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Upper triangular operator matrices, asymptotic intertwining and Browder, Weyl theorems

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Abstract

Given a Banach space $\mathcal{X}$, let $M_C \in B(\mathcal{X} \oplus \mathcal{X})$ denote the upper triangular operator matrix $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$, and let $\delta_{AB} \in B(B(\mathcal{X}))$ denote the generalized derivation $\delta_{AB}(X) = AX - XB$. If $\lim_{n\to\infty} \|\delta_{AB}^n(C)\|^{\frac{1}{n}} = 0$, then $\sigma_x(M_C) = \sigma_x(M_0)$, where σ_x stands for the spectrum or a distinguished part thereof (but not the point spectrum); furthermore, if $R = R_1 \oplus R_2 \in B(\mathcal{X} \oplus \mathcal{X})$ is a Riesz operator which commutes with M_C, then $\sigma_x(M_C + R) = \sigma_x(M_C)$, where σ_x stands for the Fredholm essential spectrum or a distinguished part thereof. These results are applied to prove the equivalence of Browder's (a-Browder's) theorem for M_0, M_C, $M_0 + R$ and $M_C + R$. Sufficient conditions for the equivalence of Weyl's (a-Weyl's) theorem are also considered.

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1 Introduction

A Banach space operator $T \in B(\mathcal{X})$, the algebra of bounded linear transformations from a Banach space $\mathcal{X}$ into itself, satisfies Browder's theorem if the Browder spectrum $\sigma_b(T)$ of T coincides with the Weyl spectrum $\sigma_w(T)$ of T; T satisfies Weyl's theorem if the complement of $\sigma_w(T)$ in $\sigma(T)$ is the set $\Pi_0(T)$ of finite multiplicity isolated eigenvalues of T. Weyl's theorem implies Browder's theorem, but the converse is generally false (see [1–3]). Let M_0 and $M_C \in B(\mathcal{X} \oplus \mathcal{X})$ denote, respectively, the upper triangular operators $M_0 = A \oplus B$ and $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ for some operators $A, C, B \in B(\mathcal{X})$. It is well known that $\sigma_x(M_0) = \sigma_x(A) \cup \sigma_x(B) = \sigma_x(M_C) \cup \{\sigma_x(A) \cap \sigma_x(B)\}$ for $\sigma_x = \sigma$ or σ_b, and $\sigma_w(M_0) \subseteq \sigma_w(A) \cup \sigma_w(B) = \sigma_w(M_C) \cup \{\sigma_w(A) \cap \sigma_w(B)\}$. The problem of finding sufficient conditions ensuring the equality of the spectrum (and certain of its distinguished parts) of M_0 and M_C, along with the problem of finding sufficient conditions for M_0 satisfies Browder's theorem and/or Weyl's theorem to imply M_C satisfies Browder's theorem and/or Weyl's theorem (and *vice versa*), has been considered by a number of authors in the recent past (see [3], and some of the references cited there). For example, if either A^* or B has the single-valued extension property, SVEP for short, then $\sigma(M_0) = \sigma(M_C) = \sigma(A) \cup \sigma(B)$. Again, if $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then $\sigma(M_0) = \sigma(M_C) = \sigma(A) \cup \sigma(B)$ [3, Proposition 3.2] and M_0 satisfies Browder's theorem if and only if M_C satisfies Browder's theorem [3, Theorem 4.8]; furthermore, in such a case, M_0 satisfies Weyl's theorem if and only if M_C satisfies Weyl's theorem if and only if $\Pi_0(M_0) = \Pi_0(M_C)$ [3, Theorem 5.1]. The equal-

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ity $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$ may be achieved in a number of ways: if either A and A^*, or A and B, or A^* and B^*, or B and B^* have SVEP, then $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$ [3, Proposition 4.5]. In this paper we consider conditions of another kind, conditions which do not assume SVEP.

Given $S, T \in B(\mathcal{X})$, S and T are said to be *asymptotically intertwined* by $X \in B(\mathcal{X})$ if $\lim_{n\to\infty} \|\delta_{ST}^n(X)\|^{\frac{1}{n}} = 0$. Here $\delta_{ST} \in B(B(\mathcal{X}))$ is the generalized derivation $\delta_{ST}(X) = SX - XT$ and $\delta_{ST}^n = \delta_{ST}(\delta_{ST}^{n-1})$. Evidently, S and T asymptotically intertwined by X does not imply T and S asymptotically intertwined by X. Furthermore, S and T asymptotically intertwined by X does not imply $\sigma(S) = \sigma(T)$, not even $\sigma(S) \subseteq \sigma(T)$; see [4, Example 3.5.9]. However, as we shall see, if A, B, C are as in the definition of M_C above, then A and B asymptotically intertwined by C implies the equality of the spectra, and many distinguished parts thereof to spectrum of M_0 and M_C. We prove in the following that if $\lim_{n\to\infty} \|\delta_{AB}^n(C)\|^{\frac{1}{n}} = 0$, then M_C satisfies Browder's theorem if and only if M_0 satisfies Browder's theorem. If, additionally, the isolated points of $\sigma(M_0)$ are poles of the resolvent of M_0, then M_c satisfies Weyl's theorem if and only if M_0 satisfies Weyl's theorem. Extensions to a-Browder's theorem, a-Weyl's theorem and perturbations by Riesz operators are considered.

2 Notation and complementary results

For a bounded linear Banach space operator $S \in B(\mathcal{X})$, let $\sigma(S)$, $\sigma_p(S)$, $\sigma_a(S)$, $\sigma_s(S)$ and $\operatorname{iso}\sigma(S)$ denote, respectively, the spectrum, the point spectrum, the approximate point spectrum, the surjectivity spectrum and the isolated points of the spectrum of S. Let $\alpha(S)$ and $\beta(S)$ denote the nullity and the deficiency of S, defined by

$$\alpha(S) = \dim S^{-1}(0) \quad \text{and} \quad \beta(S) = \operatorname{codim} S(\mathcal{X}).$$

If the range $S(\mathcal{X})$ of S is closed and $\alpha(S) < \infty$ (resp. $\beta(S) < \infty$), then S is called an *upper semi-Fredholm* (resp. a *lower semi-Fredholm*) operator. If $S \in B(\mathcal{X})$ is either upper or lower semi-Fredholm, S is called a *semi-Fredholm* operator, and $\operatorname{ind}(S)$, the *index* of S, is then defined by $\operatorname{ind}(S) = \alpha(S) - \beta(S)$. If both $\alpha(S)$ and $\beta(S)$ are finite, then S is a *Fredholm* operator. The *ascent*, denoted $\operatorname{asc}(S)$, and the *descent*, denoted $\operatorname{dsc}(S)$, of S are given by

$$\operatorname{asc}(S) = \inf\{n : S^{-n}(0) = S^{-(n+1)}(0)\}, \qquad \operatorname{dsc}(S) = \inf\{n : S^n(\mathcal{X}) = S^{n+1}(\mathcal{X})\}$$

(where the infimum is taken over the set of non-negative integers); if no such integer n exists, then $\operatorname{asc}(S) = \infty$, respectively $\operatorname{dsc}(S) = \infty$. Let

$$\Phi_+(S) = \{\lambda \in \mathbb{C} : S - \lambda \text{ is upper semi-Fredholm}\},$$

$$\Phi_-(S) = \{\lambda \in \mathbb{C} : S - \lambda \text{ is lower semi-Fredholm}\},$$

$$\Phi(S) = \{\lambda \in \mathbb{C} : S - \lambda \text{ is Fredholm}\},$$

$$\sigma_{SF_+}(S) = \{\lambda \in \sigma_a(S) : \lambda \notin \Phi_+(S)\},$$

$$\sigma_{SF_-}(S) = \{\lambda \in \sigma_a(S) : \lambda \notin \Phi_-(S)\},$$

$$\sigma_e(S) = \{\lambda \in \sigma(S) : \lambda \notin \Phi(S)\},$$

$$\sigma_w(S) = \{\lambda \in \sigma(S) : \lambda \in \sigma_e(S) \text{ or } \operatorname{ind}(S - \lambda) \neq 0\},$$

$$\sigma_{aw}(S) = \{\lambda \in \sigma_a(S) : \lambda \in \sigma_{SF_+}(S) \text{ or } \operatorname{ind}(S-\lambda) > 0\},$$
$$\sigma_{sw}(S) = \{\lambda \in \sigma_s(S) : \lambda \in \sigma_{SF_-}(S) \text{ or } \operatorname{ind}(S-\lambda) < 0\},$$
$$\sigma_b(S) = \{\lambda \in \sigma(S) : \lambda \in \sigma_e(S) \text{ or } \operatorname{asc}(S-\lambda) \neq \operatorname{dsc}(S-\lambda)\},$$
$$\sigma_{ab}(S) = \{\lambda \in \sigma_a(S) : \lambda \in \sigma_{SF_+}(S) \text{ or } \operatorname{asc}(S-\lambda) = \infty\},$$
$$\sigma_{sb}(S) = \{\lambda \in \sigma_s(S) : \lambda \in \sigma_{SF_-}(S) \text{ or } \operatorname{dsc}(S-\lambda) = \infty\},$$
$$\Pi_0(S) = \{\lambda \in \operatorname{iso}\sigma(S) : 0 < \dim(S-\lambda)^{-1}(0) = \alpha(S-\lambda) < \infty\},$$
$$p_0(S) = \{\lambda \in \operatorname{iso}\sigma(S) : \lambda \in \Phi(S), \operatorname{asc}(S-\lambda) = \operatorname{dsc}(S-\lambda) < \infty\},$$
$$H_0(S) = \Big\{x \in \mathcal{X} : \lim_{n\to\infty} \|S^n x\|^{1/n} = 0\Big\}.$$

Here $\sigma_w(S)$ is the Weyl spectrum, $\sigma_{aw}(S)$ denotes the Weyl (essential) approximate point spectrum, $\sigma_{sw}(S)$ the Weyl (essential) surjectivity spectrum, $\sigma_b(S)$ the Browder spectrum, $\sigma_{ab}(S)$ the Browder (essential) approximate point spectrum, $\sigma_{sb}(S)$ the Browder (essential) surjectivity spectrum, and $H_0(S)$ the quasi-nilpotent part of S [1]. Recall, [1], that $H_0(S)$ and $K(S)$, where $K(S)$ denotes the *analytic core*

$$K(S) = \{x \in \mathcal{X} : \text{there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \text{ for which}$$
$$x = x_0, S(x_{n+1}) = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, \ldots\},$$

are hyper-invariant (generally non-closed) subspaces of S such that $S^{-p}(0) \subseteq H_0(S)$ for every integer $p \geq 0$ and $SK(S) = K(S)$. Recall also that if $0 \in \operatorname{iso}\sigma(S)$, then $\mathcal{X} = H_0(S) \oplus K(S)$.

We say that S has the *single valued extension property*, or SVEP, at $\lambda \in \mathbb{C}$ if for every open neighborhood U of λ, the only analytic solution f to the equation $(S-\mu)f(\mu) = 0$ for all $\mu \in U$ is the constant function $f \equiv 0$; we say that S has SVEP if S has a SVEP at every $\lambda \in \mathbb{C}$. It is well known that finite ascent implies SVEP; also, an operator has SVEP at every isolated point of its spectrum (as well as at every isolated point of its approximate point spectrum).

$S \in B(\mathcal{X})$ satisfies Browder's theorem, shortened to S satisfies Bt, if $\sigma_w(S) = \sigma_b(S)$ (if and only if $\sigma(S) \setminus \sigma_w(S) = p_0(S)$, see [1, p.156]); S satisfies Weyl's theorem, shortened to S satisfies Wt, if $\sigma(S) \setminus \sigma_w(S) = \Pi_0(S)$ (if and only if S satisfies Bt and $p_0(S) = \Pi_0(S)$) [1, p.177]. The implication Wt $\Longrightarrow$ Bt is well known.

An isolated point $\lambda \in \operatorname{iso}\sigma(S)$ is a pole (of the resolvent) of $S \in B(\mathcal{X})$ if $\operatorname{asc}(S-\lambda) = \operatorname{dsc}(S-\lambda) < \infty$. In such a case we say that S is polar at λ; we say that S is polaroid (resp., polaroid on a subset F of the set of isolated points of $\sigma(S)$) if S is polar at every $\lambda \in \operatorname{iso}\sigma(S)$ (resp., at every $\lambda \in F$). Let $p(S)$ denote the set of poles of S.

Throughout the following, $M_0 \in B(\mathcal{X} \oplus \mathcal{X})$ shall denote the diagonal operator $M_0 = A \oplus B$ and $M_C \in B(\mathcal{X} \oplus \mathcal{X})$ shall denote the upper triangular operator matrix $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$, for some operators $A, B, C \in B(\mathcal{X})$. Recall, [5, Exercise 7, p.293], that $\operatorname{asc}(A) \leq \operatorname{asc}(M_C) \leq \operatorname{asc}(A) + \operatorname{asc}(B)$ and $\operatorname{dsc}(B) \leq \operatorname{dsc}(M_C) \leq \operatorname{dsc}(A) + \operatorname{dsc}(B)$.

Lemma 2.1 *If* $\sigma(M_0) = \sigma(M_C)$, *then* $p(M_0) = p(M_C)$.

Proof Since $\sigma(M_C) = \sigma(M_0) = \sigma(A) \cup \sigma(B)$, if a complex number $\lambda \in p(M_C)$ or $p(M_0)$ then $\lambda \in \operatorname{iso}(\sigma(A) \cup \sigma(B))$. We consider the case in which $\lambda \in \operatorname{iso}\sigma(A) \cap \operatorname{iso}\sigma(B)$: the argument works just as well for the case in which $\lambda \in \rho(A)$ $(= \mathbb{C} \setminus \sigma(A))$ or $\lambda \in \rho(B)$. Let $\lambda \in p(M_C)$. Then

$$\operatorname{asc}(A-\lambda) \le \operatorname{asc}(M_C-\lambda) < \infty \quad \text{and} \quad \operatorname{dsc}(B-\lambda) \le \operatorname{dsc}(M_C-\lambda) < \infty.$$

If $\lambda \in \operatorname{iso}\sigma(B)$ and $\operatorname{dsc}(B-\lambda) < \infty$, then $\operatorname{asc}(B-\lambda) = \operatorname{dsc}(B-\lambda) < \infty$ and B is polar at λ [1, Theorem 3.81]. Now let $\lambda \in \operatorname{iso}\sigma(A)$. Since M_C is polar at λ, $H_0(M_C-\lambda) = (M_C-\lambda)^{-p}(0)$ for some integer $p \ge 1$. Observe that

$$H_0(A-\lambda) = H_0(M_C-\lambda) \cap \mathcal{X} = (M_C-\lambda)^{-p}(0) \cap \mathcal{X} = (A-\lambda)^{-p}(0).$$

Hence, if $\lambda \in \operatorname{iso}\sigma(A)$, then

$$\begin{aligned} &\mathcal{X} = H_0(A-\lambda) \oplus K(A-\lambda) = (A-\lambda)^{-p}(0) \oplus K(A-\lambda) \\ &\quad\Longrightarrow \quad (A-\lambda)^p\mathcal{X} = 0 \oplus (A-\lambda)^p K(A-\lambda) = K(A-\lambda) \\ &\quad\Longrightarrow \quad \mathcal{X} = (A-\lambda)^{-p}(0) \oplus (A-\lambda)^p\mathcal{X}, \end{aligned}$$

i.e., A is polar at λ. Now, since

$$\operatorname{asc}(M_0-\lambda) \le \operatorname{asc}(A-\lambda) + \operatorname{asc}(B-\lambda) \quad \text{and} \quad \operatorname{dsc}(M_0-\lambda) \le \operatorname{dsc}(A-\lambda) + \operatorname{dsc}(B-\lambda),$$

we have

$$\operatorname{asc}(M_0-\lambda) = \operatorname{dsc}(M_0-\lambda) < \infty,$$

i.e., M_0 is polar at λ. Conversely, if $\lambda \in p(M_0)$, then $\operatorname{asc}(M_0-\lambda) = \max\{\operatorname{asc}(A-\lambda), \operatorname{asc}(B-\lambda)\}$ and $\operatorname{dsc}(M_0-\lambda) = \max\{\operatorname{dsc}(A-\lambda), \operatorname{dsc}(B-\lambda)\}$ implies $\operatorname{asc}(M_C-\lambda) \le \operatorname{asc}(A-\lambda) + \operatorname{asc}(B-\lambda)$ and $\operatorname{dsc}(M_C-\lambda) \le \operatorname{dsc}(A-\lambda) + \operatorname{dsc}(B-\lambda)$ are both finite, hence equal. Thus M_C is polar at λ. □

Remark 2.2 A number of conditions guaranteeing (the spectral equality) $\sigma(M_C) = \sigma(M_0)$ are to be found in the literature. Thus, for example, if A^* or B has SVEP, or if $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, or $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$ [3, (I) p.5 and Proposition 3.2], then $\sigma(M_C) = \sigma(M_0)$. Compact operators have SVEP; hence, if either of A or B is compact, then $\sigma(M_C) = \sigma(M_0)$.

Lemma 2.1 shows that if B is a compact operator then $p(M_0) = p(M_C)$. A proof of the following lemma may be obtained from that of Lemma 2.1: we give here an independent proof, exploiting the additional information contained in the hypothesis.

Lemma 2.3 *If* $\sigma(M_0) = \sigma(M_C)$, *then* $p_0(M_0) = p_0(M_c)$.

Proof Once again we consider points $\lambda \in \operatorname{iso}\sigma(A) \cap \operatorname{iso}\sigma(B)$. Let $\lambda \in p_0(M_C)$. Then $\alpha(M_C - \lambda) = \beta(M_C-\lambda) < \infty$ implies $M_C - \lambda \in \Phi$, and this in turn implies $A-\lambda \in \Phi_+$ and $B-\lambda \in \Phi_-$. Since λ is isolated in $\sigma(A)$ and $\sigma(B)$, $\lambda \in p_0(A) \cap p_0(B)$ [1, Theorem 3.77]. Consequently, $\lambda \in p(M_0)$; furthermore, since $\alpha(M_0-\lambda) \le \alpha(A-\lambda) + \alpha(B-\lambda)$, $\lambda \in p_0(M_0)$. Conversely, if $\lambda \in p_0(M_0)$, then $A - \lambda$ and $B - \lambda \in \Phi$, and hence (since λ is isolated in $\sigma(A)$ and $\sigma(B)$) $\lambda \in p_0(A) \cap p_0(B)$. This, as above, implies $\lambda \in p_0(M_C)$. □

The following technical lemma will be required in the sequel.

Lemma 2.4 *If A is polaroid on $\Pi_0(M_C)$ and $\sigma(M_C) = \sigma(M_0)$, then $\Pi_0(M_C) \subseteq \Pi_0(M_0)$.*

Proof Evidently, $(M_C - \lambda)^{-1}(0) \neq \emptyset$ implies $(M_0 - \lambda)^{-1}(0) \neq \emptyset$, and $\alpha(M_C - \lambda) < \infty$ implies $\alpha(A - \lambda) < \infty$. Let $\lambda \in \Pi_0(M_C)$; then $\lambda \in \operatorname{iso}\sigma(M_0)$. We prove that $\alpha(B - \lambda) < \infty$. Suppose to the contrary that $\alpha(B - \lambda) = \infty$. Since

$$(M_C - \lambda)(x \oplus y) = \{(A - \lambda)x + Cy\} \oplus (B - \lambda)y,$$

either $\dim(C(B - \lambda)^{-1}(0)) < \infty$ or $\dim(C(B - \lambda)^{-1}(0)) = \infty$. If $\dim(C(B - \lambda)^{-1}(0)) < \infty$, then (since $\alpha(B - \lambda) = \infty$) $(B - \lambda)^{-1}(0)$ contains an orthonormal sequence $\{y_j\}$ such that $(M_C - \lambda)(0 \oplus y_j) = 0$ for all $j = 1, 2, \ldots$. But then $\alpha(M_C - \lambda) = \infty$, a contradiction. Hence $\dim(C(B - \lambda)^{-1}(0)) = \infty$. Since $\lambda \in \rho(A) \cup \operatorname{iso}\sigma(A)$ and A is (by hypothesis) polar at λ (with, as observed above, $\alpha(A - \lambda) < \infty$) $\alpha(A - \lambda) = \beta(A - \lambda) < \infty$. Thus $\dim\{C(B - \lambda)^{-1}(0) \cap (A - \lambda)\mathcal{X}\} = \infty$, and so there exists a sequence $\{x_j\}$ such that $(A - \lambda)x_j = Cy_j$ for all $j = 1, 2, \ldots$. But then $(M_C - \lambda)(x_j \oplus -y_j) = 0$ for all $j = 1, 2, \ldots$, and hence $\alpha(M_C - \lambda) = \infty$. This contradiction implies that we must have $\alpha(B - \lambda) < \infty$. Since $\alpha(M_0 - \lambda) \leq \alpha(A - \lambda) + \alpha(B - \lambda)$, we conclude that $\lambda \in \Pi_0(M_0)$. □

Let $\delta_{ST} \in B(B(\mathcal{X}))$ denote the generalized derivation $\delta_{ST}(X) = SX - XT$, and define δ_{ST}^n by $\delta_{ST}^{n-1}(\delta_{ST})$. The operators $S, T \in B(\mathcal{X})$ are said to be asymptotically intertwined by the identity operator $I \in B(\mathcal{X})$ if $\lim_{n\to\infty} \|\delta_{ST}^n(I)\|^{\frac{1}{n}} = 0$; S, T are said to be *quasi-nilpotent equivalent* if $\lim_{n\to\infty} \|\delta_{ST}^n(I)\|^{\frac{1}{n}} = \lim_{n\to\infty} \|\delta_{TS}^n(I)\|^{\frac{1}{n}} = 0$ [4, p.253]. Quasi-nilpotent equivalence preserves a number of spectral properties [4, Proposition 3.4.11]. In particular:

Lemma 2.5 *Quasi-nilpotent equivalent operators have the same spectrum, the same approximate point spectrum and the same surjectivity spectrum.*

3 Results

Let $\mathcal{K}(\mathcal{X})$ denote the ideal of compact operators in $B(\mathcal{X})$. The following construction, known in the literature as the Sadovskii/Buoni, Harte and Wickstead construction [6, p.159], leads to a representation of the Calkin algebra $B(\mathcal{X})/\mathcal{K}(\mathcal{X})$ as an algebra of operators on a suitable Banach space. Let $S \in B(\mathcal{X})$. Let $\ell^\infty(\mathcal{X})$ denote the Banach space of all bounded sequences $x = (x_n)_{n=1}^\infty$ of elements of $\mathcal{X}$ endowed with the norm $\|x\|_\infty := \sup_{n\in\mathbb{N}} \|x_n\|$, and write S_∞, $S_\infty x := (Sx_n)_{n=1}^\infty$ for all $x = (x_n)_{n=1}^\infty$, for the operator induced by S on $\ell^\infty(\mathcal{X})$. The set $m(\mathcal{X})$ of all precompact sequences of elements of $\mathcal{X}$ is a closed subspace of $\ell^\infty(\mathcal{X})$ which is invariant for S_∞. Let $\mathcal{X}_q := \ell^\infty(\mathcal{X})/m(\mathcal{X})$, and denote by S_q the operator S_∞ on $\mathcal{X}_q$. The mapping $S \mapsto S_q$ is then a unital homomorphism from $B(\mathcal{X}) \to B(\mathcal{X}_q)$ with kernel $\mathcal{K}(\mathcal{X})$ which induces a norm decreasing monomorphism from $B(\mathcal{X})/\mathcal{K}(\mathcal{X})$ to $B(\mathcal{X}_q)$ with the following properties (see [6, Section 17] for details):

(i) S is upper semi-Fredholm, $S \in \Phi_+$, if and only if S_q is injective, if and only if S_q is bounded below;
(ii) S is lower semi-Fredholm, $S \in \Phi_-$, if and only if S_q is surjective;
(iii) S is Fredholm, $S \in \Phi$, if and only if S_q is invertible.

Lemma 3.1 *For every* $S \in B(\mathcal{X})$, $\sigma_e(S) = \sigma(S_q)$, $\sigma_{SF_+}(S) = \sigma_a(S_q)$ *and* $\sigma_{SF_-}(S) = \sigma_s(S_q)$.

Proof The following implications hold:

$$\begin{aligned}
\lambda \notin \sigma_{SF_+}(S) \quad &\Longleftrightarrow \quad S - \lambda \in \Phi_+ \quad \Longleftrightarrow \quad (S-\lambda)_q \text{ is bounded below} \\
&\Longleftrightarrow \quad \lambda \notin \sigma_a(S_q), \\
\lambda \notin \sigma_{SF_-}(S) \quad &\Longleftrightarrow \quad S - \lambda \in \Phi_- \quad \Longleftrightarrow \quad (S-\lambda)_q \text{ is onto} \\
&\Longleftrightarrow \quad \lambda \notin \sigma_s(S_q) \quad \text{and}
\end{aligned}$$

$$\lambda \notin \sigma_e(S) \quad \Longleftrightarrow \quad S - \lambda \in \Phi \quad \Longleftrightarrow \quad (S-\lambda)_q \text{ is invertible} \quad \Longleftrightarrow \quad \lambda \notin \sigma(S_q). \qquad \square$$

The following theorem is essentially known [7] we provide here an alternative proof, using quasi-nilpotent equivalence and the construction above. Let Σ_0 denote either of σ_e, σ_{SF_+}, σ_{SF_-}, σ_w, σ_{aw}, σ_{sw}, σ_b, σ_{ab} and σ_{sb}.

Theorem 3.2 *Let* $S, R \in B(\mathcal{X})$. *If* R *is a Riesz operator which commutes with* S, *then* $\sigma_x(S + R) = \sigma_x(S)$, *where* $\sigma_x \in \Sigma_0$.

Proof It is clear from the definition of a Riesz operator $R \in B(\mathcal{X})$ that $R - \mu$ is Browder (*i.e.*, $\mu \notin \sigma_b(R)$), and a-Browder and s-Browder, for all non-zero $\mu \in \sigma(R)$ (see, for example, [1, Theorem 3.111]). Hence $\sigma(R_q) = \{0\}$, *i.e.*, $R_q \in B(\mathcal{X}_q)$ is quasi-nilpotent. Let $t \in [0,1]$; then S commutes with tR and $(S + tR)_q = S_q + tR_q$. It follows that

$$\lim_{n\to\infty} \left\| \delta^n_{(S+tR)_q S_q}(I_q) \right\|^{\frac{1}{n}} = \lim_{n\to\infty} \left\| \delta^n_{S_q (S+tR)_q}(I_q) \right\|^{\frac{1}{n}} = 0,$$

i.e., S_q and $S_q + tR_q$ are quasi-nilpotent equivalent operators for all $t \in [0,1]$. Thus $\sigma_x((S + R)_q) = \sigma_x(S_q)$, where $\sigma_x = \sigma$ or σ_a or σ_s. Hence

$$\sigma_x(S + R) = \sigma_x(S); \quad \sigma_x = \sigma_e \text{ or } \sigma_{ae} \text{ or } \sigma_{se}.$$

The semi-Fredholm index being a continuous function, we also have from the above that

$$\sigma_x(S + R) = \sigma_x(S); \quad \sigma_x = \sigma_w \text{ or } \sigma_{aw} \text{ or } \sigma_{sw}.$$

To complete the proof, we prove next that $\sigma_b(S + R) = \sigma_b(S)$; the proof for σ_{ab} and σ_{sb} is similar, and left to the reader. It would suffice to prove that $0 \in \sigma_b(S) \Longleftrightarrow 0 \in \sigma_b(S + R)$. Suppose that $0 \notin \sigma_b(S)$. Then $S \in \Phi$ (and $\operatorname{asc}(S) = \operatorname{dsc}(S) < \infty$), hence $S + tR \in \Phi$ for all $t \in [0,1]$. For an operator T, let $\overline{\mathcal{N}^\infty(T)}$ and $T^\infty(\mathcal{X})$ denote, respectively, the closure of the hyper kernel and the hyper range of T. Then $\overline{\mathcal{N}^\infty(S + tR)} \cap (S + tR)^\infty(\mathcal{X})$ is constant on $[0,1]$, and so, since $\overline{\mathcal{N}^\infty(S)} \cap S^\infty(\mathcal{X}) = \mathcal{N}^\infty(S) \cap S^\infty(\mathcal{X}) = \{0\}$, it follows that $\mathcal{N}^\infty(S+R) \cap (S + R)^\infty(\mathcal{X}) = \{0\}$. Consequently, $S + R$ has SVEP at 0 [1, Corollary 2.26]. But then since $S + R \in \Phi$, $S + R$ is Browder. Considering $S = (S + R) - R$ proves $0 \notin \sigma_b(S + R) \Longrightarrow 0 \notin \sigma_b(S)$. $\square$

The following lemma appears in [8, Lemma 2.3]. Let $\Pi_{0f}(S) = \{\lambda \in \operatorname{iso}\sigma(S) : \alpha(S - \lambda) < \infty\}$. Clearly, $\Pi_0(S) \subseteq \Pi_{0f}(S)$.

Lemma 3.3 *If $S, R \in B(\mathcal{X})$, and R is a Riesz operator which commutes with S, then $\Pi_{0f}(S + R) \cap \sigma(S) \subseteq \operatorname{iso}\sigma(S)$.*

Let $\Sigma = \Sigma_0 \cup \sigma \cup \sigma_a \cup \sigma_s$.

Theorem 3.4 *If $\lim_{n\to\infty} \|\delta_{AB}^n(C)\|^{\frac{1}{n}} = 0$, then $\sigma_x(M_C) = \sigma_x(M_0)$, where $\sigma_x \in \Sigma$.*

Proof A straightforward calculation shows that

$$\delta_{M_C M_0}^n(I) = -\delta_{M_0 M_C}^n(I) = \begin{pmatrix} 0 & \delta_{AB}^{n-1}(C) \\ 0 & 0 \end{pmatrix}.$$

Hence

$$\lim_{n\to\infty} \left\|\delta_{M_C M_0}^n(I)\right\|^{\frac{1}{n}} = \lim_{n\to\infty} \left\|\delta_{M_0 M_C}^n(I)\right\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \left\|\delta_{AB}^{n-1}(C)\right\|^{\frac{1}{n}} = 0,$$

i.e., M_C and M_0 are quasi-nilpotent equivalent. Similarly, writing $M_{C(q)}$ for $(M_C)_q$ and $M_{0(q)}$ for $(M_0)_q$,

$$\begin{aligned} \lim_{n\to\infty} \left\|\delta_{M_{C(q)} M_{0(q)}}^n(I_q)\right\|^{\frac{1}{n}} &= \lim_{n\to\infty} \left\|\delta_{M_{0(q)} M_{C(q)}}^n(I_q)\right\|^{\frac{1}{n}} \\ &\leq \lim_{n\to\infty} \left\|\delta_{A_q B_q}^{n-1}(C_q)\right\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \left\|\delta_{AB}^{n-1}(C)\right\|^{\frac{1}{n}} = 0, \end{aligned}$$

i.e., $M_{C(q)}$ and $M_{0(q)}$ are quasi-nilpotent equivalent (in $B((\mathcal{X} \oplus \mathcal{X})_q)$). Hence $\sigma_x(M_C) = \sigma_x(M_0)$, where $\sigma_x = \sigma$ or σ_a or σ_s or σ_e or σ_{SF_+} or σ_{SF_-}. Since

$$M_0 = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

and

$$M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix},$$

where $\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)$ is invertible, and since $\lambda \notin \sigma_e(M_C) \Longleftrightarrow \lambda \notin \sigma_e(M_0) \Longrightarrow A - \lambda, B - \lambda \in \Phi$ (similarly, $\lambda \notin \sigma_{SF_+}(M_C) \Longrightarrow A - \lambda, B - \lambda \in \Phi_+$ and $\lambda \notin \sigma_{SF_-}(M_C) \Longrightarrow A - \lambda, B - \lambda \in \Phi_-$), $\operatorname{ind}(M_C - \lambda) = \operatorname{ind}(A - \lambda) + \operatorname{ind}(B - \lambda) = \operatorname{ind}(M_0 - \lambda)$. Hence $\sigma_x(M_C) = \sigma_x(M_0)$, where $\sigma_x = \sigma_w$ or σ_{aw} or σ_{sw}. Observe that

$$\begin{aligned} \sigma_b(M_C) &= \left\{\lambda \in \sigma(M_C) : \lambda \in \sigma_w(M_C) \text{ or } \lambda \notin \operatorname{iso}\sigma(M_C)\right\} \\ &= \left\{\lambda \in \sigma(M_0) : \lambda \in \sigma_w(M_0) \text{ or } \lambda \notin \operatorname{iso}\sigma(M_0)\right\}, \\ \sigma_{ab}(M_C) &= \left\{\lambda \in \sigma_a(M_C) : \lambda \in \sigma_{aw}(M_C) \text{ or } \lambda \notin \operatorname{iso}\sigma_a(M_C)\right\} \\ &= \left\{\lambda \in \sigma_a(M_0) : \lambda \in \sigma_{aw}(M_0) \text{ or } \lambda \notin \operatorname{iso}\sigma_a(M_0)\right\} \end{aligned}$$

and

$$\begin{aligned}\sigma_{sb}(M_C) &= \{\lambda \in \sigma_s(M_C) : \lambda \in \sigma_{sw}(M_C) \text{ or } \lambda \notin \operatorname{iso}\sigma_s(M_C)\} \\ &= \{\lambda \in \sigma_s(M_0) : \lambda \in \sigma_{sw}(M_0) \text{ or } \lambda \notin \operatorname{iso}\sigma_s(M_0)\}\end{aligned}$$

[1, Corollary 3.23, Theorem 3.23 and Theorem 3.27]. Hence $\sigma_x(M_C) = \sigma_x(M_0)$, where $\sigma_x = \sigma_b$ or σ_{ab} or σ_{sb}. □

Remark 3.5 If $M \in B(\mathcal{X} \oplus \mathcal{X})$ is the operator $M = \begin{pmatrix} A & C \\ D & B \end{pmatrix}$ such that the entries A, B, C and D mutually commute, then $\sigma_x(M) = \{\lambda \in \mathbb{C} : 0 \in \sigma_x((A-\lambda)(B-\lambda) - CD)\}$ [9, Theorem 2.3], where $\sigma_x = \sigma$ or σ_e. Dispensing with the mutual commutativity hypothesis and assuming instead that $CD = DC = 0$, C commutes with A and B, and $\lim_{n\to\infty} \|\delta^n_{AB}(D)\|^{\frac{1}{n}} = 0$, an argument similar to that used to prove Theorem 3.4 shows that $\sigma_x(M) = \sigma_x(M_C)$, where $\sigma_x = \sigma$ or σ_a or σ_s or σ_e or $\sigma_{SF_\pm}$.

Theorem 3.6 *Suppose that* $\lim_{n\to\infty} \|\delta^n_{AB}(C)\|^{\frac{1}{n}} = 0$. *Then*:

(a) *M_C satisfies Bt if and only if M_0 satisfies Bt.*

(b) *Let $R_i \in B(\mathcal{X})$, $i = 1, 2$, be Riesz operators such that $R = R_1 \oplus R_2$ commutes with M_C. Then M_0 satisfies Bt $\Longleftrightarrow$ $M_C + R$ satisfies Bt $\Longleftrightarrow$ $M_0 + R$ satisfies Bt $\Longleftrightarrow$ M_C satisfies Bt.*

Proof The hypothesis R commutes with M_C implies R commutes with M_0, $R_1C = CR_2$ and $\delta^n_{(M_C+R)(M_0+R)}(I) = \delta^n_{M_CM}(I)$.

(a) Recall that an operator S satisfies Bt if and only if $\sigma_w(S) = \sigma_b(S)$. Hence the following implications hold:

$$\begin{aligned} M_0 \text{ satisfies Bt} \quad &\Longleftrightarrow \quad \sigma_w(M_0) = \sigma_b(M_0) \\ &\Longleftrightarrow \quad \sigma_w(M_c) = \sigma_b(M_C) \quad \text{(Theorem 3.4)} \\ &\Longleftrightarrow \quad M_C \text{ satisfies Bt.}\end{aligned}$$

(b) The hypothesis $\lim_{n\to\infty} \|\delta^n_{AB}(C)\|^{\frac{1}{n}} = 0$ implies that $M_C + R$ and $M_0 + R$ are quasinilpotent equivalent ($\Longrightarrow$ by Theorem 3.4 that $\sigma_x(M_C + R) = \sigma_x(M_0 + R)$, where $\sigma_x \in \Sigma$). The operator R being Riesz, Theorem 3.2 implies $\sigma_x(T + R) = \sigma_x(T)$, where $T = M_C$ or M_0 and $\sigma_x = \sigma_w$ or σ_b. The (two way) implications

$$\begin{aligned} M_0 \text{ satisfies Bt} \quad &\Longleftrightarrow \quad \sigma_w(M_0) = \sigma_b(M_0) \quad \Longleftrightarrow \quad \sigma_w(M_0 + R) = \sigma_b(M_0 + R) \\ &\qquad (\Longleftrightarrow M_0 + R \text{ satisfies Bt}) \\ &\Longleftrightarrow \quad \sigma_w(M_C + R) = \sigma_b(M_C + R) \quad \Longleftrightarrow \quad M_C + R \text{ satisfies Bt} \\ &\Longleftrightarrow \quad \sigma_w(M_C) = \sigma_b(M_C) \quad \Longleftrightarrow \quad M_C \text{ satisfies Bt}\end{aligned}$$

now complete the proof. □

Remark 3.7 (i) $S \in B(\mathcal{X})$ satisfies a-Browder's theorem, a-Bt, if and only if $\sigma_{aw}(S) = \sigma_{ab}(S)$ (equivalently, if and only if $\sigma_a(S) \setminus \sigma_{aw}(S) = p^a_0(S) = \{\lambda \in \operatorname{iso}\sigma_a(S) : S - \lambda \in \Phi_+\} = \{\lambda \in \sigma_a(S) :$

$S-\lambda \in \Phi_+, \operatorname{asc}(S-\lambda) < \infty\}$ [2, Theorem 3.3]). Theorem 3.6 holds with Bt replaced by a-Bt. (Thus, if either M_0 or M_C satisfies a-Bt, then M_0, M_C, M_0+R and M_C+R all satisfy a-Bt.) Furthermore, since S satisfies generalized Browder's theorem, gBt, if and only if it satisfies Bt and S satisfies generalized a-Browder's theorem, a-gBt, if and only if it satisfies a-Bt [10], Bt may be replaced by gBt or a-gBt in Theorem 3.6. Here, we refer the interested reader to consult [2, 10] for information about gBt and a-gBt.

(ii) The equivalence S satisfies Bt $\Longleftrightarrow S^*$ satisfies Bt is well known. This does not hold for a-Bt: S satisfies a-Bt does not imply S^* satisfies a-Bt (or *vice versa*). We say that S satisfies s-Bt if S^* satisfies a-Bt (equivalently, if $\sigma_{sb}(S) = \sigma_{sw}(S)$). It is easily seen, we leave the verification to the reader, if either M_0 or M_C satisfies s-Bt, then (in Theorem 3.6) M_0, M_C, M_0+R and M_C+R all satisfy s-Bt.

We consider next a sufficient condition for the equivalence of Weyl's theorem for operators M_0 and M_C such that $\lim_{n\to\infty} \|\delta^n_{AB}(C)\|^{\frac{1}{n}} = 0$. We say in the following that an operator S is finitely polaroid on a subset $F \subseteq \operatorname{iso}\sigma(S)$ if every $\lambda \in F$ is a finite rank pole of S. Evidently, M_0 is finitely polaroid if and only if A and B are finitely polaroid.

Theorem 3.8 *Suppose that* $\lim_{n\to\infty} \|\delta^n_{AB}(C)\|^{\frac{1}{n}} = 0$.

(a) *If A is polaroid, then M_C satisfies Wt if and only if M_0 satisfies Wt.*

(b) *Let $R_i \in B(\mathcal{X})$, $i = 1, 2$, be Riesz operators such that $R = R_1 \oplus R_2$ commutes with M_C. A sufficient condition for the equivalence $M_C + R$ satisfies Wt $\Longleftrightarrow M_0 + R$ satisfies Wt is that M_0 is finitely polaroid.*

Proof (a) If M_C satisfies Wt, then $\sigma(M_C) \setminus \sigma_w(M_C) = p_0(M_C) = \Pi_0(M_C)$. Since $\sigma(M_0) = \sigma(M_C)$ and $\sigma_w(M_C) = \sigma_w(M_0)$ (Theorem 3.4) and since Wt implies Bt, Theorem 3.6(a) implies $\sigma(M_0) \setminus \sigma_w(M_0) = p_0(M_0) \subseteq \Pi_0(M_0)$. Consequently, $\Pi_0(M_C) \subseteq \Pi_0(M_0)$. Let $\lambda \in \Pi_0(M_0)$. Then $\lambda \in \operatorname{iso}\sigma(M_C)$, $\alpha(A-\lambda) < \infty$ and $\alpha(B-\lambda) < \infty$. Hence, since $\alpha(A-\lambda) \le \alpha(M_C-\lambda) \le \alpha(A-\lambda) + \alpha(B-\lambda)$, $\alpha(M_C-\lambda) < \infty$. Evidently, $\lambda \in \operatorname{iso}\sigma(A) \cup \rho(A)$. If $\lambda \in \operatorname{iso}\sigma(A)$, then A polaroid implies $0 < \alpha(A-\lambda)$, and hence $0 < \alpha(M_C-\lambda)$. If instead $\lambda \in \rho(A)$, then $-(A-\lambda)^{-1}Cx \oplus x \in (M_C-\lambda)^{-1}(0)$ for every $x \in (B-\lambda)^{-1}(0)$; once again, $0 < \alpha(M_C-\lambda)$. Consequently, $\lambda \in \Pi_0(M_C-\lambda) = p_0(M_C-\lambda) = p_0(M_0-\lambda)$ and hence $\Pi_0(M_0) = p_0(M_0) \Longrightarrow M_0$ satisfies Wt. Conversely, if M_0 satisfies Wt, then $\sigma(M_C) \setminus \sigma_w(M_C) = p_0(M_C) = p_0(M_0) = \Pi_0(M_0) = \sigma(M_0) \setminus \sigma_w(M_0)$ and $\Pi_0(M_0) \subseteq \Pi_0(M_C)$. Since A is polaroid (hence polar on $\Pi_0(M_C)$) and $\sigma(M_0) = \sigma(M_C)$, Lemma 2.4 implies $\Pi_0(M_0) = \Pi_0(M_C)$. Thus M_C satisfies Wt.

(b) Start by observing that $\sigma(M_0) = \sigma(M_C)$, and hence M_C is finitely polaroid if and only if M_0 is finitely polaroid (Lemma 2.3). Suppose $M_0 + R$ satisfies Wt. Then the implication Wt $\Longrightarrow$ Bt combined with Theorem 3.6(b) implies that both $M_0 + R$ and $M_C + R$ satisfy Bt. As noted in the proof of Theorem 3.6(b), $\sigma_w(T+R) = \sigma_w(T)$, $T = M_0$ or M_C. Furthermore, since $M_0 + R$ and $M_C + R$ are quasi-nilpotent equivalent, $\sigma_x(M_0+R) = \sigma_x(M_C+R)$, $\sigma_x = \sigma$ or σ_w (Theorem 3.4). Hence

$$\begin{aligned}\Pi_0(M_0+R) &= \sigma(M_0+R) \setminus \sigma_w(M_0+R) = \sigma(M_C+R) \setminus \sigma_w(M_C+R)\\ &= p_0(M_C+R) \subseteq \Pi_0(M_C+R).\end{aligned}$$

If $\lambda \in \Pi_0(M_C+R)$ and $\lambda \notin \sigma(M_C)$, then $(M_C-\lambda)$ is invertible and so $M_C - \lambda \in \Phi \Longrightarrow M_C + R - \lambda \in \Phi$. Hence, since $\lambda \in \operatorname{iso}\sigma(M_C+R)$, $\lambda \in p_0(M_C+R)$. If, instead, $\lambda \in \sigma(M_C)$, then $\lambda \in \operatorname{iso}\sigma(M_C)$ (Lemma 3.3) $\Longrightarrow \lambda \in \operatorname{iso}\sigma(M_0) \Longrightarrow \lambda \in p_0(M_0)$ (since M_0 is finitely polaroid) $\Longrightarrow \lambda \in p_0(M_C)$ (Lemma 2.3) $\Longrightarrow M_C - \lambda \in \Phi$, and this as above implies $\lambda \in p_0(M_c + R)$. Hence $\Pi_0(M_C + R) = p_0(M_C + R)$, and $M_C + R$ satisfies Wt. The converse, $M_C + R$ satisfies Wt $\Longrightarrow M_0 + R$ satisfies Wt follows from a similar argument (recall that M_C is finitely polaroid follows from the hypothesis that M_0 is finitely polaroid). □

Remark 3.9 The equivalence of Theorem 3.8(b) extends to

$$\begin{aligned} M_0 \text{ satisfies Bt} \quad &\Longleftrightarrow \quad M_0 + R \text{ satisfies Wt} \quad \Longleftrightarrow \quad M_C + R \text{ satisfies Wt} \\ &\Longleftrightarrow \quad M_C \text{ satisfies Bt.} \end{aligned}$$

This is seen as follows. The implication $M_0 + R$ satisfies Wt $\Longrightarrow M_0$ satisfies Bt and $M_C + R$ satisfies Wt $\Longrightarrow M_C$ satisfies Bt are clear from Theorem 3.6(b). If M_0 satisfies Bt, then the hypothesis M_0 is finitely polaroid implies M_0 satisfies Wt. By Theorem 3.6(b), $M_0 + R$ satisfies Bt, *i.e.*, $\sigma(M_0 + R) \setminus \sigma_w(M_0 + R) = p_0(M_0 + R) \subseteq \Pi_0(M_0 + R)$. Let $\lambda \in \Pi_0(M_0 + R)$. If $\lambda \notin \sigma(M_0)$, then $(M_0 - \lambda \in \Phi \Longrightarrow) M_0 + R - \lambda \in \Phi \Longrightarrow \lambda \in p_0(M_0 + R)$ (since $\lambda \in \operatorname{iso}\sigma(M_0 + R)$); if $\lambda \in \sigma(M_0)$, then $\lambda \in \operatorname{iso}\sigma(M_0)$ (by Lemma 3.3) and so (since M_0 is finitely polaroid) $\lambda \in p_0(M_0) \Longrightarrow M_0 - \lambda \in \Phi \Longrightarrow M_0 + R - \lambda \in \Phi \Longrightarrow \lambda \in p_0(M_0 + R)$. Thus, in either case, $\Pi_0(M_0 + R) \subseteq p_0(M_0 + R)$, and hence $M_0 + R$ satisfies Wt. The proof for M_C satisfies Bt $\Longrightarrow M_C + R$ satisfies Wt is similar: recall from Lemma 2.3 that M_0 finitely polaroid implies M_C finitely polaroid.

a-Wt. $T \in B(\mathcal{X})$ satisfies *a*-Weyl's theorem, *a*-Wt for short, if T satisfies *a*-Bt and $p_0^a(T) = \Pi_0^a(T)$ (equivalently, if $\sigma_a(T) \setminus \sigma_{aw}(T) = p_0^a(T) = \Pi_0^a(T)$), where $\Pi_0^a(T) = \{\lambda \in \operatorname{iso}\sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\}$ [1]. We say in the following that T is *a*-polaroid if T is polar at every $\lambda \in \operatorname{iso}\sigma_a(T)$. Trivially, *a*-polaroid implies polaroid (indeed, $p_0^a(T) = p_0(T)$ in such a case), but the converse is not true in general. Theorem 3.8 has an *a*-Wt analogue, which we prove below. We note, however, that the perturbation of an operator by a commuting Riesz operator preserves neither its spectrum nor its approximate point spectrum: this will, *per se*, force us into making an assumption on the approximate point spectrum of M_0 and $M_0 + R$ in the analogue of Theorem 3.8(b).

Theorem 3.10 *Suppose that* $\lim_{n\to\infty} \|\delta_{AB}^n(C)\|^{\frac{1}{n}} = 0$.

(a) *If M_0 is a-polaroid, then M_C satisfies a-Wt if and only if M_0 satisfies a-Wt.*
(b) *Let $R_i \in B(\mathcal{X})$, $i = 1, 2$, be Riesz operators such that $R = R_1 \oplus R_2$ commutes with M_C. If $\sigma_a(M_0) = \sigma_a(M_0 + R)$, then a sufficient condition for the equivalence $M_C + R$ satisfies a-Wt $\Longleftrightarrow M_0 + R$ satisfies a-Wt is that M_0 is finitely a-polaroid.*

Proof (a) We prove $\Pi_0^a(M_0) = \Pi_0^a(M_C)$: the proof of (a) would then follow from the fact that if M_0 satisfies *a*-Wt ($\Longrightarrow M_0$ satisfies *a*-Bt $\Longleftrightarrow M_C$ satisfies *a*-Bt), then

$$\Pi_0^a(M_0) = \sigma_a(M_0) \setminus \sigma_{aw}(M_0) = \sigma_a(M_C) \setminus \sigma_{aw}(M_C) = p_0^a(M_C) \subseteq \Pi_0^a(M_C)$$

and if M_C satisfies *a*-Wt, then

$$\Pi_0^a(M_C) = \sigma_a(M_C) \setminus \sigma_{aw}(M_C) = \sigma_a(M_0) \setminus \sigma_{aw}(M_0) = p_0^a(M_0) \subseteq \Pi_0^a(M_0).$$

If $\lambda \in \Pi_0^a(M_0)$, then

$$\begin{aligned}
&\lambda \in \operatorname{iso}\sigma_a(M_0), \quad 0 < \alpha(M_0 - \lambda) < \infty \\
&\iff \quad \lambda \in p_0(M_0) \quad (\text{since } M_0 \text{ is } a\text{-polaroid}) \\
&\iff \quad \lambda \in \big(p_0(A) \cup p_0(B)\big) \cup \big(p_0(A) \cup \rho(B)\big) \cup \big(\rho(A) \cup p_0(B)\big) \\
&\implies \quad \alpha(M_C - \lambda) \leq \alpha(A - \lambda) + \alpha(B - \lambda) < \infty, \\
&\qquad \operatorname{asc}(M_C - \lambda) \leq \operatorname{asc}(A - \lambda) + \operatorname{asc}(B - \lambda) < \infty, \\
&\qquad \operatorname{dsc}(M_C - \lambda) \leq \operatorname{dsc}(A - \lambda) + \operatorname{dsc}(B - \lambda) < \infty \\
&\implies \quad \operatorname{asc}(M_C - \lambda) = \operatorname{dsc}(M_C - \lambda) < \infty, \quad 0 < \alpha(M_C - \lambda) < \infty \\
&\implies \quad \lambda \in p_0(M_C) \subseteq \Pi_0(M_C) \subseteq \Pi_0^a(M_C);
\end{aligned}$$

if instead $\lambda \in \Pi_0^a(M_C)$, then

$$\begin{aligned}
&\lambda \in \operatorname{iso}\sigma_a(M_C), \quad 0 < \alpha(M_C - \lambda) < \infty \\
&\iff \quad \lambda \in \operatorname{iso}\sigma_a(M_0), \quad 0 < \alpha(M_C - \lambda) < \infty \\
&\implies \quad \lambda \in p(M_0), \quad 0 < \alpha(M_C - \lambda) < \infty \\
&\iff \quad \lambda \in p_0(M_c) \quad (\text{Lemma 2.4}) \\
&\iff \quad \lambda \in p_0(M_0) \quad (\text{Lemma 2.1}) \\
&\implies \quad \lambda \in \Pi_0(M_0) \subseteq \Pi_0^a(M_C).
\end{aligned}$$

(b) If $\sigma_a(M_0 + R) = \sigma_a(M_0)$, then it follows from Lemma 2.4 and Theorem 3.4 that

$$\sigma_x(M_0) = \sigma_x(M_0 + R) = \sigma_x(M_C + R) = \sigma_x(M_C); \quad \sigma_x = \sigma_a \text{ or } \sigma_{aw}.$$

Recall from Remark 3.7 that if either of $M_0 + R$ or $M_C + R$ satisfies a-Bt, then M_0, $M_0 + R$, M_C and $M_C + R$ all satisfy a-Bt. Hence, in view of the spectral equalities above,

$$p_0^a(M_0) = p_0^a(M_C) = p_0^a(M_C + R) = p_0^a(M_0 + R),$$

whenever either of M_0, $M_0 + R$, M_C and $M_C + R$ satisfies a-Bt. Observe that the hypothesis M_0 is finitely a-polaroid implies $p_0^a(M_0) = p_0(M_0) = p_0(M_C) = p_0^a(M_0 + R)$; hence (since $p_0^a(M_0) = p_0^a(M_C) = p_0^a(M_C + R) = p_0^a(M_0 + R)$) $p_0^a(S) = p_0^a(T)$ for every choice of $S, T = M_0$ or M_C or $M_0 + R$ or $M_C + R$. We prove now that if either of $M_0 + R$ and $M_C + R$ satisfies a-Wt, then $\Pi_0^a(M_0 + R) = \Pi_0^a(M_C + R)$: this would then imply that if one satisfies a-Wt, then so does the other.

Suppose $M_0 + R$ satisfies a-Wt. Then $p_0(M_0 + R) = p_0^a(M_0 + R) = \Pi_0^a(M_0 + R)$ ($\Longrightarrow \Pi_0^a(M_0 + R) = \Pi_0(M_0 + R)$) and $\Pi_0^a(M_0 + R) \subseteq \Pi_0^a(M_C + R)$. Let $\lambda \in \Pi_0^a(M_c + R)$; then $\lambda \in \operatorname{iso}\sigma_a(M_C + R) = \operatorname{iso}\sigma_a(M_0)$ implies $\lambda \in p_0(M_0) = p_0^a(M_C + R)$. Thus $\Pi_0^a(M_C + R) \subseteq p_0^a(M_C + R) = p_0^a(M_0 + R) = \Pi_0^a(M_0 + R)$. Consequently, $\Pi_0^a(M_0 + R) = \Pi_0^a(M_C + R)$ in this case. Suppose next that $M_C + R$ satisfies a-Wt. Then $p_0(M_C + R) = p_0^a(M_C + R) = \Pi_0^a(M_C + R)$

and $\Pi_0^a(M_C + R) \subseteq \Pi_0^a(M_0 + R)$. Let $\lambda \in \Pi_0^a(M_0 + R)$; then $\lambda \in \operatorname{iso}\sigma_a(M_0)$ implies $\lambda \in p_0^a(M_0) = p_0^a(M_C + R)$. As above, this implies $\Pi_0^a(M_0 + R) = \Pi_0^a(M_C + R)$. □

The following corollary is immediate from Theorem 3.10(b).

Corollary 3.11 *Suppose that* $\lim_{n\to\infty} \|\delta_{AB}^n(C)\|^{\frac{1}{n}} = 0$. *If* $R_i \in B(\mathcal{X})$, $i = 1, 2$, *are quasi-nilpotent operators such that* $R = R_1 \oplus R_2$ *commutes with* M_C, *then a sufficient condition for the equivalence* $M_C + R$ *satisfies a-Wt* $\Longleftrightarrow$ $M_0 + R$ *satisfies a-Wt is that* M_0 *is finitely a-polaroid.*

Competing interests
The authors declare that they have no competing interests.

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