right| + r\exp(Ar)\right).$$

For $\mu \in \mathbb{R}$, if $f \in \mathcal{S}_b^*(\mu, A, B)$ then

$$H_{\varphi 1} \leq \left|f'(z)\right| \leq H_{\varphi 2}.$$

Corollary 3.4 *Let $\varphi(z) = zh'_{\varphi}(z)/h_{\varphi}(z)$,*

$$H_{\varphi 1} = \frac{|\mu||1-z|^{\mu}}{2r(1+r)^{\mu(1-\beta)}}\left(-\left|\frac{1+z}{1-z}\right| + \frac{1-(1-2\beta)r}{1+r}\right)$$

and

$$H_{\varphi 2} = \frac{|\mu||1-z|^{\mu}}{2r(1-r)^{\mu(1-\beta)}}\left(\left|\frac{1+z}{1-z}\right| + \frac{1+(1-2\beta)r}{1-r}\right).$$

For $\mu \in \mathbb{R}$, if $f \in \mathcal{S}_b^*(\mu, \beta)$ then

$$H_{\varphi 1} \leq \left|f'(z)\right| \leq H_{\varphi 2}.$$

4 Necessary and sufficient condition

Theorem 4.1 *Let φ be a convex univalent function defined on $\mathbb{D}$. The function $f \in \mathcal{S}_b^*(\mu, \varphi)$ if and only if for all $|s| \leq 1$, $|t| \leq 1$,*

$$\frac{s}{t}\left(\frac{1-tz}{1-sz}\right)^2 \left(\frac{f(sz)}{f(tz)}\right)^{\frac{2}{\mu}} \prec \frac{h_{\varphi}(sz)}{h_{\varphi}(tz)},$$

where $h_{\varphi}(z) = z\exp(\int_0^z ((\varphi(\zeta)-1)/\zeta)\, d\zeta)$.

Proof Ruscheweyh [20, Theorem 1] showed that for φ a convex univalent function, F as in the hypothesis and $h \in \mathcal{A}$

$$\frac{zh'(z)}{h(z)} \prec \varphi(z)$$

if and only if for all $|s| \leq 1$, $|t| \leq 1$,

$$\frac{h(sz)}{h(tz)} \prec \frac{h_{\varphi}(sz)}{h_{\varphi}(tz)}. \tag{4.1}$$

From the relation (3.3), we know that $f \in \mathcal{S}_b^*(\mu,\varphi)$ if and only if $h \in \mathcal{S}^*(\varphi)$. Substituting (3.2) in (4.1), we have

$$\frac{\frac{sz}{(1-sz)^2} f(sz)^{\frac{2}{\mu}}}{\frac{tz}{(1-tz)^2} f(tz)^{\frac{2}{\mu}}} \prec \frac{h_\varphi(sz)}{h_\varphi(tz)}$$

and hence the desired result follows. □

The following corollaries hold for $\varphi(z) = \frac{1+Az}{1+Bz}$ is convex univalent on $\mathbb{D}$.

Corollary 4.1 *The function $f \in \mathcal{S}_b^*(\mu,A,B)$ if and only if for all $|s| \le 1$, $|t| \le 1$,*

$$\left(\frac{1-tz}{1-sz}\right)^{\mu} \left(\frac{f(sz)}{f(tz)}\right) \prec \left(\frac{1+Bsz}{1+Btz}\right)^{\frac{\mu(A-B)}{2B}} \quad \textit{for } B \neq 0,$$

$$\left(\frac{1-tz}{1-sz}\right)^{\mu} \left(\frac{f(sz)}{f(tz)}\right) \prec \exp\left(\frac{\mu Az(s-t)}{2}\right) \quad \textit{for } B = 0.$$

Let $0 \le \beta < 1$, $A = 1 - 2\beta$ and $B = -1$ in Corollary 4.1 and hence we have the result.

Corollary 4.2 [18] *The function $f \in \mathcal{S}_b^*(\mu,\beta)$ if and only if for all $|s| \le 1$, $|t| \le 1$,*

$$\left(\frac{1-tz}{1-sz}\right)^{\mu} \frac{f(sz)}{f(tz)} \prec \left(\frac{1-tz}{1-sz}\right)^{\mu(1-\beta)}.$$

Theorem 4.2 as well as Corollaries 4.3 and 4.4 below are respectively special cases of Theorem 4.1 and Corollaries 4.1 and 4.2 when $s = 1$ and $t = 0$. However, we prove the below without the convexity assumption on φ.

Theorem 4.2 *If $f \in \mathcal{S}_b^*(\mu,\varphi)$, then*

$$\frac{f(z)^{\frac{2}{\mu}}}{(1-z)^2} \prec \frac{h_\varphi(z)}{z},$$

where $h_\varphi(z) = z\exp(\int_0^z ((\varphi(\zeta)-1)/\zeta)\,d\zeta)$.

Proof Clearly $zh'_\varphi(z)/h_\varphi(z) = \varphi(z)$. If $h \in \mathcal{S}^*(\varphi)$, then

$$\frac{zh'(z)}{h(z)} \prec \frac{zh'_\varphi(z)}{h_\varphi(z)}.$$

Therefore by [19, Theorem 1′]

$$\frac{h(z)}{z} \prec \frac{h_\varphi(z)}{z}.$$

Let $h(z)$ be defined as in (3.2) and hence we arrive at the desired conclusion. □

Corollary 4.3 *If* $f \in \mathcal{S}_b^*(\mu, A, B)$ *then*

$$\frac{f(z)}{(1-z)^\mu} \prec (1+Bz)^{\frac{\mu(A-B)}{2B}} \quad \text{for } B \neq 0$$

and

$$\frac{f(z)}{(1-z)^\mu} \prec \exp\left(\frac{\mu A z}{2}\right) \quad \text{for } B = 0.$$

When $0 \leq \beta < 1$, $A = 1 - 2\beta$ and $B = -1$, the above corollary reduces to the following result.

Corollary 4.4 [18] *If* $f \in \mathcal{S}_b^*(\mu, \beta)$ *then*

$$\frac{f(z)}{(1-z)^\mu} \prec \frac{1}{(1-z)^{\mu(1-\beta)}}.$$

5 Coefficient estimate for $f \in \mathcal{S}_b^*(\varphi)$

In particular, when $\mu = 1$, (1.2) becomes

$$2\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z} \prec \varphi(z), \quad z \in \mathbb{D}.$$

We denote the class satisfying the above subordination as $\mathcal{S}_b^*(\varphi)$.

Theorem 5.1 *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. *If* $f \in \mathcal{S}_b^*(\varphi)$, *then the coefficients* d_1, d_2, d_3 *satisfy the following inequalities*:

$$|d_1| \leq \frac{B_1}{2} + 1,$$
$$|d_2| \leq \frac{B_1}{4} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1}{2}\right|\right\} + \frac{B_1}{2},$$
$$|d_3| \leq \frac{B_1}{6} H\left(\frac{6B_1^2 + 16B_2}{8B_1}, \frac{B_1^3 + 6B_1B_2 + 8B_3}{8B_1}\right) + \frac{B_1}{4} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1}{2}\right|\right\},$$

where $H(q_1, q_2)$[a] *is as defined in* [21] (*see also* [22, Lemma 3]) *and*

$$\left|d_2 - \nu d_1^2\right| \leq \begin{cases} \frac{B_1}{4}(\frac{B_2}{B_1} - (2\nu - 1)\frac{B_1}{2}) + (2\nu + 1)\frac{B_1}{2} + 2\nu, & \nu \leq \sigma_1, \\ \frac{B_1}{4} + (2\nu + 1)\frac{B_1}{2} + 2\nu, & \sigma_1 \leq \nu \leq \sigma_2, \\ \frac{B_1}{4}((2\nu - 1)\frac{B_1}{2} - \frac{B_2}{B_1}) + (2\nu + 1)\frac{B_1}{2} + 2\nu, & \nu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{1}{B_1}\left(\frac{B_2}{B_1} - 1\right) + \frac{1}{2}, \qquad \sigma_2 = \frac{1}{B_1}\left(\frac{B_2}{B_1} + 1\right) + \frac{1}{2}.$$

Proof Define the function $g(z) = 1 + g_1 z + g_2 z^2 + \cdots$ by

$$g(z) = \frac{f(z)}{(1-z)}, \quad z \in \mathbb{D}.$$

Then a computation shows that

$$2\frac{zg'(z)}{g(z)} + 1 = 2\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}.$$

Since $f \in \mathcal{S}_b^*(\varphi)$, we have

$$2\frac{zg'(z)}{g(z)} + 1 \prec \varphi(z),$$

or there is an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots$ such that

$$2\frac{zg'(z)}{g(z)} + 1 = \varphi\bigl(w(z)\bigr).$$

Since

$$2\frac{zg'(z)}{g(z)} + 1 = 1 + 2g_1 z + \bigl(-2g_1^2 + 4g_2\bigr)z^2 + \bigl(2g_1^3 - 6g_1 g_2 + 6g_3\bigr)z^3 + \cdots$$

and

$$\varphi\bigl(w(z)\bigr) = 1 + B_1 w_1 z + \bigl(B_2 w_1^2 + B_1 w_2\bigr)z^2 + \bigl(B_3 w_1^3 + 2B_2 w_1 w_2 + b_1 w_3\bigr)z^3 + \cdots,$$

we see that

$$\begin{aligned} g_1 &= \frac{B_1 w_1}{2}, \\ g_2 &= \frac{B_1}{4}\left(w_2 + \left(\frac{B_2}{B_1} + \frac{B_1}{2}\right)w_1^2\right), \\ g_3 &= \frac{B_1}{6}\left(w_3 + \left(\frac{6B_1^2 + 16B_2}{8B_1}\right)w_1 w_2 + \left(\frac{B_1^3 + 6B_1 B_2 + 8B_3}{8B_1}\right)w_1^3\right). \end{aligned}$$

In view of the well-known inequality $|w_1| \leq 1$, we have

$$|g_1| \leq \frac{B_1}{2}.$$

Applying [23, inequality 7, p.10] and [22, Lemma 3] (see also [21]), we get

$$|g_2| \leq \frac{B_1}{4}\max\left\{1, \left|\frac{B_2}{B_1} + \frac{B_1}{2}\right|\right\}$$

and

$$|g_3| \leq \frac{B_1}{6} H\left(\frac{6B_1^2 + 16B_2}{8B_1}, \frac{B_1^3 + 6B_1 B_2 + 8B_3}{8B_1}\right),$$

respectively. Also, we see that applying [22, Lemma 1] (see also [19]) to inequality

$$g_2 - \nu g_1^2 = \frac{B_1}{4}\left(w_2 - \left((2\nu - 1)\frac{B_1}{2} - \frac{B_2}{B_1}\right)w_1^2\right)$$

yields

$$\left|g_2 - \nu g_1^2\right| \le \begin{cases} \frac{B_1}{4}(\frac{B_2}{B_1} - (2\nu - 1)\frac{B_1}{2}), & \nu \le \sigma_1, \\ \frac{B_1}{4}, & \sigma_1 \le \nu \le \sigma_2, \\ \frac{B_1}{4}((2\nu - 1)\frac{B_1}{2} - \frac{B_2}{B_1}), & \nu \ge \sigma_2 \end{cases}$$

for σ_1 and σ_2 as in the hypothesis. Todorov in [3] shows that for

$$g(z) = 1 + \sum_1^\infty g_n z^n,$$

the coefficient

$$g_n = 1 + d_1 + d_2 + \cdots + d_n,$$

and hence from the above relation the desired results are obtained. □

Corollary 5.1 *When* $\varphi(z) = (1 + z)/(1 - z)$, *our results coincide with* [3, Corollary 2.3].

Remark 5.1 All the results for the special case when $\mu = 1$ or the class starlike with respect to a boundary point defined by subordination were presented at the 8th International Symposium on GFTA, 27-31 August 2012, Ohrid, Republic of Macedonia and thereafter published as [24].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The first author MHM is currently a PhD student under supervision of the second author MD and jointly worked on deriving the results. All authors read and approved the final manuscript.

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Endnote
[a] The expression for *H* is too lengthy to be reproduced here. See [21] or [22] for the full expression.

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References

1. Robertson, MS: Univalent functions starlike with respect to a boundary point. J. Math. Anal. Appl. **81**(2), 327-345 (1981)
2. Lyzzaik, A: On a conjecture of M.S. Robertson. Proc. Am. Math. Soc. **91**(1), 108-110 (1984)
3. Todorov, PG: On the univalent functions starlike with respect to a boundary point. Proc. Am. Math. Soc. **97**(4), 602-604 (1986)
4. Silverman, H, Silvia, EM: Subclasses of univalent functions starlike with respect to a boundary point. Houst. J. Math. **16**(2), 289-299 (1990)
5. Abdullah, AS, Ali, RM, Singh, V: On functions starlike with respect to a boundary point. Ann. Univ. Mariae Curie-Skłodowska, Sect. A **50**, 7-15 (1996)
6. Chên, MP, Owa, S: Generalization of Robertson functions. In: Complex Analysis and Its Applications (Hong Kong, 1993). Pitman Res. Notes Math. Ser., vol. 305, pp. 159-165. Longman, Harlow (1994)
7. Dong, J, Liu, L: Distortion properties of a class of functions starlike with respect of a boundary point. Heilongjiang Daxue Ziran Kexue Xuebao **15**(4), 1-6 (1998)
8. Elin, M, Reich, S, Shoikhet, D: Dynamics of inequalities in geometric function theory. J. Inequal. Appl. **6**(6), 651-664 (2001)
9. Lecko, A: On the class of functions starlike with respect to a boundary point. J. Math. Anal. Appl. **261**(2), 649-664 (2001)

10. Lecko, A, Lyzzaik, A: A note on univalent functions starlike with respect to a boundary point. J. Math. Anal. Appl. **282**(2), 846-851 (2003)
11. Xu, Q-H, Lv, C-B, Srivastava, HM: Coefficient estimates for the inverses of a certain general class of spirallike functions. Appl. Math. Comput. **219**(12), 7000-7011 (2013)
12. Elin, M, Reich, S, Shoikhet, D: Holomorphically accretive mappings and spiral-shaped functions of proper contractions. Nonlinear Anal. Forum **5**, 149-161 (2000)
13. Elin, M, Shoikhet, D: Angle distortion theorems for starlike and spirallike functions with respect to a boundary point. Int. J. Math. Math. Sci. **2006**, Art. ID 81615 (2006)
14. Lecko, A: The class of functions spirallike with respect to a boundary point. Int. J. Math. Math. Sci. **2004**(37-40), 2133-2143 (2004)
15. Shoikhet, D: Representations of holomorphic generators and distortion theorems for spirallike functions with respect to a boundary point. Int. J. Pure Appl. Math. **5**(3), 335-361 (2003)
16. Aharonov, D, Elin, M, Shoikhet, D: Spiral-like functions with respect to a boundary point. J. Math. Anal. Appl. **280**(1), 17-29 (2003)
17. Elin, M, et al.: Dynamics of spirallike functions. In: Complex Analysis and Dynamical Systems. Contemp. Math., vol. 364, pp. 41-57. Am. Math. Soc., Providence (2004)
18. Elin, M: Covering and distortion theorems for spirallike functions with respect to a boundary point. Int. J. Pure Appl. Math. **28**(3), 387-400 (2006)
19. Ma, WC, Minda, D: A unified treatment of some special classes of univalent functions. In: Proceedings of the Conference on Complex Analysis (Tianjin, 1992). Conf. Proc. Lecture Notes Anal., vol. I, pp. 157-169. Int. Press, Cambridge (1994)
20. Ruscheweyh, S: A subordination theorem for F-like functions. J. Lond. Math. Soc. **13**(2), 275-280 (1976)
21. Prokhorov, DV, Szynal, J: Inverse coefficients for (α, β)-convex functions. Ann. Univ. Mariae Curie-Skłodowska, Sect. A **35**(1981), 125-143 (1984)
22. Ali, RM, Ravichandran, V, Seenivasagan, N: Coefficient bounds for p-valent functions. Appl. Math. Comput. **187**(1), 35-46 (2007)
23. Keogh, FR, Merkes, EP: A coefficient inequality for certain classes of analytic functions. Proc. Am. Math. Soc. **20**, 8-12 (1969). MR0232926
24. Mohd, MH, Darus, M: Starlike function with respect to a boundary point defined by subordination. Adv. Math. Sci. J. **1**(1), 15-21 (2012)

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