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General Toeplitz operators on weighted Bloch-type spaces in the unit ball of $\mathbb{C}^n$

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Abstract

In this paper, we consider the weighted Bloch-type spaces $\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$ with $\alpha > 0$ and $\beta \geq 0$ in the unit ball of $\mathbb{C}^n$. We present some basic properties of the spaces $\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$, then we consider the Toeplitz operator $T^{\alpha,\beta;\omega}_{\mu}$ acting between $\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$ spaces, where μ is a positive Borel measure in the unit ball $\mathbb{B}_n$. Moreover, we characterize complex measures μ for which the Toeplitz operator $T^{\alpha,\beta;\omega}_{\mu}$ is bounded or compact on $\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$.

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1 Introduction

We start here with some terminology, notations and definitions of various classes of analytic functions defined on the unit ball of $\mathbb{C}^n$.

Let $\mathbb{B}_n$ be the unit ball of the n-dimensional complex Euclidean space $\mathbb{C}^n$. The boundary of $\mathbb{B}_n$ is denoted by $\mathbb{S}_n$ and is called the unit sphere in $\mathbb{C}^n$. Occasionally, we will also need the closed unit ball $\overline{\mathbb{B}}_n$. We denote the class of all holomorphic functions on the unit ball $\mathbb{B}_n$ by $\mathcal{H}(\mathbb{B}_n)$. The ball centered at $\mathbf{z} \in \mathbb{C}^n$ with radius r is denoted by $B(\mathbf{z}, r)$. For $\alpha > -1$, let $d\nu_\alpha(\mathbf{z}) = c_\alpha(1 - |\mathbf{z}|^2)^\alpha\, d\nu$, where $d\nu$ is the normalized Lebesgue volume measure on $\mathbb{B}_n$ and $c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$ (where Γ denotes the gamma function) so that $\nu_\alpha(\mathbb{B}_n) \equiv 1$. The surface measure on $\mathbb{S}_n$ is denoted by $d\sigma$. Once again, we normalize σ so that $\sigma_\alpha(\mathbb{S}_n) \equiv 1$. For any $\mathbf{z} = (z_1, z_2, \dots, z_n), \mathbf{w} = (w_1, w_2, \dots, w_n) \in \mathbb{C}^n$, the inner product is defined by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{k=1}^{n} z_k \overline{w}_k.$$

For every point $\mathbf{a} \in \mathbb{B}_n$, the Möbius transformation $\varphi_{\mathbf{a}} : \mathbb{B}_n \to \mathbb{B}_n$ is defined by

$$\varphi_{\mathbf{a}}(\mathbf{z}) = \frac{\mathbf{a} - P_{\mathbf{a}}(\mathbf{z}) - S_{\mathbf{a}} Q_{\mathbf{a}}(\mathbf{z})}{1 - \langle \mathbf{z}, \mathbf{a} \rangle}, \quad \mathbf{z} \in \mathbb{B}_n,$$

where $S_{\mathbf{a}} = \sqrt{1 - |\mathbf{a}|^2}$, $P_{\mathbf{a}}(\mathbf{z}) = \frac{\mathbf{a}\langle \mathbf{z}, \mathbf{a}\rangle}{|\mathbf{a}|^2}$, $P_0 = 0$ and $Q_{\mathbf{a}} = I - P_{\mathbf{a}}$. The map $\varphi_{\mathbf{a}}$ has the following properties that $\varphi_{\mathbf{a}}(0) = \mathbf{a}$, $\varphi_{\mathbf{a}}(\mathbf{a}) = 0$, $\varphi_{\mathbf{a}} = \varphi_{\mathbf{a}}^{-1}$ and

$$1 - \langle \varphi_{\mathbf{a}}(\mathbf{z}), \varphi_{\mathbf{a}}(\mathbf{w}) \rangle = \frac{(1 - |\mathbf{a}|^2)(1 - \langle \mathbf{z}, \mathbf{w} \rangle)}{(1 - \langle \mathbf{z}, \mathbf{a} \rangle)(1 - \langle \mathbf{a}, \mathbf{w} \rangle)},$$

where $\mathbf{z}$ and $\mathbf{w}$ are arbitrary points in $\mathbb{B}_n$. In particular,

$$1-\left|\varphi_{\mathbf{a}}(\mathbf{z})\right|^2=\frac{(1-|\mathbf{a}|^2)(1-|\mathbf{z}|^2)}{|1-\langle\mathbf{z},\mathbf{a}\rangle|^2}.$$

For $f\in\mathcal{H}(\mathbb{B}_n)$, the holomorphic gradient of f at $\mathbf{z}$ is defined by

$$\nabla f(\mathbf{z})=\left(\frac{\partial f}{\partial z_1}(\mathbf{z}),\frac{\partial f}{\partial z_2}(\mathbf{z}),\dots,\frac{\partial f}{\partial z_n}(\mathbf{z})\right)$$

and the radial derivative of f at $\mathbf{z}$ is defined by

$$\Re f(\mathbf{z})=\langle\nabla f,\overline{\mathbf{z}}\rangle=\sum_{j=1}^{n}z_j\frac{\partial f(\mathbf{z})}{\partial z_j}.$$

Similarly, the Möbius invariant complex gradient of f at $\mathbf{z}$ is defined by

$$\widetilde{\nabla}f(\mathbf{z})=\nabla(f\circ\varphi_{\mathbf{z}})(0).$$

For $\alpha>0$, a function $f\in\mathcal{H}(\mathbb{B}_n)$ is said to belong to the α-Bloch spaces $\mathcal{B}^\alpha(\mathbb{B}_n)$ if (see [1])

$$b_\alpha(f)(\mathbb{B}_n)=\sup_{\mathbf{z}\in\mathbb{B}_n}\left|\nabla f(\mathbf{z})\right|\left(1-|\mathbf{z}|^2\right)^\alpha<\infty.$$

The little Bloch space $\mathcal{B}_0^\alpha(\mathbb{B}_n)$ consists of all $f\in\mathcal{B}^\alpha(\mathbb{B}_n)$ such that

$$\lim_{|\mathbf{z}|\to1^-}\left|\nabla f(\mathbf{z})\right|\left(1-|\mathbf{z}|^2\right)^\alpha=0.$$

With the norm $\|f\|_{\mathcal{B}^\alpha}=|f(0)|+b_\alpha(f)(\mathbb{B}_n)$, we know that $\mathcal{B}^\alpha(\mathbb{B}_n)$ becomes a Banach space and $\mathcal{B}_0^\alpha(\mathbb{B}_n)$ is its closed subspace (see [1]). For $\alpha=1$, the spaces $\mathcal{B}^1(\mathbb{B}_n)$ and $\mathcal{B}_0^1(\mathbb{B}_n)$ become the Bloch and the little Bloch space, respectively (see, for example, [2–5]). Zhu in [5] says that the norm $\|f\|_{\mathcal{B}(\mathbb{B}_n)}$ is equivalent to

$$\left|f(0)\right|+\sup_{\mathbf{z}\in\mathbb{B}_n}\left|\Re f(\mathbf{z})\right|\left(1-|\mathbf{z}|^2\right).$$

For $\alpha>-1$ and $0<p<\infty$, the weighted Bergman space $A_\alpha^p(\mathbb{B}_n)$ consists of holomorphic functions $f\in L^p(\mathbb{B}_n,d\nu_\alpha)$ such that

$$\|f\|^p_{A_\alpha^p(\mathbb{B}_n)}:=\int_{\mathbb{B}_n}\left|f(\mathbf{z})\right|^p\,d\nu_\alpha(\mathbf{z})<\infty,$$

that is, $A_\alpha^p(\mathbb{B}_n)=L^p(\mathbb{B}_n,d\nu_\alpha)\cap\mathcal{H}(\mathbb{B}_n)$. When the weight $\alpha=0$, we simply write $A^p(\mathbb{B}_n)$ for $A_0^p(\mathbb{B}_n)$. In the special case when $p=2$, $A_\alpha^2(\mathbb{B}_n)$ is a Hilbert space. It is well known that for $\alpha>-1$, the Bergman kernel of $A_\alpha^2(\mathbb{B}_n)$ is given by

$$K^\alpha(\mathbf{z},\mathbf{w})=\frac{1}{(1-\langle\mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}},\quad \mathbf{z},\mathbf{w}\in\mathbb{B}_n.$$

For $\alpha > -1$, a complex measure μ is such that

$$\left|\int_{\mathbb{B}_n} (1-|\mathbf{w}|^2)^\alpha \, d\mu(\mathbf{w})\right| = \left|\int_{\mathbb{B}_n} d\mu_\alpha(\mathbf{w})\right| < \infty.$$

The general Bergman projection P_α is the orthogonal projection of the measure μ from $L^2(\mathbb{B}_n, d\nu_\alpha)$ into $A^2_\alpha(\mathbb{B}_n)$ defined by

$$P_\alpha(\mu)(\mathbf{z}) = c_\alpha \int_{\mathbb{B}_n} \frac{(1-|\mathbf{w}|^2)^\alpha}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}} \, d\mu(\mathbf{w}) = c_\alpha \int_{\mathbb{B}_n} \frac{d\mu_\alpha(\mathbf{w})}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}}.$$

The general Bergman projection of the function f is

$$P_\alpha f(\mathbf{z}) = c_\alpha \int_{\mathbb{B}_n} \frac{f(\mathbf{w})(1-|\mathbf{w}|^2)^\alpha}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}} \, d\nu(\mathbf{w}) = c_\alpha \int_{\mathbb{B}_n} \frac{f(\mathbf{w})\, d\nu_\alpha(\mathbf{w})}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}}.$$

Let $\omega : (0,1] \to (0,\infty)$ be a right-continuous and nondecreasing function. For a complex measure $\mu, \alpha > -1$, $\beta \geq 0$, and $f \in L^1(\mathbb{B}_n, d\nu_{\alpha+\beta})$, define weighted general Toeplitz operator as follows:

$$\begin{aligned} T^{\alpha,\beta;\omega}_\mu f(\mathbf{z}) &= \frac{c_{\alpha+\beta}}{(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1-|\mathbf{w}|)} \int_{\mathbb{B}_n} \frac{(1-|\mathbf{w}|^2)^{\alpha+\beta} f(\mathbf{w})}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+\beta+1}} \, d\mu(\mathbf{w}) \\ &= \frac{c_{\alpha+\beta}}{(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1-|\mathbf{w}|)} \int_{\mathbb{B}_n} \frac{f(\mathbf{w})\, d\mu_{\alpha+\beta}(\mathbf{w})}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+\beta+1}}. \end{aligned}$$

Thus $P_{\alpha+\beta;\omega}(\mu)(\mathbf{z}) = T^{\alpha,\beta;\omega}_\mu(1)(\mathbf{z})$, where 1 stands for a constant function.

Toeplitz operators have been studied extensively on the Bergman spaces by many authors. For references, see [6] and [7]. Boundedness and compactness of the general Toeplitz operators T^α_μ on the α-Bloch $\mathcal{B}^\alpha(\mathbb{D})$ spaces have been investigated in [8] on the unit disk $\mathbb{D}$ for $0 < \alpha < \infty$. Also, in [9], the authors extended the general Toeplitz operator T^α_μ to $\mathcal{B}^\alpha(\mathbb{B}_n)$ with $1 \leq \alpha < 2$. Recently, in [10], the general Toeplitz operators T^α_μ on the analytic Besov $B_p(\mathbb{D})$ spaces with $1 \leq p < \infty$ have been investigated. Under a prerequisite condition, the authors characterized a complex measure μ on the unit disk for which T^α_μ is bounded or compact on the Besov space $B_p(\mathbb{D})$. For more studies on the Toeplitz operator, we refer to [11–17].

In this paper, we consider the weighted Bloch-type spaces $\mathcal{B}^{\alpha,\beta}_\omega(\mathbb{B}_n)$ with $\alpha > 0$ and $\beta \geq 0$ in the unit ball of $\mathbb{C}^n$. We prove a certain integral representation theorem that is used to determine the degree of growth of the functions in the space $\mathcal{B}^{\alpha,\beta}_\omega(\mathbb{B}_n)$. It is also proved that the space $\mathcal{B}^{\alpha,\beta}_\omega(\mathbb{B}_n)$ is a Banach space for each weight $\alpha > 0$, $\beta \geq 0$, and the Banach dual of the Bergman space $A^1(\mathbb{B}_n)$ is $\mathcal{B}^{\alpha,\beta}_\omega(\mathbb{B}_n)$ for each $\alpha \geq 1$, $\beta \geq 0$. Further, we extend the Toeplitz operator $T^{\alpha,\beta;\omega}_\mu$ to $\mathcal{B}^{\alpha,\beta}_\omega(\mathbb{B}_n)$ in the unit ball of $\mathbb{C}^n$ and completely characterize the positive Borel measure μ such that $T^{\alpha,\beta;\omega}_\mu$ is bounded or compact in $\mathcal{B}^{\alpha,\beta}_\omega(\mathbb{B}_n)$ spaces with $\alpha + \beta \geq 1$.

Throughout the paper, we say that the expressions A and B are equivalent, and write $A \approx B$ whenever there exist positive constants C_1 and C_2 such that $C_1 A \leq B \leq C_2 A$. As usual, the letter C denotes a positive constant, possibly different on each occurrence. Hereafter, ω stands for a right-continuous and nondecreasing function.

Theorem 1.1 (see [5, Theorem 1.12]) *Suppose b is real and $s > -1$. Then the integrals*

$$I_b(\mathbf{z}) = \int_{\mathbb{S}_n} \frac{d\sigma(\boldsymbol{\zeta})}{|1 - \langle \mathbf{z}, \boldsymbol{\zeta} \rangle|^{n+b}}, \quad \mathbf{z} \in \mathbb{B}_n$$

and

$$J_{b,s}(\mathbf{z}) = \int_{\mathbb{B}_n} \frac{(1 - |\mathbf{w}|^2)^s \, d\nu(\mathbf{w})}{|1 - \langle \mathbf{z}, \mathbf{w} \rangle|^{n+1+s+b}}, \quad \mathbf{z} \in \mathbb{B}_n,$$

have the following asymptotic properties.

(1) *If $b < 0$, then $I_b(\mathbf{z})$ and $J_{b,s}(\mathbf{z})$ are both bounded in $\mathbb{B}_n$.*

(2) *If $b = 0$, then*

$$I_b(\mathbf{z}) \approx I_{b,s}(\mathbf{z}) \approx \log \frac{1}{1 - |\mathbf{z}|^2} \quad \text{as } |\mathbf{z}| \to 1^{-1}.$$

(3) *If $b > 0$, then*

$$I_b(\mathbf{z}) \approx J_{b,s}(\mathbf{z}) \approx \left(1 - |\mathbf{z}|^2\right)^{-b} \quad \text{as } |\mathbf{z}| \to 1^{-1}.$$

Lemma 1.1 (see [5, Lemma 3.3]) *Suppose γ is a real constant and $g \in L^1(\mathbb{B}_n, d\nu)$. If*

$$u(\mathbf{z}) = \left(1 - \left|\varphi_{\mathbf{a}}(\mathbf{z})\right|^2\right)^\beta \int_{\mathbb{B}_n} \frac{g(\mathbf{w}) \, d\nu(\mathbf{w})}{(1 - \langle \mathbf{z}, \mathbf{w} \rangle)^\gamma}, \quad \mathbf{z} \in \mathbb{B}_n,$$

then

$$\left|\widetilde{\nabla} u(\mathbf{z})\right| \le \sqrt{2} |\gamma| \left(1 - |\mathbf{z}|^2\right)^{\frac{1}{2}} \int_{\mathbb{B}_n} \frac{g(\mathbf{w}) \, d\nu(\mathbf{w})}{|1 - \langle \mathbf{z}, \mathbf{w} \rangle|^{\gamma + \frac{1}{2}}}, \quad \forall \mathbf{z} \in \mathbb{B}_n.$$

Let $\beta(\cdot, \cdot)$ be the Bergman metric on $\mathbb{B}_n$. Denote the Bergman metric ball at $\mathbf{w}^{(j)}$ by $B(\mathbf{w}^{(j)}, r) = \{\mathbf{z} \in \mathbb{B}_n : \beta(\mathbf{w}^{(j)}, \mathbf{z}) < r\}$, where $\mathbf{w}^{(j)} \in \mathbb{B}_n$ and $r > 0$.

Lemma 1.2 (see [5, Theorem 2.23]) *For fixed $r > 0$, there is a sequence $\{\mathbf{w}^{(j)}\} \in \mathbb{B}_n$ such that*:

- $\bigcup_{j=1}^{\infty} B(\mathbf{w}^{(j)}, r) = \mathbb{B}_n$;
- *there is a positive integer N such that each $\mathbf{z} \in \mathbb{B}_n$ is contained in at most N of the sets $B(\mathbf{w}^{(j)}, 2r)$.*

The following characterization of Carleson measures can be found in [6], or in [5].

A positive Borel measure μ on the unit ball $\mathbb{B}_n$ is said to be a Carleson measure for the Bergman space $A^p_\alpha(\mathbb{B}_n)$ if

$$\int_{\mathbb{B}_n} \left|f(\mathbf{z})\right|^p \, d\nu_\alpha(\mathbf{z}) \le C \|f\|^p_{A^p_\alpha(\mathbb{B}_n)}, \quad \forall f \in A^p_\alpha(\mathbb{B}_n).$$

It is well known that a positive Borel measure μ is a Carleson measure if and only if there is a positive constant C such that

$$\sup_{\mathbf{w}^{(j)} \in \mathbb{B}_n} \frac{\mu(B(\mathbf{w}^{(j)}, r))}{\nu(B(\mathbf{w}^{(j)}, r))} < \infty,$$

where $\{\mathbf{w}^{(j)}\}$ is the sequence in Lemma 1.2. If μ satisfies that

$$\lim_{j\to\infty}\frac{\mu(B(\mathbf{w}^{(j)},r))}{\nu(B(\mathbf{w}^{(j)},r))}=0,$$

then μ is called a vanishing Carleson measure.

For a given reasonable function $\omega:(0,1]\to(0,\infty)$, the weighted Bloch space $\mathcal{B}_\omega$ of several complex variables is defined as the set of all analytic functions f on $\mathbb{B}_n$ satisfying

$$\big(1-|\mathbf{z}|\big)^\alpha\big|\nabla f(\mathbf{z})\big|\leq C\omega\big(1-|\mathbf{z}|\big),\quad \mathbf{z}\in\mathbb{B}_n,\ \text{where } \alpha\in(0,\infty),$$

for some fixed $C=C_f>0$. In the special case where $\omega\equiv 1$, $\mathcal{B}_\omega$ reduces to the classical Bloch space $\mathcal{B}$ in $\mathbb{C}^n$. This class of functions extends and generalizes the well known Bloch space. Now, we define the space $\mathcal{B}_{\alpha,\beta;\omega}(\mathbb{B}_n)$ in the unit ball $\mathbb{B}_n$. For $\alpha>0$ and $\beta\geq 0$, a function $f\in\mathcal{H}(\mathbb{B}_n)$ is said to belong to the $(\alpha,\beta;\omega)$-Bloch space $\mathcal{B}_{\alpha,\beta;\omega}(\mathbb{B}_n)$ if

$$b_{\alpha,\beta;\omega}(f)(\mathbb{B}_n)=\sup_{\mathbf{a},\mathbf{z}\in\mathbb{B}_n}\frac{(1-|\mathbf{z}|^2)^{\alpha+\beta}}{(1-|\varphi_{\mathbf{a}}(\mathbf{z})|^2)^\beta\omega(1-|\mathbf{z}|)}\big|\nabla f(\mathbf{z})\big|<\infty.$$

The little $(\alpha,\beta;\omega)$-Bloch space $\mathcal{B}_{\alpha,\beta;\omega,0}(\mathbb{B}_n)$ is a subspace of $\mathcal{B}_{\alpha,\beta;\omega}(\mathbb{B}_n)$ consisting of all $f\in\mathcal{B}_{\alpha,\beta;\omega}(\mathbb{B}_n)$ such that

$$\lim_{|\mathbf{a}|\to 1^-}\lim_{|\mathbf{z}|\to 1^-}\frac{(1-|\mathbf{z}|^2)^{\alpha+\beta}}{(1-|\varphi_{\mathbf{a}}(\mathbf{z})|^2)^\beta\omega(1-|\mathbf{z}|)}\big|\nabla f(\mathbf{z})\big|=0.$$

If $\beta=0$, $\omega(1-|\mathbf{z}|)=1$, then we get the α-Bloch space $\mathcal{B}^\alpha(\mathbb{B}_n)$ and the little α-Bloch space $\mathcal{B}_0^\alpha(\mathbb{B}_n)$. If $\omega(1-|\mathbf{z}|)=1$, $\alpha=1$ and $\beta=0$, then we get the classical Bloch space $\mathcal{B}(\mathbb{B}_n)$ and $\mathcal{B}_0(\mathbb{B}_n)$. These classes extend the weighted Bloch spaces defined in [18] to the setting of several complex variables.

The logarithmic $(\alpha,\beta;\omega)$-Bloch space $\mathcal{LB}_\omega^{\alpha,\beta}(\mathbb{B}_n)$ is the space of holomorphic functions f such that

$$\sup_{\mathbf{a},\mathbf{z}\in\mathbb{B}_n}\frac{(1-|\mathbf{z}|^2)^{\alpha+\beta}}{(1-|\varphi_{\mathbf{a}}(\mathbf{z})|^2)^\beta\omega(1-|\mathbf{z}|)}\left(\ln\frac{2}{1-|\mathbf{z}|^2}\right)\big|\nabla f(\mathbf{z})\big|<\infty.$$

Correspondingly, the little logarithmic $(\alpha,\beta;\omega)$-Bloch space $\mathcal{LB}_{\omega;0}^{\alpha,\beta}(\mathbb{B}_n)$ is a subspace of $\mathcal{LB}_\omega^{\alpha,\beta}(\mathbb{B}_n)$ consisting of all functions f such that

$$\lim_{|\mathbf{a}|\to 1^-}\lim_{|\mathbf{z}|\to 1^-}\frac{(1-|\mathbf{z}|^2)^{\alpha+\beta}}{(1-|\varphi_{\mathbf{a}}(\mathbf{z})|^2)^\beta\omega(1-|\mathbf{z}|)}\left(\ln\frac{2}{1-|\mathbf{z}|^2}\right)\big|\nabla f(\mathbf{z})\big|=0.$$

If $\omega(1-|\mathbf{z}|)=1$ and $\beta=0$, then we get the logarithmic α-Bloch space $\mathcal{LB}^\alpha(\mathbb{B}_n)$ and the little logarithmic α-Bloch space $\mathcal{LB}_0^\alpha(\mathbb{B}_n)$. If $\omega(1-|\mathbf{z}|)=1$, $\alpha=1$ and $\beta=0$, then we get the logarithmic Bloch space $\mathcal{LB}(\mathbb{B}_n)$ and $\mathcal{LB}_0(\mathbb{B}_n)$ (see [19]).

2 Holomorphic $(\alpha,\beta;\omega)$-Bloch space in the unit ball

In this section, we study the general $(\alpha,\beta;\omega)$-Bloch space $\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$ in the unit ball of $\mathbb{C}^n$ by giving some characterizations of $(\alpha,\beta;\omega)$-Bloch space, then we present several auxiliary results, which play important roles in the proofs of our main results.

Lemma 2.1 *Let $\alpha, \beta \in (0, \infty)$ and $f \in \mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$. Suppose that*

$$\int_0^1 \frac{\omega(1 - t|\mathbf{z}|)|\mathbf{z}|\,dt}{(1 - t^2|\mathbf{z}|^2)^{\alpha+\beta}} < \infty. \tag{1}$$

Then

$$\left|f(z)\right| \le \left|f(0)\right| + \|f\|_{\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)}.$$

Proof Let $\mathbf{z} \in \mathbb{B}_n$, $0 \le t < 1$ and $f \in \mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$. By the definition of $\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$ and $|\mathbf{z}| > \frac{1}{2}$, we have that

$$\begin{aligned}
\left|f(\mathbf{z}) - f\left(\frac{\mathbf{z}}{2}\right)\right| &= \left|\int_{\frac{1}{2}}^1 \langle \nabla f(t\mathbf{z}), \mathbf{z}\rangle\,dt\right| \\
&\le \left|\int_{\frac{1}{2}}^1 \Re f(t\mathbf{z})\frac{dt}{t}\right| \\
&\le b_{\alpha,\beta;\omega}(f)\int_0^1 \frac{(1 - |\varphi_{\mathbf{a}}(t\mathbf{z})|^2)^\beta \omega(1 - t|\mathbf{z}|)}{(1 - t^2|\mathbf{z}|^2)^{\alpha+\beta}}|\mathbf{z}|\,dt \\
&\le b_{\alpha,\beta;\omega}(f)\int_0^1 \frac{(1 - |\mathbf{a}|^2)^\beta \omega(1 - t|\mathbf{z}|)}{|1 - \langle t\mathbf{z}, \mathbf{a}\rangle|^{2\beta}(1 - t^2|\mathbf{z}|^2)^\alpha}|\mathbf{z}|\,dt.
\end{aligned}$$

Since $(1 - |\mathbf{a}|) \le |1 - \langle t\mathbf{z}, \mathbf{a}\rangle|$ and $(1 - t|\mathbf{z}|) \le |1 - \langle t\mathbf{z}, \mathbf{a}\rangle|$, $\mathbf{a}, \mathbf{z} \in \mathbb{B}_n$, we get

$$\begin{aligned}
\left|f(\mathbf{z}) - f\left(\frac{\mathbf{z}}{2}\right)\right| &\le b_{\alpha,\beta;\omega}(f)\int_0^1 \frac{(1 - |\mathbf{a}|^2)^\beta \omega(1 - |\mathbf{z}|)}{(1 - |\mathbf{a}|)^\beta(1 - t|\mathbf{z}|)^\beta(1 - t^2|\mathbf{z}|^2)^\alpha}|\mathbf{z}|\,dt \\
&\le 4^\beta b_{\alpha,\beta}(f)\int_0^1 \frac{\omega(1 - t|\mathbf{z}|)|\mathbf{z}|\,dt}{(1 - t^2|\mathbf{z}|^2)^{\alpha+\beta}},
\end{aligned}$$

from which the result follows. □

Theorem 2.1 *For each $0 < \alpha, \beta < \infty$, $\gamma > -1$ and $f \in \mathcal{H}(\mathbb{B}_n)$. Then the following conditions are equivalent*:

(i) $f \in \mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$;

(ii) *The function* $\frac{(1-|\mathbf{z}|^2)^{\alpha+\beta}}{(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1-|\mathbf{w}|)}|\Re f(\mathbf{z})|$ *is bounded in* $\mathbb{B}_n$;

(iii) *There exists a function $g \in L^\infty(\mathbb{B}_n)$ such that*

$$f(\mathbf{z}) = \left(1 - \left|\varphi_{\mathbf{z}}(\mathbf{w})\right|^2\right)^\beta \omega\left(1 - |\mathbf{w}|\right)\int_{\mathbb{B}_n} \frac{g(\mathbf{w})\,d\nu_\gamma(\mathbf{w})}{(1 - \langle \mathbf{z}, \mathbf{w}\rangle)^{n+\alpha+\beta+\gamma}}, \quad \mathbf{z} \in \mathbb{B}_n.$$

Proof By the Cauchy-Schwarz inequality in $\mathbb{C}^n$, we have

$$\left|\Re f(\mathbf{z})\right| \le |\mathbf{z}|\left|\nabla f(\mathbf{z})\right| \le \left|\nabla f(\mathbf{z})\right|.$$

This proves that (i) $\Rightarrow$ (ii).

If (ii) holds, then the function

$$g(\mathbf{z}) = \frac{c_{\alpha+\beta+\gamma}}{c_\gamma}\frac{(1 - |\mathbf{z}|^2)^{\alpha+\beta}}{(1 - |\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1 - |\mathbf{w}|)}\left(f(\mathbf{z}) + \frac{\Re f(\mathbf{z})}{n + \alpha + \beta + \gamma}\right)$$

is bounded in $\mathbb{B}_n$. For $\mathbf{z} \in \mathbb{B}_n$ consider the holomorphic function

$$\begin{aligned} F(\mathbf{z}) &= \int_{\mathbb{B}_n} \frac{g(\mathbf{w})(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1-|\mathbf{w}|)\,d\nu_\gamma(\mathbf{w})}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+\beta+\gamma}} \\ &= \int_{\mathbb{B}_n} \frac{\omega(1-|\mathbf{w}|)}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+\beta+\gamma}} \left(f(\mathbf{w}) + \frac{\Re f(\mathbf{w})}{n+\alpha+\beta+\gamma} \right) d\nu_{\alpha+\beta+\gamma}(\mathbf{w}). \end{aligned}$$

As in the proof of Theorem 7.1 in [5], we have $F = f$.

This shows that (ii) implies (iii). That (iii) implies (i) follows from differentiating under the integral sign and then applying Theorem 1.12 in [5]. □

Theorem 2.2 *For each $\alpha > 0$, $\beta \geq 0$, $\alpha + \beta > 0$ and $s = \alpha + \beta - 1$. If $s > -1$, then the Banach dual of $A^1(\mathbb{B}_n)$ can be identified with $\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$ (with equivalent norms) under the following integral pairing*:

$$\langle f, g\rangle_s = \int_{\mathbb{B}_n} f(\mathbf{z})\overline{g(\mathbf{z})}\,d\nu_s(\mathbf{z}), \quad f \in A^1(\mathbb{B}_n), g \in \mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n). \tag{2}$$

Proof It is easy to see that $1-(\alpha+\beta)+s > -1$. If $g \in \mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$, then by Theorem 2.1, there exists a function $h \in L^\infty(\mathbb{B}_n)$ such that

$$g(\mathbf{z}) = \frac{1}{(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1-|\mathbf{w}|)} \int_{\mathbb{B}_n} \frac{h(\mathbf{w})\,d\nu_{1-(\alpha+\beta)+s}(\mathbf{w})}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+1+s}}, \quad \mathbf{z},\mathbf{w} \in \mathbb{B}_n,$$

and $\|h\|_\infty \leq C\|g\|_{\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)}$, where C is a positive constant independent of g. By Fubini's theorem,

$$\begin{aligned} \langle f, g\rangle_s &= \int_{\mathbb{B}_n} f(\mathbf{z})\overline{h(\mathbf{z})}\big(1-|\mathbf{z}|^2\big)\,d\nu_{1-(\alpha+\beta)+s}(\mathbf{z}) \\ &= c_{1-(\alpha+\beta)+s} \int_{\mathbb{B}_n} f(\mathbf{z})\overline{h(\mathbf{z})}\,d\nu(\mathbf{z}). \end{aligned}$$

Applying Lemma 2.15 in [5] for all $f \in A^1(\mathbb{B}_n)$, we have

$$\int_{\mathbb{B}_n} \big|f(\mathbf{z})\big|\,d\nu(\mathbf{z}) \leq \|f\|_{A^1(\mathbb{B}_n)}.$$

Combining this, we see that

$$\big|\langle f, g\rangle_s\big| \leq \|h\|_\infty \|f\|_{A^1(\mathbb{B}_n)} \leq C\|g\|_{\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)} \|f\|_{A^1(\mathbb{B}_n)}.$$

Conversely, if F is a bounded linear functional on $A^1(\mathbb{B}_n)$ and $f \in A^1(\mathbb{B}_n)$, then

$$f_r(\mathbf{z}) = \frac{1}{(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1-|\mathbf{w}|)} \int_{\mathbb{B}_n} \frac{f_r(\mathbf{w})\,d\nu_s(\mathbf{w})}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+1+s}} \quad \text{for } 0 < r < 1.$$

It is easy to verify (using the homogeneous expansion of the kernel function) that

$$F(f_r) = \int_{\mathbb{B}_n} f_r(\mathbf{w}) F_{\mathbf{z}}\left[\frac{1}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+1+s}(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1-|\mathbf{w}|)}\right] d\nu_s(\mathbf{w}).$$

Define a function g on $\mathbb{B}_n$ by

$$\overline{g(\mathbf{w})} = F_{\mathbf{z}}\left[\frac{1}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+1+s}(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^{\beta}\omega(1-|\mathbf{w}|)}\right].$$

Then

$$F(f_r) = \int_{\mathbb{B}_n} f_r(\mathbf{w})\overline{g(\mathbf{w})}\,dv_s(\mathbf{w}) = \langle f, g\rangle_s.$$

It remains for us to show that $g \in \mathcal{B}_{\omega}^{\alpha,\beta}$.

We interchange differentiation and the application of F, which can be justified by using the homogeneous expansion of the kernel. The result is

$$\overline{\Re g(\mathbf{w})} = (n+1+s)F_{\mathbf{z}}\left[\frac{1}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+1+s}(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^{\beta}\omega(1-|\mathbf{w}|)}\right].$$

Since F is bounded on $A^1(\mathbb{B}_n)$, we have

$$\left|\Re g(\mathbf{w})\right| \le \frac{C\|F\|}{(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^{\beta}\omega(1-|\mathbf{w}|)}\int_{\mathbb{B}_n}\frac{dv(\mathbf{w})}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n+2+s}}.$$

An application of Theorem 1.1 for $s+1=\alpha+\beta$ then shows that

$$\left|\Re g(\mathbf{w})\right| \le \frac{C\|F\|}{(1-|\mathbf{z}|^2)^{\alpha+\beta}(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^{\beta}\omega(1-|\mathbf{w}|)}.$$

This shows that $g \in \mathcal{B}_{\omega}^{\alpha,\beta}(\mathbb{B}_n)$ and completes the proof of the theorem. □

Lemma 2.2 *If $n>1$, $\alpha+\beta>\frac{1}{2}$, then $f \in \mathcal{B}_{\omega}^{\alpha,\beta}(\mathbb{B}_n)$ if and only if the function*

$$\frac{(1-|\mathbf{z}|^2)^{\alpha+\beta-1}}{(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta}\omega(1-|\mathbf{z}|)}\left|\widetilde{\nabla} f(\mathbf{z})\right|$$

is bounded in $\mathbb{B}_n$.

Proof Recall from Lemma 2.14 in [5] that

$$\left(1-|\mathbf{z}|^2\right)\left|\nabla f(\mathbf{z})\right| \le \left|\widetilde{\nabla} f(\mathbf{z})\right|, \quad \mathbf{z}\in\mathbb{B}_n.$$

So, the boundedness of

$$\frac{(1-|\mathbf{z}|^2)^{\alpha+\beta-1}}{(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta}\omega(1-|\mathbf{z}|)}\left|\widetilde{\nabla} f(\mathbf{z})\right|$$

implies that of

$$\frac{(1-|\mathbf{z}|^2)^{\alpha+\beta}}{(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta}\omega(1-|\mathbf{z}|)}\left|\nabla f(\mathbf{z})\right|.$$

On the other hand, if $f \in \mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$, then by Theorem 2.1,

$$f(\mathbf{z}) = \left(1 - \left|\varphi_{\mathbf{w}}(\mathbf{z})\right|^2\right)^{\beta} \omega\left(1 - |\mathbf{z}|\right) \int_{\mathbb{B}_n} \frac{g(\mathbf{w})\, d\nu(\mathbf{w})}{(1 - \langle \mathbf{z}, \mathbf{w}\rangle)^{n+\alpha+\beta}}, \quad \mathbf{z} \in \mathbb{B}_n,$$

where g is a function in $L^{\infty}(\mathbb{B}_n)$. Now we let $f(\mathbf{z}) = h(\mathbf{z})u(\mathbf{z})$, where $h(\mathbf{z}) = (1 - |\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta} \times \omega(1 - |\mathbf{z}|)$ and

$$u(\mathbf{z}) = \int_{\mathbb{B}_n} \frac{g(\mathbf{w})\, d\nu(\mathbf{w})}{(1 - \langle \mathbf{z}, \mathbf{w}\rangle)^{n+\alpha+\beta}}.$$

An application of Lemma 1.1 gives

$$\left|\widetilde{\nabla} u(\mathbf{z})\right| \le |n + \alpha + \beta| \sqrt{2}\left(1 - |\mathbf{z}|^2\right)^{\frac{1}{2}} \int_{\mathbb{B}_n} \frac{g(\mathbf{w})\, d\nu(\mathbf{w})}{|1 - \langle \mathbf{z}, \mathbf{w}\rangle|^{n+\alpha+\beta+\frac{1}{2}}}, \quad \forall \mathbf{z} \in \mathbb{B}_n.$$

Since $g(\mathbf{z})$ is bounded, by Theorem 1.1 we have

$$\int_{\mathbb{B}_n} \frac{g(\mathbf{w})\, d\nu(\mathbf{w})}{|1 - \langle \mathbf{z}, \mathbf{w}\rangle|^{n+\alpha+\beta+\frac{1}{2}}} \approx \left(1 - |\mathbf{z}|^2\right)^{\frac{1}{2} - (\alpha+\beta)}.$$

So,

$$\left|\widetilde{\nabla} u(\mathbf{z})\right| \le C\left(1 - |\mathbf{z}|^2\right)^{1-(\alpha+\beta)}.$$

It is easy to check that $\widetilde{\nabla} h(\mathbf{z}) = \nabla(h \circ \varphi_{\mathbf{z}})(0) = 0$.

Using the product rule, we have

$$\left|\widetilde{\nabla} f(\mathbf{z})\right| \le \left|\widetilde{\nabla} h(\mathbf{z})\right|\left|u(\mathbf{z})\right| + \left|h(\mathbf{z})\right|\left|\widetilde{\nabla} u(\mathbf{z})\right| \le \left|\widetilde{\nabla} h(\mathbf{z})\right|\left|u(\mathbf{z})\right|$$

and we have

$$\left|\widetilde{\nabla} f(\mathbf{z})\right| \le |n + \alpha + \beta| \sqrt{2}\left(1 - |\mathbf{z}|^2\right)^{\frac{1}{2}} \left(1 - \left|\varphi_{\mathbf{a}}(\mathbf{z})\right|^2\right)^{\beta} \int_{\mathbb{B}_n} \frac{g(\mathbf{w})\, d\nu(\mathbf{w})}{|1 - \langle \mathbf{z}, \mathbf{w}\rangle|^{n+\alpha+\beta+\frac{1}{2}}}$$

for all $\mathbf{z} \in \mathbb{B}_n$.

Hence,

$$\frac{(1 - |\mathbf{z}|^2)^{\alpha+\beta-1}}{(1 - |\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta} \omega(1 - |\mathbf{z}|)} \left|\widetilde{\nabla} f(\mathbf{z})\right|$$

is bounded in $\mathbb{B}_n$. This completes the proof. □

Lemma 2.3 *Let* $0 < \alpha + \beta \le 2$. *Let* λ *be any real number satisfying the following properties*:

- $0 \le \lambda \le \alpha + \beta$ *if* $0 < \alpha + \beta < 1$;
- $0 < \lambda < 1$ *if* $\alpha + \beta = 1$;
- $\alpha + \beta - 1 \le \lambda \le 1$ *if* $1 < \alpha + \beta \le 2$.

Then, for all $\mathbf{z}, \mathbf{w} \in \mathbb{B}_n$ *a holomorphic function* $f \in \mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$ *if and only if*

$$\sup_{\mathbf{z} \neq \mathbf{w}} \frac{(1 - |\mathbf{z}|^2)^{\lambda}(1 - |\mathbf{w}|)^{\alpha+\beta-\lambda}}{(1 - |\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta} \omega(1 - |\mathbf{z}|)} \frac{|f(\mathbf{z}) - f(\mathbf{w})|}{|\mathbf{z} - \mathbf{w}|} < \infty. \tag{3}$$

Proof Let $f \in \mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$. By a similar proof to the one for Theorem 3.1 in [20], we have

$$\big|f(\mathbf{z}) - f(\mathbf{w})\big| = \sqrt{n}|\mathbf{z} - \mathbf{w}| \int_0^1 \big|\nabla f\big(t\mathbf{z} - (1-t)\mathbf{w}\big)\big|\, dt$$

for any $\mathbf{z}, \mathbf{w} \in \mathbb{B}_n$ with $\mathbf{z} \neq \mathbf{w}$. We know that

$$\|f\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)} \approx \sup_{\mathbf{w},\mathbf{z}\in\mathbb{B}_n} \frac{(1-|\mathbf{z}|^2)^{\alpha+\beta}}{(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta}\omega(1-|\mathbf{z}|)}\big|\nabla f(\mathbf{z})\big|.$$

Thus, there is a constant $C > 0$ such that

$$\frac{|f(\mathbf{z}) - f(\mathbf{w})|}{|\mathbf{z} - \mathbf{w}|} \le C\|f\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)} \int_0^1 \frac{(1-|\varphi_{\mathbf{a}}(t\mathbf{z} - (1-t)\mathbf{a})|^2)^{\beta}\omega(1-|t\mathbf{z} - (1-t)\mathbf{w}|)}{(1-|t\mathbf{z} - (1-t)\mathbf{w}|^2)^{\alpha+\beta}}\, dt.$$

Since $(1-|\mathbf{z}|) \le |1 - \langle \mathbf{w}, \mathbf{z}\rangle|$ and

$$1 - \big|t\mathbf{z} + (1-t)\mathbf{w}\big|^2 \ge 1 - \big|t\mathbf{z} + (1-t)\mathbf{w}\big| \ge 1 - |\mathbf{w}| + \big(|\mathbf{w}| - |\mathbf{z}|\big)t,$$

we get

$$\begin{aligned}
\big(1 - \big|\varphi_{\mathbf{w}}\big(t\mathbf{z} - (1-t)\mathbf{w}\big)\big|^2\big)^{\beta} &= \frac{(1-|\mathbf{w}|^2)^{\beta}(1-|t\mathbf{z} - (1-t)\mathbf{w}|^2)^{\beta}}{|1 - \langle t\mathbf{z} - (1-t)\mathbf{w}, \mathbf{w}\rangle|^{2\beta}} \\
&\le \frac{(1-|\mathbf{w}|^2)^{\beta}(1-|t\mathbf{z} - (1-t)\mathbf{w}|^2)^{\beta}}{(1-|\mathbf{w}|)^{2\beta}} \\
&\le \frac{(1+|\mathbf{w}|)^{\beta}(1-|t\mathbf{z} - (1-t)\mathbf{w}|^2)^{\beta}}{(1-|\mathbf{w}|)^{\beta}} \\
&\le \frac{(1-|t\mathbf{z} - (1-t)\mathbf{w}|^2)^{\beta}}{(1-|\mathbf{w}|)^{\beta}}.
\end{aligned}$$

Thus

$$\frac{|f(\mathbf{z}) - f(\mathbf{w})|}{|\mathbf{z} - \mathbf{w}|} \le C\|f\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)} \int_0^1 \frac{1}{(1-|\mathbf{w}| + (|\mathbf{w}| - |\mathbf{z}|)t)^{\alpha}(1-|\mathbf{w}|)^{\beta}}\, dt. \tag{4}$$

If $|\mathbf{z}| = |\mathbf{a}|$, then

$$\frac{|f(\mathbf{z}) - f(\mathbf{w})|}{|\mathbf{z} - \mathbf{w}|} \le C\|f\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)} \int_0^1 \frac{1}{(1-|\mathbf{w}|)^{\alpha+\beta}}\, dt \le \frac{C\|f\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)}}{(1-|\mathbf{z}|^2)^{\lambda}(1-|\mathbf{w}|^2)^{\alpha+\beta-\lambda}}. \tag{5}$$

Now suppose $|\mathbf{z}| \neq |\mathbf{w}|$ as in [21], there is a constant $C > 0$ such that this integral in (4) is dominated by

$$\frac{C}{(1-|\mathbf{z}|^2)^{\lambda}(1-|\mathbf{w}|^2)^{\alpha+\beta-\lambda}}.$$

Combining with (5), we get that whenever $\mathbf{z} \neq \mathbf{w}$,

$$\frac{|f(\mathbf{z}) - f(\mathbf{w})|}{|\mathbf{z} - \mathbf{w}|} \le \frac{C\|f\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)}}{(1-|\mathbf{z}|^2)^{\lambda}(1-|\mathbf{w}|^2)^{\alpha+\beta-\lambda}}.$$

This proves the necessity. The proof of the sufficiency condition is much akin to the corresponding one in [21], so the proof is omitted. □

Proposition 2.1 *Suppose* $f \in \mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$ *and* $1 \le \alpha + \beta \le 2$. *Let* λ *be any real number satisfying*:

- $0 < \lambda < 1$ *if* $\alpha + \beta = 1$;
- $\alpha + \beta - 1 \le \lambda \le 1$ *if* $1 < \alpha + \beta \le 2$.

Then

$$\sup_{\mathbf{z},\mathbf{w}\in\mathbb{B}_n} \frac{(1-|\mathbf{z}|^2)^{2\alpha+2\beta-\lambda-1}(1-|\mathbf{w}|^2)^{\alpha+\beta-\lambda}|f(\mathbf{z})-f(\mathbf{w})|}{(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta}\omega(1-|\mathbf{z}|)|1-\langle\mathbf{w},\mathbf{z}\rangle|^{2(\alpha+\beta)-(2\lambda+1)}|\mathbf{z}-P_{\mathbf{z}}(\mathbf{w})-S_{\mathbf{z}}Q_{\mathbf{z}}(\mathbf{w})|} \le C\|f\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)}. \tag{6}$$

Proof Let $\mathbf{z} = 0$ in (3), then we have

$$\left(1-|\mathbf{w}|^2\right)^{\alpha+\beta-\lambda}\frac{|f(\mathbf{0})-f(\mathbf{w})|}{|\mathbf{w}|} \le Cb_{\alpha,\beta;\omega}(f)(\mathbb{B}_n), \quad \mathbf{w}\in\mathbb{B}_n\backslash\{\mathbf{0}\}.$$

Now, replacing f by $f\circ\varphi_{\mathbf{w}}$, we get

$$\left(1-|\mathbf{u}|^2\right)^{\alpha+\beta-\lambda}\frac{|f\circ\varphi_{\mathbf{w}}(\mathbf{0})-f\circ\varphi_{\mathbf{w}}(\mathbf{u})|}{|\mathbf{u}|} \le Cb_{\alpha,\beta;\omega}(f\circ\varphi_{\mathbf{w}})(\mathbb{B}_n), \quad \mathbf{u}\in\mathbb{B}_n\backslash\{\mathbf{0}\}. \tag{7}$$

Since

$$\left|\widetilde{\nabla}(f\circ\varphi_{\mathbf{w}})(\mathbf{z})\right| = \left|\widetilde{\nabla}f\left(\varphi_{\mathbf{w}}(\mathbf{z})\right)\right|,$$

by Lemma 2.2, we obtain that

$$\begin{aligned} b_{\alpha,\beta;\omega}(f\circ\varphi_{\mathbf{w}})(\mathbb{B}_n) &\approx \sup_{\mathbf{z},\mathbf{w}\in\mathbb{B}_n}\frac{(1-|\mathbf{w}|^2)^{\alpha+\beta-1}}{(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta}\omega(1-|\mathbf{z}|)}\left|\widetilde{\nabla}(f\circ\varphi_{\mathbf{w}})(\mathbf{z})\right| \\ &= \sup_{\mathbf{z},\mathbf{w}\in\mathbb{B}_n}\frac{(1-|\mathbf{w}|^2)^{\alpha+\beta-1}}{(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta}\omega(1-|\mathbf{z}|)}\left|\widetilde{\nabla}f\left(\varphi_{\mathbf{w}}(\mathbf{z})\right)\right| \\ &= \sup_{\mathbf{z},\mathbf{w}\in\mathbb{B}_n}\frac{(1-|\mathbf{w}|^2)^{\alpha+\beta-1}(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\alpha+\beta-1}}{(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\alpha+\beta-1}(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta}\omega(1-|\mathbf{z}|)}\left|\widetilde{\nabla}f\left(\varphi_{\mathbf{w}}(\mathbf{z})\right)\right|. \end{aligned}$$

Then

$$\begin{aligned} b_{\alpha,\beta;\omega}(f\circ\varphi_{\mathbf{z}})(\mathbb{B}_n) &\le C\|f\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)}\frac{(1-|\mathbf{w}|^2)^{\alpha+\beta-1}}{(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^{\alpha+\beta-1}\omega(1-|\mathbf{w}|)} \\ &\le \frac{C\|f\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)}}{(1-|\mathbf{z}|^2)^{\alpha+\beta-1}}. \end{aligned}$$

Letting $u = \varphi_{\mathbf{w}}(\mathbf{z})$ and $\mathbf{w} \neq \mathbf{z}$ in (7), we obtain

$$\frac{|f(\mathbf{z})-f(\mathbf{w})|}{|\varphi_{\mathbf{w}}(\mathbf{z})|\omega(1-|\mathbf{z}|)(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\alpha+\beta-\lambda}} \le \frac{C\|f\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)}}{(1-|\mathbf{z}|^2)^{\alpha+\beta-1}}.$$

Since

$$1-\left|\varphi_{\mathbf{z}}(\mathbf{w})\right|^2=\frac{(1-|\mathbf{z}|^2)(1-|\mathbf{w}|^2)}{|1-\langle\mathbf{w},\mathbf{z}\rangle|^2}$$

and

$$\varphi_{\mathbf{z}}(\mathbf{w})=\frac{\mathbf{z}-P_{\mathbf{z}}(\mathbf{w})-S_{\mathbf{z}}Q_{\mathbf{z}}(\mathbf{w})}{1-\langle\mathbf{w},\mathbf{z}\rangle}.$$

Consequently,

$$\frac{(1-|\mathbf{z}|^2)^{2\alpha+2\beta-\lambda-1}(1-|\mathbf{w}|^2)^{\alpha+\beta-\lambda}|f(\mathbf{z})-f(\mathbf{w})|}{\omega(1-|\mathbf{z}|)(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^{\alpha+\beta-\lambda}|1-\langle\mathbf{w},\mathbf{z}\rangle|^{2(\alpha+\beta)-(2\lambda+1)}|\mathbf{z}-P_{\mathbf{z}}(\mathbf{w})-S_{\mathbf{z}}Q_{\mathbf{z}}(\mathbf{w})|}$$
$$\leq C\|f\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)}.$$ □

3 Boundedness of general Toeplitz operators

In this section, we study the boundedness of general Toeplitz operators acting on the weighted Bloch-type spaces $\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$ in the unit ball of $\mathbb{C}^n$.

Theorem 3.1 *Let μ be a positive Borel measure on $\mathbb{B}_n$. Then we have*

(1) *if $\alpha+\beta=1$, then $T^{\alpha,\beta;\omega}_{\mu}$ is bounded on $\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$ if and only if $P_{\alpha+\beta-1;\omega}(\mu)\in\mathcal{LB}_{\omega}(\mathbb{B}_n)$ and μ is a Carleson measure;*

(2) *if $\alpha=\beta=1$, then $T^{\alpha,\beta;\omega}_{\mu}$ is bounded on $\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$ if and only if $P_{\alpha+\beta-1;\omega}(\mu)\in\mathcal{B}_{\omega}(\mathbb{B}_n)\cap\mathcal{LB}^2_{\omega}(\mathbb{B}_n)$ and μ is a Carleson measure;*

(3) *if $\alpha>1$, $\beta>1$, then $T^{\alpha,\beta;\omega}_{\mu}$ is bounded on $\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$ if and only if $P_{\alpha+\beta-1;\omega}(\mu)\in\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$ and μ is a Carleson measure.*

Proof Since the Banach dual of $A^1(\mathbb{B}_n)$ is $\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$ under the pairing (2), to prove the boundedness of $T^{\alpha,\beta;\omega}_{\mu}$, it suffices to show that

$$\left|\left\langle f,T^{\alpha,\beta;\omega}_{\mu}(g)\right\rangle_{\alpha}\right|\leq C\|f\|_{A^1(\mathbb{B}_n)}\|g\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)}$$

for all $f\in A^1(\mathbb{B}_n)$ and $g\in\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$, where C is a positive constant that does not depend on f or g.

For $s=\alpha+\beta-1$, by Fubini's theorem, we get

$$\begin{aligned}\left\langle f,T^{\alpha,\beta;\omega}_{\mu}g\right\rangle_s&=\int_{\mathbb{B}_n}f(\mathbf{z})\overline{T^{\alpha,\beta;\omega}_{\mu}g(\mathbf{z})}\,d\nu_s(\mathbf{z})\\&=c_{\alpha+\beta-1}\int_{\mathbb{B}_n}f(\mathbf{z})\overline{g(\mathbf{z})}\left(1-|\mathbf{z}|^2\right)^{\alpha+\beta-1}d\mu(\mathbf{z}).\end{aligned}$$

Using the operator $P_{\alpha+\beta;\omega}$, we have

$$\begin{aligned}\left\langle f,T^{\alpha,\beta;\omega}_{\mu}g\right\rangle_s&=c_{\alpha+\beta-1}\int_{\mathbb{B}_n}(I_{\mathbf{z},\mathbf{w};\boldsymbol{\omega}}-P_{\alpha+\beta;\omega})(f\overline{g})(\mathbf{z})\left(1-|\mathbf{z}|^2\right)^{\alpha+\beta-1}d\mu(\mathbf{z})\\&\quad+c_{\alpha+\beta-1}\int_{\mathbb{B}_n}P_{\alpha+\beta;\omega}(f\overline{g})(\mathbf{z})\left(1-|\mathbf{z}|^2\right)^{\alpha+\beta-1}d\mu(\mathbf{z})\\&=I_1+I_2,\end{aligned}$$

where $I_{\mathbf{z},\mathbf{w};\omega} = \frac{I}{(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1-|\mathbf{w}|)}$, and I is the identity operator. Also,

$$\begin{aligned}
&(I_{\mathbf{z},\mathbf{w};\omega} - P_{\alpha+\beta;\omega})(f\overline{g})(\mathbf{z}) \\
&\quad = \frac{f(\mathbf{z})\overline{g(\mathbf{z})}}{(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1-|\mathbf{w}|)} \\
&\qquad - \frac{c_{\alpha+\beta}}{(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1-|\mathbf{w}|)} \int_{\mathbb{B}_n} \frac{f(\mathbf{w})\overline{g(\mathbf{w})}(1-|\mathbf{w}|^2)^{\alpha+\beta}}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+\beta+1}}\,d\nu(\mathbf{w}) \\
&\quad = \frac{c_{\alpha+\beta}}{(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1-|\mathbf{w}|)} \\
&\qquad \times \int_{\mathbb{B}_n} \frac{(\overline{g(\mathbf{z})}-\overline{g(\mathbf{w})})f(\mathbf{w})(1-|\mathbf{w}|^2)^{\alpha+\beta}}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+\beta+1}(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1-|\mathbf{w}|)}\,d\nu(\mathbf{z}).
\end{aligned}$$

By Proposition 2.1, we have

$$\begin{aligned}
|I_1| &= c_{\alpha+\beta-1}\left|\int_{\mathbb{B}_n} (I_{\mathbf{z},\mathbf{w};\omega} - P_{\alpha+\beta;\omega})(f\overline{g})(\mathbf{z})\big(1-|\mathbf{z}|^2\big)^{\alpha+\beta-1}\,d\mu(\mathbf{z})\right| \\
&= c_{\alpha+\beta-1}c_{\alpha+\beta}\left|\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{(\overline{g(\mathbf{z})}-\overline{g(\mathbf{w})})f(\mathbf{w})(1-|\mathbf{w}|^2)^{\alpha+\beta}(1-|\mathbf{z}|^2)^{\alpha+\beta-1}}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+\beta+1}(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1-|\mathbf{w}|)}\,d\nu(\mathbf{w})\,d\mu(\mathbf{z})\right| \\
&= c_{\alpha+\beta-1}c_{\alpha+\beta}\left|\int_{\mathbb{B}_n} f(\mathbf{w})\big(1-|\mathbf{w}|^2\big)^{\alpha+\beta}\right. \\
&\qquad \left.\times \int_{\mathbb{B}_n} \frac{(\overline{g(\mathbf{z})}-\overline{g(\mathbf{w})})(1-|\mathbf{z}|^2)^{\alpha+\beta-1}}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+\beta+1}(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1-|\mathbf{w}|)}\,d\mu(\mathbf{z})\,d\nu(\mathbf{w})\right| \\
&\le c_{\alpha+\beta-1}c_{\alpha+\beta}\int_{\mathbb{B}_n} \big|f(\mathbf{w})\big|\big(1-|\mathbf{w}|^2\big)^{\lambda} \\
&\qquad \times \int_{\mathbb{B}_n} \frac{(1-|\mathbf{z}|^2)^{2(\alpha+\beta)-\lambda-1}(1-|\mathbf{w}|^2)^{\alpha+\beta-\lambda}|f(\mathbf{z})-f(\mathbf{w})|}{|1-\langle \mathbf{w},\mathbf{z}\rangle|^{2(\alpha+\beta)-(2\lambda+1)}|\mathbf{z}-P_{\mathbf{z}}(\mathbf{w})-S_{\mathbf{z}}Q_{\mathbf{z}}(\mathbf{w})|} \\
&\qquad \times \frac{|\mathbf{z}-P_{\mathbf{z}}(\mathbf{w})-S_{\mathbf{z}}Q_{\mathbf{z}}(\mathbf{w})|(1-|\mathbf{z}|^2)^{\lambda-(\alpha+\beta)}}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n-(\alpha+\beta)+2\lambda+2}(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^\beta \omega(1-|\mathbf{w}|)}\,d\mu(\mathbf{z})\,d\nu(\mathbf{w}) \\
&\le C\int_{\mathbb{B}_n} \|g\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)}\big|f(\mathbf{w})\big|\big(1-|\mathbf{w}|^2\big)^{\lambda}\int_{\mathbb{B}_n} \frac{(1-|\mathbf{z}|^2)^{\lambda-(\alpha+\beta)}}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n-(\alpha+\beta)+2\lambda+1}}\,d\mu(\mathbf{z})\,d\nu(\mathbf{w}).
\end{aligned}$$

Since μ is a Carleson measure, taking $\lambda-(\alpha+\beta) > -1$, then as in [9] or in [22, Proposition 1.4.10], for fixed $r > 0$, we get

$$\begin{aligned}
&\int_{\mathbb{B}_n} \frac{(1-|\mathbf{z}|^2)^{\lambda-(\alpha+\beta)}}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n-(\alpha+\beta)+2\lambda+1}}\,d\mu(\mathbf{z}) \\
&\quad \le \sum_{j=1}^{\infty} \frac{\mu(B(\mathbf{z}^{(j)},r))}{\nu(B(\mathbf{z}^{(j)},r))}\int_{B(\mathbf{z}^{(j)},r)} \frac{(1-|\mathbf{z}|^2)^{\lambda-(\alpha+\beta)}}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n-(\alpha+\beta)+2\lambda+1}}\,d\nu(\mathbf{z}) \le C.
\end{aligned}$$

Therefore,

$$|I_1| \le C\|f\|_{A^1(\mathbb{B}_n)}\|g\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)}.$$

Next considering I_2, we have

$$\begin{aligned}
|I_2| &= c_{\alpha+\beta-1}\left|\int_{\mathbb{B}_n} P_{\alpha+\beta;\omega}(f\overline{g})(\mathbf{z})\bigl(1-|\mathbf{z}|^2\bigr)^{\alpha+\beta-1}\,d\mu(\mathbf{z})\right| \\
&= c_{\alpha+\beta-1}c_{\alpha+\beta}\left|\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{f(\mathbf{w})\overline{g(\mathbf{w})}(1-|\mathbf{w}|^2)^{\alpha+\beta}(1-|\mathbf{z}|^2)^{\alpha+\beta-1}}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+\beta+1}(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^{\beta}\omega(1-|\mathbf{w}|)}\,d\nu(\mathbf{w})\,d\mu(\mathbf{z})\right| \\
&\le c_{\alpha+\beta}\int_{\mathbb{B}_n}\bigl|f(\mathbf{w})\bigr|\bigl(1-|\mathbf{w}|^2\bigr)^{\alpha+\beta}\bigl|g(\mathbf{w})\bigr| \\
&\quad\times\left(\frac{c_{\alpha+\beta-1}}{(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^{\beta}\omega(1-|\mathbf{w}|)}\int_{\mathbb{B}_n}\frac{(1-|\mathbf{z}|^2)^{\alpha+\beta-1}\,d\mu(\mathbf{z})}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n+\alpha+\beta+1}}\right)d\nu(\mathbf{w}) \\
&\le C\int_{\mathbb{B}_n}\|f\|_{A^1(\mathbb{B}_n)}\bigl(1-|\mathbf{w}|^2\bigr)^{\alpha+\beta}\bigl|g(\mathbf{w})\bigr|Q_{\mu}^{\alpha,\beta;\omega}(\mathbf{w})\,d\nu(\mathbf{w}),
\end{aligned}$$

where

$$Q_{\mu}^{\alpha,\beta;\omega}(\mathbf{w}) = \frac{c_{\alpha+\beta-1}}{(1-|\varphi_{\mathbf{z}}(\mathbf{w})|^2)^{\beta}\omega(1-|\mathbf{w}|)}\int_{\mathbb{B}_n}\frac{(1-|\mathbf{z}|^2)^{\alpha+\beta-1}\,d\mu(\mathbf{z})}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n+\alpha+\beta+1}}.$$

As in [9], by simple calculation, we have

$$Q_{\mu}^{\alpha,\beta;\omega}(\mathbf{w}) = P_{\alpha+\beta-1;\omega}(\mu)(\mathbf{w}) + \frac{1}{n+\alpha+\beta}\Re P_{\alpha+\beta-1;\omega}(\mu)(\mathbf{w}). \tag{8}$$

It is easy to see that

(1) if $\alpha+\beta=1$ and $P_{\alpha+\alpha-1;\omega}(\mu)\in\mathcal{LB}_{\omega}^{\alpha,\beta}(\mathbb{B}_n)$, then

$$\bigl(1-|\mathbf{w}|^2\bigr)Q_{\mu}^{\alpha,\beta;\omega}(\mathbf{w})\left(\ln\frac{2}{1-|\mathbf{w}|^2}\right)\in L^{\infty}(\mathbb{B}_n);$$

(2) if $\alpha=\beta=1$, $P_{\alpha+\alpha-1;\omega}(\mu)\in\mathcal{B}_{\omega}(\mathbb{B}_n)\cap\mathcal{LB}-\omega^2(\mathbb{B}_n)$, then

$$\bigl(1-|\mathbf{w}|^2\bigr)Q_{\mu}^{\alpha,\beta;\omega}(\mathbf{w})\in L^{\infty}(\mathbb{B}_n)$$

and

$$\bigl(1-|\mathbf{w}|^2\bigr)^2Q_{\mu}^{\alpha,\beta;\omega}(\mathbf{w})\left(\ln\frac{2}{1-|\mathbf{w}|^2}\right)\in L^{\infty}(\mathbb{B}_n);$$

(3) if $\alpha>1$, $\beta>1$, and $P_{\alpha+\alpha-1;\omega}(\mu)\in\mathcal{B}-\omega^{\alpha,\beta}(\mathbb{B}_n)$, then

$$\bigl(1-|\mathbf{w}|^2\bigr)^{\alpha+\beta}Q_{\mu}^{\alpha,\beta;\omega}(\mathbf{w})\in L^{\infty}(\mathbb{B}_n).$$

This implies that $|I_2|\le C\|f\|_{A^1(\mathbb{B}_n)}\|g\|_{\mathcal{B}_{\omega}^{\alpha,\beta}(\mathbb{B}_n)}$. Hence, $T_{\mu}^{\alpha,\beta;\omega}$ is a bounded operator on $\mathcal{B}_{\omega}^{\alpha,\beta}(\mathbb{B}_n)$ with $\alpha>0$, $\beta\ge 0$.

Conversely, suppose that $T_{\mu}^{\alpha,\beta;\omega}$ is a bounded operator on $\mathcal{B}_{\omega}^{\alpha,\beta}(\mathbb{B}_n)$. Take

$$f_{\mathbf{w}}(\mathbf{z}) = \frac{(1-|\mathbf{w}|^2)^t}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+t+1}}\quad\text{for } t>0.$$

It is clear that $\|f_{\mathbf{w}}\|_{A^1(\mathbb{B}_n)} \le C$. On the other hand, take

$$g_{\mathbf{w}}(\mathbf{z}) = \frac{(1-|\mathbf{w}|^2)^{n+2+t-(\alpha+\beta)}}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+t+1}}; \qquad \varphi_{\mathbf{w}}(\mathbf{z}) \equiv 1 \quad \text{and} \quad \omega\bigl(1-|\mathbf{z}|\bigr) \equiv 1 \quad \text{for } t > 0.$$

Then, we have $\|g_{\mathbf{w}}\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)} \le C$. Therefore

$$\begin{aligned}\bigl|\langle f, T^{\alpha}_{\mu} g\rangle_s\bigr| &= c_{\alpha+\beta-1}\bigl(1-|\mathbf{w}|^2\bigr)^{n+2+2t-(\alpha+\beta)} \int_{\mathbb{B}_n} \frac{(1-|\mathbf{z}|^2)^{\alpha+\beta-1}\, d\mu(\mathbf{z})}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{2(n+t+1)}} \\ &\le C \bigl\| T^{\alpha,\beta;\omega}_{\mu}\bigr\| \|f_{\mathbf{w}}\|_{A^1(\mathbb{B}_n)} \|g_{\mathbf{w}}\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)} \le C.\end{aligned}$$

Thus,

$$\bigl(1-|\mathbf{w}|^2\bigr)^{n+2+2t-(\alpha+\beta)} \int_{B(\mathbf{w},r)} \frac{(1-|\mathbf{z}|^2)^{\alpha+\beta-1}\, d\mu(\mathbf{z})}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{2(n+t+1)}} \le C$$

for every $\mathbf{w} \in \mathbb{B}_n$. This implies that

$$\sup_{\mathbf{w}\in\mathbb{B}_n} \frac{\mu(B(\mathbf{w},r))}{\nu(B(\mathbf{w},r))} < \infty.$$

Hence μ is a Carleson measure on $\mathbb{B}_n$.

From the proof of the sufficient condition, we find that there exists a constant C such that

$$\begin{aligned}|I_2| &= c_{\alpha+\beta} \left| \int_{\mathbb{B}_n} f(\mathbf{w})\bigl(1-|\mathbf{w}|^2\bigr)^{\alpha+\beta} \overline{g(\mathbf{w}) Q^{\alpha,\beta;\omega}_{\mu}(\mathbf{w})}\, d\nu(\mathbf{w})\right| \\ &\le C\|f\|_{A^1(\mathbb{B}_n)} \|g\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)}.\end{aligned}$$

This implies that

$$\bigl|g(\mathbf{w}) Q^{\alpha,\beta;\omega}_{\mu}(\mathbf{w})\bigr| \bigl(1-|\mathbf{w}|^2\bigr)^{\alpha+\beta} \le C\|g\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)}.$$

If $\alpha+\beta = 1$, we have

$$\bigl|g(\mathbf{w}) Q^{\alpha,\beta;\omega}_{\mu}(\mathbf{w})\bigr| \bigl(1-|\mathbf{w}|^2\bigr) \le C\|g\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)}.$$

Take $g_{\mathbf{w}}(\mathbf{z}) = \ln \frac{2}{1-\langle \mathbf{z},\mathbf{w}\rangle}$; $\varphi_{\mathbf{w}}(\mathbf{z}) \equiv 1$ and $\omega(1-|\mathbf{z}|) \equiv 1$. It is clear that $\|g_{\mathbf{w}}\|_{\mathcal{LB}_{\omega}(\mathbb{B}_n)} \le C$. Taking $\mathbf{z} = \mathbf{w}$, then

$$\bigl|Q^{\alpha,\beta;\omega}_{\mu}(\mathbf{w})\bigr| \bigl(1-|\mathbf{w}|^2\bigr) \left(\ln \frac{2}{1-|\mathbf{w}|^2}\right) \le C.$$

From (8) we have $P_{\alpha+\beta-1}(\mu) \in \mathcal{LB}_{\omega}(\mathbb{B}_n)$. Let $\alpha = \beta = 1$, we have

$$\bigl|g(\mathbf{w}) Q^{\alpha,\beta;\omega}_{\mu}(\mathbf{w})\bigr| \bigl(1-|\mathbf{w}|^2\bigr)^2 \le C\|g\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)}.$$

Take $g_{\mathbf{w}}(\mathbf{z}) = \frac{1}{1-\langle \mathbf{z},\mathbf{w}\rangle} + \ln \frac{2}{1-\langle \mathbf{z},\mathbf{w}\rangle}$; $\varphi_{\mathbf{w}}(\mathbf{z}) \equiv 1$ and $\omega(1-|\mathbf{z}|) \equiv 1$. It is clear that

$$\|g_{\mathbf{w}}\|_{\mathcal{B}_{\omega}(\mathbb{B}_n)\cap\mathcal{LB}^2_{\omega}(\mathbb{B}_n)} \le C.$$

Taking $\mathbf{z} = \mathbf{w}$, then

$$\begin{aligned}
&\left|g(\mathbf{w})Q_\mu^{\alpha,\beta;\omega}(\mathbf{w})\right|\left(1-|\mathbf{w}|^2\right)^2 \\
&\quad= \left(\frac{1}{1-|\mathbf{w}|^2} + \ln\frac{2}{1-|\mathbf{w}|^2}\right)\left|Q_\mu^{\alpha,\beta}(\mathbf{w})\right|\left(1-|\mathbf{w}|^2\right)^2 \\
&\quad\le \left|Q_\mu^{\alpha,\beta;\omega}(\mathbf{w})\right|\left(1-|\mathbf{w}|^2\right) + \left|Q_\mu^{\alpha,\beta;\omega}(\mathbf{w})\right|\left(1-|\mathbf{w}|^2\right)^2\left(\ln\frac{2}{1-|\mathbf{w}|^2}\right) \\
&\quad\le C.
\end{aligned}$$

By (8), then $P_{\alpha+\beta-1;\omega}(\mu) \in \mathcal{LB}_\omega^2(\mathbb{B}_n)$ and $P_{\alpha+\beta-1;\omega}(\mu) \in \mathcal{B}_\omega(\mathbb{B}_n)$.

When $\alpha, \beta > 1$, taking $g_{\mathbf{w}}(\mathbf{z}) = (1-\langle\mathbf{z},\mathbf{w}\rangle)^{1-(\alpha+\beta)}$, we have $\|g_{\mathbf{w}}\|_{\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)} \le C$.

From Lemma 2.1, we get

$$\left|Q_\mu^{\alpha,\beta;\omega}(\mathbf{w})\right|\left(1-|\mathbf{w}|^2\right)^{\alpha+\beta} \le C \quad \text{for } \mathbf{w} \in \mathbb{B}_n.$$

By (8) it is obvious that $P_{\alpha+\beta-1;\omega}(\mu) \in \mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$.

This completes the proof of Theorem 3.1. □

4 Compactness of general Toeplitz operators

In this section, we study the compactness of Toeplitz operators on the weighted Bloch-type spaces $\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$ in the unit ball of $\mathbb{C}^n$. We need the following lemma.

Lemma 4.1 *Let $0 < \alpha < \infty$, $0 \le \beta < \infty$ and $T_\mu^{\alpha,\beta;\omega}$ be a bounded linear operator from $\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$ into $\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$. When $0 < \alpha < 1$, $0 \le \beta < 1$ and $\alpha + \beta < 1$, then $T_\mu^{\alpha,\beta;\omega}$ is compact if and only if*

$$\lim_{j\to\infty}\left\|T_\mu^{\alpha,\beta;\omega} f_j\right\|_{\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)} = 0,$$

whenever (f_j) is a bounded sequence in $\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$ that converges to 0 uniformly on $\overline{\mathbb{B}}_n$.

Proof This lemma can be proved by Montel's theorem and Lemma 2.1. □

Theorem 4.1 *Let μ be a positive Borel measure on $\mathbb{B}_n$. We have the following*:

1. *if $\alpha + \beta = 1$, then $T_\mu^{\alpha,\beta;\omega}$ is compact on $\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$ if and only if $P_{\alpha+\beta-1;\omega}(\mu) \in \mathcal{LB}_{\omega;0}(\mathbb{B}_n)$ and μ is a vanishing Carleson measure*;
2. *if $\alpha = \beta = 1$, then $T_\mu^{\alpha,\beta;\omega}$ is compact on $\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$ if and only if $P_{\alpha+\beta-1;\omega}(\mu) \in \mathcal{B}_{\omega;0}(\mathbb{B}_n) \cap \mathcal{LB}_{\omega;0}^2(\mathbb{B}_n)$ and μ is a vanishing Carleson measure*;
3. *if $\alpha > 1$, $\beta > 1$, then $T_\mu^{\alpha,\beta;\omega}$ is compact on $\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$ if and only if $P_{\alpha+\beta-1}(\mu) \in \mathcal{B}_{\omega;0}^{\alpha,\beta}(\mathbb{B}_n)$ and μ is a vanishing Carleson measure.*

Proof For $\alpha + \beta \ge 1$, let (g_j) be a sequence in $\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$ satisfying $\|g_j\|_{\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)} \le 1$ and g_j converges to 0 uniformly as $j \to \infty$ on $\overline{\mathbb{B}}_n$. Suppose $f \in A^1(\mathbb{B}_n)$. By duality, we have that $T_\mu^{\alpha,\beta;\omega}$ is compact on $\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$ if and only if

$$\lim_{j\to\infty}\sup_{\|f\|_{A^1(\mathbb{B}_n)}\le 1}\left|\left\langle f, T_\mu^{\alpha,\beta;\omega}(g_j)\right\rangle\right| = 0.$$

Similarly, as in the proof of Theorem 3.1 for $s = \alpha + \beta - 1$, we have

$$\begin{aligned}\langle f, T_\mu^{\alpha,\beta;\omega} g_j\rangle_s &= c_{\alpha+\beta-1}\int_{\mathbb{B}_n} f(\mathbf{z})\overline{g_j(\mathbf{z})}\big(1-|\mathbf{z}|^2\big)^{\alpha+\beta-1}\,d\mu(\mathbf{z})\\ &= c_{\alpha+\beta-1}\int_{\mathbb{B}_n}\big[(I_{\mathbf{z},\mathbf{w};\omega} - P_{\alpha+\beta;\omega})(f\overline{g}_j)\big](\mathbf{z})\big(1-|\mathbf{z}|^2\big)^{\alpha+\beta-1}\,d\mu(\mathbf{z})\\ &\quad + c_{\alpha+\beta-1}\int_{\mathbb{B}_n} P_{\alpha+\beta;\omega}(f\overline{g}_j)(\mathbf{z})\big(1-|\mathbf{z}|^2\big)^{\alpha+\beta-1}\,d\mu(\mathbf{z})\\ &= J_1 + J_2.\end{aligned}$$

For fixed $0 < \varepsilon < 1$, since μ is a vanishing Carleson measure, there exists $0 < \eta < 1$ such that

$$\big(1-|\mathbf{z}|^2\big)^{\lambda}\int_{\mathbb{B}_n\setminus\eta\mathbb{B}_n}\frac{(1-|\mathbf{w}|^2)^{\lambda-(\alpha+\beta)}}{|1-\langle\mathbf{z},\mathbf{w}\rangle|^{n+2\lambda+1-(\alpha+\beta)}}\,d\mu(\mathbf{w}) < \varepsilon,$$

where $\eta\mathbb{B}_n = \{\mathbf{z}\in\mathbb{C}^n, |\mathbf{z}| < \eta\}$ and $\lambda - (\alpha+\beta) > -1$. For a positive constant $0 < \delta < 1$, as in the proof of Theorem 3.1, by Proposition 2.1, we obtain

$$\begin{aligned}|J_1| &= c_{\alpha+\beta-1}c_{\alpha+\beta}\left|\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{(\overline{g_j(\mathbf{z})}-\overline{g_j(\mathbf{w})})f(\mathbf{w})(1-|\mathbf{w}|^2)^{\alpha+\beta}(1-|\mathbf{z}|^2)^{\alpha+\beta-1}}{(1-\langle\mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta}\omega(1-|\mathbf{z}|)}\,d\nu(\mathbf{w})\,d\mu(\mathbf{z})\right|\\ &= c_{\alpha+\beta-1}c_{\alpha+\beta}\left|\int_{\mathbb{B}_n} f(\mathbf{w})\big(1-|\mathbf{w}|^2\big)^{\alpha+\beta}\right.\\ &\quad\left.\times\int_{\mathbb{B}_n}\frac{(\overline{g_j(\mathbf{z})}-\overline{g_j(\mathbf{w})})(1-|\mathbf{z}|^2)^{\alpha+\beta-1}}{(1-\langle\mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta}\omega(1-|\mathbf{z}|)}\,d\mu(\mathbf{z})\,d\nu(\mathbf{w})\right|\\ &\le c_{\alpha+\beta-1}c_{\alpha+\beta}\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n}\big|f(\mathbf{w})\big|\big(1-|\mathbf{w}|^2\big)^{\alpha+\beta}\\ &\quad\times\int_{\mathbb{B}_n}\frac{|g_j(\mathbf{z})-g_j(\mathbf{w})|(1-|\mathbf{z}|^2)^{\alpha+\beta-1}}{|1-\langle\mathbf{z},\mathbf{w}\rangle|^{n+\alpha+1}(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta}\omega(1-|\mathbf{z}|)}\,d\mu(\mathbf{z})\,d\nu(\mathbf{w})\\ &\quad + c_{\alpha+\beta-1}c_{\alpha+\beta}\int_{\delta\mathbb{B}_n}\big|f(\mathbf{w})\big|\big(1-|\mathbf{w}|^2\big)^{\alpha+\beta}\\ &\quad\times\int_{\mathbb{B}_n}\frac{|g_j(\mathbf{z})-g_j(\mathbf{w})|(1-|\mathbf{z}|^2)^{\alpha+\beta-1}}{|1-\langle\mathbf{z},\mathbf{w}\rangle|^{n+\alpha+1}(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta}\omega(1-|\mathbf{z}|)}\,d\mu(\mathbf{z})\,d\nu(\mathbf{w})\\ &= L_1 + L_2.\end{aligned}$$

Since $g_j \to 0$ as $j \to \infty$ on compact subsets of $\mathbb{B}_n$, we can choose j big enough so that

$$\big|f(\mathbf{w})\big|\big(1-|\mathbf{w}|^2\big)^{\alpha+\beta} < \varepsilon.$$

Therefore,

$$\begin{aligned}L_2 &\le \varepsilon C\int_{\delta\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{|g_j(\mathbf{z})-g_j(\mathbf{w})|(1-|\mathbf{z}|^2)^{\alpha+\beta-1}}{|1-\langle\mathbf{z},\mathbf{w}\rangle|^{n+\alpha+1}(1-|\varphi_{\mathbf{w}}(\mathbf{z})|^2)^{\beta}\omega(1-|\mathbf{z}|)}\,d\mu(\mathbf{z})\,d\nu(\mathbf{w})\\ &\le \varepsilon C\|g_j\|_{\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)}.\end{aligned}$$

Now, taking δ such that $1-[\varepsilon(1-\eta)^{n+1+\lambda}]^{\frac{1}{\lambda}} \leq \delta < 1$, then

$$\begin{aligned}
L_1 &\leq C\|g_j\|_{\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)} \int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n} \left|f(\mathbf{w})\right|\left(1-|\mathbf{w}|^2\right)^{\lambda} \int_{\mathbb{B}_n} \frac{(1-|\mathbf{z}|^2)^{\lambda-(\alpha+\beta)}}{|1-\langle\mathbf{z},\mathbf{w}\rangle|^{n+2\lambda+1-(\alpha+\beta)}}\,d\mu(\mathbf{z})\,d\nu(\mathbf{w}) \\
&\leq C\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n} \left|f(\mathbf{w})\right|\left(1-|\mathbf{w}|^2\right)^{\lambda} \int_{\mathbb{B}_n\setminus\eta\mathbb{B}_n} \frac{(1-|\mathbf{z}|^2)^{\lambda-(\alpha+\beta)}}{|1-\langle\mathbf{z},\mathbf{w}\rangle|^{n+2\lambda+1-(\alpha+\beta)}}\,d\mu(\mathbf{z})\,d\nu(\mathbf{w}) \\
&\quad + C\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n} \left|f(\mathbf{w})\right|\left(1-|\mathbf{w}|^2\right)^{\lambda} \int_{\eta\mathbb{B}_n} \frac{(1-|\mathbf{z}|^2)^{\lambda-(\alpha+\beta)}}{|1-\langle\mathbf{z},\mathbf{w}\rangle|^{n+2\lambda+1-(\alpha+\beta)}}\,d\mu(\mathbf{z})\,d\nu(\mathbf{w}) \\
&\leq C\varepsilon\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n} \left|f(\mathbf{w})\right|\,d\nu(\mathbf{w}) + C\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n} \left|f(\mathbf{w})\right|\frac{(1-\delta)^{\lambda}}{(1-\eta)^{n+1+\lambda}}\,d\nu(\mathbf{w}) \\
&\leq C\varepsilon\|f\|_{A^1(\mathbb{B}_n)}.
\end{aligned}$$

Hence $|J_1| < C\varepsilon$, where C does not depend on $f(\mathbf{z})$, and so

$$\lim_{j\to\infty} \sup_{\|f\|_{A^1(\mathbb{B}_n)}\leq 1} |J_1| = 0.$$

Thus, $T^{\alpha,\beta;\omega}_{\mu}$ is compact on $\mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$ if and only if

$$\lim_{j\to\infty} \sup_{\|f\|_{A^1(\mathbb{B}_n)}\leq 1} |J_2| = 0.$$

Again, as in the proof of Theorem 3.1, we have

$$|J_2| \leq C\int_{\mathbb{B}_n} \|f\|_{A^1(\mathbb{B}_n)}\left(1-|\mathbf{w}|^2\right)^{\alpha+\beta}\left|g(\mathbf{w})\right| Q^{\alpha,\beta;\omega}_{\mu}(\mathbf{w})\,d\nu(\mathbf{w}).$$

From (8) it is easy to see that

(1) if $\alpha+\beta=1$ and $P_{\alpha+\alpha-1;\omega}(\mu) \in \mathcal{LB}^{\alpha,\beta}_{\omega,0}(\mathbb{B}_n)$, then

$$\lim_{|\mathbf{w}|\to 1}\left(1-|\mathbf{w}|^2\right) Q^{\alpha,\beta;\omega}_{\mu}(\mathbf{w})\left(\ln\frac{2}{1-|\mathbf{w}|^2}\right) = 0;$$

(2) if $\alpha=\beta=1$, $P_{\alpha+\alpha-1;\omega}(\mu) \in \mathcal{B}_{\omega;0}(\mathbb{B}_n) \cap \mathcal{LB}^2_{\omega;0}(\mathbb{B}_n)$, then

$$\lim_{|\mathbf{w}|\to 1}\left(1-|\mathbf{w}|^2\right) Q^{\alpha,\beta;\omega}_{\mu}(\mathbf{w}) = 0$$

and

$$\lim_{|\mathbf{w}|\to 1}\left(1-|\mathbf{w}|^2\right)^2 Q^{\alpha,\beta;\omega}_{\mu}(\mathbf{w})\left(\ln\frac{2}{1-|\mathbf{w}|^2}\right) = 0;$$

(3) if $\alpha > 1$, $\beta > 1$, and $P_{\alpha+\alpha-1;\omega}(\mu) \in \mathcal{B}^{\alpha,\beta}_{\omega;0}(\mathbb{B}_n)$, then

$$\lim_{|\mathbf{w}|\to 1}\left(1-|\mathbf{w}|^2\right)^{\alpha+\beta} Q^{\alpha,\beta;\omega}_{\mu}(\mathbf{w}) = 0.$$

Combined with $g_j \to 0$ as $j \to \infty$ on compact subsets of $\mathbb{B}_n$, we have

$$\lim_{j\to\infty} \sup_{\|f\|_{A^1(\mathbb{B}_n)}\le 1} |J_2| = 0.$$

Therefore,

$$\lim_{j\to\infty} \| T_\mu^{\alpha,\beta;\omega} g_j \|_{\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)} = 0,$$

which implies that $T_\mu^{\alpha,\beta;\omega}$ is a compact operator.

Next assume that $T_\mu^{\alpha,\beta;\omega}$ is a compact operator on $\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$. Again, as in the proof of Theorem 3.1, we take

$$f_{\mathbf{w}}(\mathbf{z}) = \frac{(1-|\mathbf{w}|^2)^t}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+t+1}} \quad \text{for } t > 0.$$

We know that $\|f_{\mathbf{w}}\|_{A^1(\mathbb{B}_n)} \le C$. On the other hand, take

$$g_{\mathbf{w}}(\mathbf{z}) = \frac{(1-|\mathbf{w}|^2)^{n+2+t-(\alpha+\beta)}}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+t+1}}; \qquad \varphi_{\mathbf{w}}(\mathbf{z}) \equiv 1 \quad \text{and} \quad \omega\big(1-|\mathbf{z}|\big) \equiv 1 \quad \text{for } t > 0.$$

Then $\|g_{\mathbf{w}}\|_{\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)} \le C$ and $g_{\mathbf{w}} \to 0$ uniformly on compact subsets of $\mathbb{B}_n$, as $|\mathbf{w}| \to 1$,

$$\begin{aligned} &\big|\langle f, T_\mu^{\alpha,\beta;\omega} g\rangle_s\big| \\ &\quad = c_{\alpha+\beta-1}\big(1-|\mathbf{w}|^2\big)^{n+2+2t-(\alpha+\beta)} \int_{\mathbb{B}_n} \frac{(1-|\mathbf{z}|^2)^{\alpha+\beta-1}\, d\mu(\mathbf{z})}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{2(n+t+1)}} \\ &\quad \le C\|f_{\mathbf{w}}\|_{A^1(\mathbb{B}_n)} \| T_\mu^{\alpha,\beta} g_{\mathbf{w}} \|_{\mathcal{B}^{\alpha,\beta}(\mathbb{B}_n)}. \end{aligned}$$

From Lemma 4.1, we have

$$\lim_{|\mathbf{w}|\to 1} \|f_{\mathbf{w}}\|_{A^1(\mathbb{B}_n)} \| T_\mu^{\alpha,\beta;\omega} g_{\mathbf{w}} \|_{\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)} = 0, \quad \forall \mathbf{w} \in \mathbb{B}_n.$$

This implies that μ is a vanishing Carleson measure on $\mathbb{B}_n$.

Next let

$$f_{\mathbf{w}}(\mathbf{z}) = \frac{(1-|\mathbf{w}|^2)^{\alpha+\beta}}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+\beta+1}}.$$

Then, we have $\|f_{\mathbf{w}}\|_{A^1(\mathbb{B}_n)} \le C$. Let $\{g_j\}$ be a bounded sequence in $\mathcal{B}_\omega^{\alpha,\beta}(\mathbb{B}_n)$ that converges to zero uniformly as $j \to \infty$ on $\overline{\mathbb{B}}_n$. By the compactness of $T_\mu^{\alpha,\beta;\omega}$, we have

$$\begin{aligned} 0 &= \lim_{j\to\infty} J_2 = \lim_{j\to\infty} c_{\alpha+\beta} \int_{\mathbb{B}_n} f_{\mathbf{w}}(\mathbf{z})\big(1-|\mathbf{z}|^2\big)^{\alpha+\beta} \overline{g_j(\mathbf{z}) Q_\mu^{\alpha,\beta;\omega}(\mathbf{z})}\, d\nu(\mathbf{z}) \\ &= \lim_{j\to\infty} c_{\alpha+\beta}\big(1-|\mathbf{w}|^2\big)^{\alpha+\beta} \int_{\mathbb{B}_n} \frac{(1-|\mathbf{z}|^2)^{\alpha+\beta} \overline{g_j(\mathbf{z}) Q_\mu^{\alpha,\beta;\omega}(\mathbf{z})}}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+\beta+1}}\, d\nu(\mathbf{z}) \\ &= \lim_{j\to\infty} \big(1-|\mathbf{w}|^2\big)^{\alpha+\beta} \overline{g_j(\mathbf{w}) Q_\mu^{\alpha,\beta;\omega}(\mathbf{w})}. \end{aligned}$$

When $\alpha + \beta = 1$, taking

$$g_{\mathbf{w}}(\mathbf{z}) = \left(\ln \frac{2}{1 - \langle \mathbf{z}, \mathbf{w} \rangle}\right)^2 \left(\ln \frac{1}{1 - |\mathbf{w}|^2}\right)^{-1}; \qquad \varphi_{\mathbf{w}}(\mathbf{z}) \equiv 1 \quad \text{and} \quad \omega\bigl(1 - |\mathbf{z}|\bigr) \equiv 1,$$

with $|\mathbf{w}| \geq \frac{1}{2}$, we have $P_{\alpha+\beta-1;\omega}(\mu) \in \mathcal{LB}_{\omega;0}(\mathbb{B}_n)$.

When $\alpha = \beta = 1$, taking

$$g_{\mathbf{w}}(\mathbf{z}) = \frac{1 - |\mathbf{w}|^2}{(1 - \langle \mathbf{z}, \mathbf{w} \rangle)^2} + \ln \frac{2}{1 - \langle \mathbf{z}, \mathbf{w} \rangle}; \qquad \varphi_{\mathbf{w}}(\mathbf{z}) \equiv 1 \quad \text{and} \quad \omega\bigl(1 - |\mathbf{z}|\bigr) \equiv 1,$$

we have $P_{\alpha+\beta-1;\omega}(\mu) \in \mathcal{LB}^2_{\omega}(\mathbb{B}_n) \cap \mathcal{B}_{\omega}(\mathbb{B}_n)$.

Finally, when $\alpha, \beta > 1$, take

$$g_{\mathbf{w}}(\mathbf{z}) = \frac{1 - |\mathbf{w}|^2}{(1 - \langle \mathbf{z}, \mathbf{w} \rangle)^{\alpha+\beta}}; \qquad \varphi_{\mathbf{w}}(\mathbf{z}) \equiv 1 \quad \text{and} \quad \omega\bigl(1 - |\mathbf{z}|\bigr) \equiv 1.$$

Then, it is obvious that $P_{\alpha+\beta-1;\omega}(\mu) \in \mathcal{B}^{\alpha,\beta}_{\omega}(\mathbb{B}_n)$.

This completes the proof of Theorem 4.1. □

Remark 4.1 It is still an open problem to study the properties of radial Toeplitz operators on the studied spaces of this paper. For more information on radial Toeplitz operators, we refer to [23, 24].

Competing interests
The author declares that they have no competing interests.

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