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A new iteration process for equilibrium, variational inequality, fixed point problems, and zeros of maximal monotone operators in a Banach space

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Abstract
In this article, a new iterative process is introduced to approximate a common element of a fixed point set, the solutions of equilibrium problems, the solution set of variational inequality problems, and the set of zeros of maximal monotone operators in a uniformly smooth and strictly convex Banach space by using a hybrid projection method. Also, we prove new strong convergence theorems for this proposed iterative precess in a Banach space.

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1 Introduction

Let E be a real Banach space, E^* be the dual space of E. A set-valued mapping $A : D(A) \subset E \to E^*$ with graph $G(A) = \{(x, x^*) : x^* \in Ax\}$, domain $D(A) = \{x \in E : Ax \neq \emptyset\}$, and range $R(A) = \cup\{Ax : x \in D(A)\}$. A is said to be *monotone* if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever $x^* \in Ax$, $y^* \in Ay$. A monotone operator A is said to be *maximal monotone* if its graph is not properly contained in the graph of any other monotone operator. Let $A \subset E \times E^*$ be a maximal monotone operator. We consider the problem for finding $x \in E$

$$0 \in Ax, \tag{1.1}$$

a point $x \in E$ is called a *zero point* of A. Denote by $A^{-1}0$ the set of all points $x \in E$ such that $0 \in Ax$. We know that if A is maximal monotone, then the solution set $A^{-1}0 = \{x \in D(A) : 0 \in Tx\}$ is closed and convex. One popular algorithm for approximating a solution of this problem is called the proximal point algorithm which was first proposed by Martinet [1] and studied further by Rockafellar [2] in Hilbert spaces. Since the proximal point algorithm weakly converges in general which is the proximal point algorithm is defined by $x_0 \in E$ and

$$x_{n+1} = J_{r_n} x_n, \quad \text{for } n = 0, 1, 2, 3, \ldots, \tag{1.2}$$

where $\{r_n\} \subset (0, \infty)$ and J_{r_n} are the resolvent of A. Solovov and Svaitor [3] proposed a modified proximal point algorithm which converges strongly to a solution of the equation $A^{-1}0$ by using the projection method. Many problems in nonlinear analysis and optimization can be formulated by the proximal point algorithm (see [4–9]).

Let E be a real Banach space with dual E^* and let C be a nonempty closed and convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction. The *equilibrium problem* is to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The equilibrium problem is very general in the sense that it includes, as special cases, optimization problems, variational inequality problems, min-max problems, saddle point problem, fixed point problem, Nash EP. In 2008, Takahashi and Zembayashi [10, 11] introduced iterative sequences for finding a common solution of an equilibrium problem and a fixed point problem.

A mapping $A : D(A) \subset E \to E^*$ is said to be *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

If A is α-inverse strongly monotone, then it is $\frac{1}{\alpha}$-Lipschitz continuous, *i.e.*,

$$\|Ax - Ay\| \leq \frac{1}{\alpha} \|x - y\|, \quad \forall x, y \in C.$$

Let C be a nonempty closed and convex subset of a real Banach space E. Let A be a monotone operator from C into E . *The variational inequality problem* for an operator A is to find $\hat{z} \in C$ such that

$$\langle y - \hat{z}, A\hat{z} \rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $VI(A, C)$.

Let C be a nonempty closed and convex subset of E. A mapping T from C into itself is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

T is said to be *total asymptotically nonexpansive* if there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\left\| T^n x - T^n y \right\| \leq \|x - y\| + \mu_n \psi\left(\|x - y\|\right) + \nu_n, \quad \forall x, y \in C, \forall n \geq 1.$$

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the fixed point set of T; that is, $F(T) = \{x \in C : Tx = x\}$. A point p in C is called an *asymptotic fixed point* of

T [12] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The asymptotic fixed point set of T is denoted by $\widehat{F}(T)$.

The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^*\rangle$ or $x^*(x)$. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\right\}$$

for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x, y \in E. \tag{1.5}$$

If E is a Hilbert space, then $\phi(y,x) = \|y - x\|^2$. It is obvious from the definition of ϕ that

$$\left(\|y\| - \|x\|\right)^2 \le \phi(y,x) \le \left(\|y\| + \|x\|\right)^2, \quad \forall x, y \in E. \tag{1.6}$$

T is said to be *ϕ-nonexpansive* [13, 14] if

$$\phi(Tx, Ty) \le \phi(x,y), \quad \forall x, y \in C.$$

T is said to be *quasi-ϕ-nonexpansive* [13, 14] if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \le \phi(p,x), \quad \forall x \in C \text{ and } p \in F(T).$$

T is said to be *asymptotically ϕ-nonexpansive* [14] if there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi\left(T^n x, T^n y\right) \le k_n \phi(x,y), \quad \forall x, y \in C.$$

T is said to be *quasi-ϕ-asymptotically nonexpansive* [14] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi\left(p, T^n x\right) \le k_n \phi(p,x), \quad \forall x \in C, p \in F(T), \forall n \ge 1.$$

T is said to be *total quasi-ϕ-asymptotically nonexpansive* if $F(T) \neq \emptyset$ and there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\phi\left(p, T^n x\right) \le \phi(p,x) + \nu_n \varphi\left(\phi(p,x)\right) + \mu_n, \quad \forall n \ge 1, \forall x \in C, p \in F(T).$$

A mapping T is said to be *uniformly L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\left\|T^n x - T^n y\right\| \le L\|x - y\|, \quad \forall x, y \in C. \tag{1.7}$$

T is said to be *closed* if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, $Tx_0 = y_0$.

Remark 1.1 Every quasi-ϕ-nonexpansive mapping implies a quasi-ϕ-asymptotically nonexpansive mapping and a quasi-ϕ-asymptotically nonexpansive mapping implies a total quasi-ϕ-asymptotically nonexpansive mapping, but the converse is not true.

On the other hand, Alber [15] introduced that the *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution of the minimization problem

$$\phi(\bar{x},x) = \inf_{y\in C} \phi(y,x). \tag{1.8}$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J. Let Π_C be the generalized projection from a smooth strictly convex and reflexive Banach space E onto a nonempty closed convex subset C of E. Then Π_C is a closed relatively quasi-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

Matsushita and Takahashi [16] proposed the following hybrid iteration method with a generalized projection for a relatively nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z,y_n) \leq \phi(z,x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n\cap Q_n} x_0. \end{cases} \tag{1.9}$$

They proved that $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$. Many authors studied the methods for approximating fixed points of a countable family of (relatively quasi-) nonexpansive mappings (see [17–19]).

Recently, Qin *et al.* [20] considered a pair of asymptotically quasi-ϕ-nonexpansive mappings. To be more precise, they proved the following results.

Theorem QCK *Let E be a uniformly smooth and uniformly convex Banach space and C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1,\infty)$ such that $k_n^{(t)} \to 1$ as $n \to \infty$ and $S : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_s^{(t)}\} \subset [1,\infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be real number sequences in $[0,1]$. Assume that T and S are uniformly asymptotically regular on C and $\Omega = F(T) \cap F(S)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the* *following manner*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ z_n = J^{-1}(\beta_n J x_n + \gamma_n J(T^n x_n) + \delta_n J(S^n x_n)), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \\ C_{n+1} = \{w \in C_n : \phi(w, y_n) \le \phi(w, x_n) + (k_n - 1) M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{1.10}$$

where $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ *for each* $n \ge 1$, J *is the duality mapping on* E, $M_n = \sup\{\phi(z, x_n) : z \in \Omega\}$ *for each* $n \ge 1$. *Assume that the control sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\delta_n\}$ *satisfy the following restrictions*:

(a) $\beta_n + \gamma_n + \delta_n = 1$, $\forall n \ge 1$;
(b) $\liminf_{n\to\infty} \gamma_n \delta_n$, $\lim_{n\to\infty} \beta_n = 0$;
(c) $0 \le \alpha_n < 1$ *and* $\limsup_{n\to\infty} \alpha_n < 1$.

In 2008, Alber *et al.* [21] proved the strong convergence theorems to approximate a fixed point of a total asymptotically nonexpansive mapping in a Hilbert space. In 2011, Chang *et al.* [22, 23] proved the strong convergence theorems for finding the set of fixed points of a total quasi-ϕ-asymptotically nonexpansive mapping in the framework of Banach spaces.

Motivated and inspired by the work mentioned above, in this paper, we introduce a new hybrid projection algorithm for a pair of total quasi-ϕ-asymptotically nonexpansive mappings for finding a set of solutions of the equilibrium problem, a zero point of maximal monotone operators, and a set of solutions of the variation inequality in a uniformly smooth and strictly convex Banach space.

2 Preliminaries

In this article, we denote the strong convergence and weak convergence of a sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

A Banach space E with the norm $\|\cdot\|$ is called *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \ne y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is called *smooth* if the limit $\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for each $x, y \in U$. It is also called *uniformly smooth* if the limit exists uniformly for all $x, y \in U$. The *modulus of convexity* of E is the function $\delta : [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \ge \varepsilon\right\}.$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let p be a fixed real number with $p \ge 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \ge c\varepsilon^p$ for all $\varepsilon \in [0,2]$. Observe that every p-uniformly convex is uniformly convex. One should note that no Banach space is p-uniformly convex for $1 < p < 2$.

Remark 2.1 The basic properties of E, J, and J^{-1} are as follows (see [24]).

- If E is an arbitrary Banach space, then J is monotone and bounded;
- If E is strictly convex, then J is strictly monotone;
- If E is smooth, then J is single-valued and semi-continuous;
- If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;
- If E is reflexive smooth and strictly convex, then the normalized duality mapping J is single-valued, one-to-one, and onto;
- If E is a reflexive strictly convex and smooth Banach space and J is the duality mapping from E into E^*, then J^{-1} is also single-valued, bijective and is also the duality mapping from E^* into E, and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$;
- If E is uniformly smooth, then E is smooth and reflexive;
- If E is a reflexive and strictly convex Banach space, then J^{-1} is norm-weak*-continuous.

Remark 2.2 If E is a reflexive strictly convex and smooth Banach space, then $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$, then $x = y$. From (1.5), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [24–26] for more details).

Recall that a Banach space E has the Kadec-Klee property [24, 25, 27] if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach space, then E has the Kadec-Klee property.

The generalized projection [15] from E into C is defined by $\Pi_C(x) = \operatorname{argmin}_{y\in C} \phi(y,x)$. The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(y,x)$ and the strict monotonicity of the mapping J (see, for example, [4, 15, 24, 25, 28]). If E is a Hilbert space, then $\phi(x,y) = \|x - y\|^2$ and Π_C becomes the metric projection $P_C : H \to C$. If C is a nonempty closed and convex subset of a Hilbert space H, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. We also need the following lemmas for the proof of our main results.

Lemma 2.3 (Alber [15]) *Let C be a nonempty closed convex subset of a smooth Banach space E and let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.4 (Alber [15]) *Let E be a reflexive strictly convex and smooth Banach space, C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y,x), \quad \forall y \in C.$$

Lemma 2.5 (Change *et al.* [22]) *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $S : C \to C$ be a closed and total quasi-ϕ-asymptotically nonexpansive mapping with non-negative real sequences ν_n and μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$. If $\mu_n = 0$, then the fixed point set $F(S)$ is a closed convex subset of C.*

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, *i.e.*, $f(x,y) + f(y,x) \le 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} f\bigl(tz + (1-t)x, y\bigr) \le f(x,y);$$

(A4) for each $x \in C$, $y \mapsto f(x,y)$ is convex and lower semi-continuous.

The following result is in Blum and Oettli [8].

Lemma 2.6 (Blum and Oettli [8]) *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *and let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C.$$

Lemma 2.7 (Takahashi and Zembayashi [11]) *Let C be a closed convex subset of a uniformly smooth strictly convex and reflexive Banach space E and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions* (A1)-(A4). *For all $r > 0$ and $x \in E$, define a mapping $K_r : E \to C$ as follows*:

$$K_r x = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \forall y \in C \right\}.$$

Then the following hold:

(1) *K_r is single-valued*;

(2) *K_r is a firmly nonexpansive-type mapping* [29], *that is, for all $x, y \in E$,*

$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \le \langle K_r x - K_r y, Jx - Jy\rangle;$$

(3) *$F(K_r) = EP(f)$*;

(4) *$EP(f)$ is closed and convex.*

Lemma 2.8 (Takahashi and Zembayashi [11]) *Let C be a closed convex subset of a smooth strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and let $r > 0$. Then, for $x \in E$ and $q \in F(K_r)$,*

$$\phi(q, K_r x) + \phi(K_r x, x) \le \phi(q, x).$$

Lemma 2.9 [30] *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \le r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \le \|\lambda x\|^2 + |\mu y\|^2 + \|\gamma z\|^2 - \lambda\mu g\bigl(\|x - y\|\bigr)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0,1]$ with $\lambda + \mu + \gamma = 1$.

Let E be a smooth strictly convex and reflexive Banach space, C be a nonempty closed convex subset of E and $A \subset E \times E^*$ be a monotone operator satisfying $D(A) \subset C \subset J^{-1}(\bigcap_{\lambda>0} R(J + \lambda A))$. Then the *resolvent* $J_\lambda : C \to D(A)$ of A is defined by

$$J_\lambda x = \{z \in D(A) : Jx \in Jz + \lambda Az, \forall x \in C\}.$$

J_λ is a single-valued mapping from E to $D(A)$. For any $\lambda > 0$, the *Yosida approximation* $A_\lambda : C \to E^*$ of A is defined by $A_\lambda x = \frac{Jx - JJ_\lambda x}{\lambda}$ for all $x \in C$. We know that $A_\lambda x \in A(J_\lambda x)$ for all $\lambda > 0$ and $x \in E$.

Lemma 2.10 (Kohsaka and Takahashi [29]) *Let E be a smooth strictly convex and reflexive Banach space, C be a nonempty closed convex subset of E and $A \subset E \times E^*$ be a monotone operator satisfying $D(A) \subset C \subset J^{-1}(\bigcap_{\lambda>0} R(J + \lambda A))$. For any $\lambda > 0$, let J_λ and A_λ be the resolvent and the Yosida approximation of A, respectively. Then the following hold*:

(a) $\phi(p, J_\lambda x) + \phi(J_\lambda x, x) \le \phi(p, x)$ *for all $x \in C$ and $p \in A^{-1}0$*;
(b) $(J_\lambda x, A_\lambda x) \in A$ *for all $x \in C$*;
(c) $F(J_\lambda) = A^{-1}0$.

Lemma 2.11 (Rockafellar [31]) *Let E be a reflexive strictly convex and smooth Banach space. Then an operator $A \subset E \times E^*$ is maximal monotone if and only if $R(J + \lambda A) = E^*$ for all $\lambda > 0$.*

3 Main result

Theorem 3.1 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly uniformly convex Banach space E with the Kadec-Klee property. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A1)-(A4) *and let $A \subset E \times E^*$ be a maximal monotone operator satisfying $D(A) \subset C$ and $J_{r_n} = (J + r_n A)^{-1} J$ for all $r_n > 0$. Let $S : C \to C$ be a closed and total quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequences ν_n^S, μ_n^S with $\nu_n^S \to 0$, $\mu_n^S \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi^S : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi^S(0) = 0$. Let $T : C \to C$ be a closed and total quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequences ν_n^T, μ_n^T with $\nu_n^T \to 0$, $\mu_n^T \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi^T : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi^T(0) = 0$. Assume that S and T are uniformly L-Lipschitz continuous and $F = F(S) \cap F(T) \cap EP(f) \cap A^{-1}0 \neq \emptyset$. For an initial point $x_1 \in E$, $C_1 = C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} z_n = J_{r_n} x_n, \\ u_n = K_{r_n} x_n, \\ y_n = J^{-1}(\alpha_n J x_n + \beta_n J S^n z_n + \gamma_n J T^n u_n), \\ C_{n+1} = \{\nu \in C_n : \phi(\nu, y_n) \le \phi(\nu, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\{r_n\} \subset [d, \infty)$ for some $d > 0$ $\mu_n = \sup\{\mu_n^S, \mu_n^T\}$, $\nu_n = \sup\{\nu_n^S, \nu_n^T\}$, $\psi = \sup\{\psi^S, \psi^T\}$ for all $n \ge 1$, $\zeta = \nu_n \sup_{q \in \mathcal{F}} \psi(\phi(q, x_n)) + \mu_n$. If $\lim_{n\to\infty} \alpha_n \beta_n = 0$ and $\liminf_{n\to\infty} \alpha_n \gamma_n < 1$, then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Proof First, we show that C_n is closed and convex for all $n \in \mathbb{N}$ since $C_1 = C$ is convex. Suppose that C_n is convex for all $n \in \mathbb{N}$. For any $v \in C_n$, we know that $\phi(v, y_n) \le \phi(v, x_n) + \zeta_n$ is equivalent to

$$2\langle v, Jx_n - Jy_n\rangle \le \|x_n\|^2 - \|y_n\|^2 + \zeta_n.$$

That is, C_{n+1} is convex for all $n \in \mathbb{N}$. By the definition of C_n, it is obvious that C_n is closed for all $n \in \mathbb{N}$.

We show that $\{x_n\}$ is well defined. It is obvious that $F \subset C_1 = C$. Suppose $F \subset C_n$ for $n \in \mathbb{N}$, from Lemma 2.8 and Lemma 2.10, S, T are total quasi-ϕ-asymptotically nonexpansive mappings. For each $q \in F \subset C_n$, it follows that

$$\begin{aligned}
\phi(q, y_n) &= \phi\bigl(q, J^{-1}\bigl(\alpha_n Jx_n + \beta_n JS^n z_n + \gamma_n JT^n u_n\bigr)\bigr) \\
&= \|q\|^2 - 2\bigl\langle q, \alpha_n Jx_n + \beta_n JS^n z_n + \gamma_n JT^n u_n\bigr\rangle + \bigl\|\alpha_n Jx_n + \beta_n JS^n z_n + \gamma_n JT^n u_n\bigr\|^2 \\
&\le \alpha_n \phi(q, x_n) + \beta_n \phi\bigl(q, S^n z_n\bigr) + \gamma_n \phi\bigl(q, T^n u_n\bigr) \\
&\le \alpha_n \phi(q, x_n) + \beta_n \bigl(\phi(q, z_n) + v_n^S \psi^S\bigl(\phi(q, z_n)\bigr) + \mu_n^S\bigr) \\
&\quad + \gamma_n \bigl(\phi(q, u_n) + v_n^T \psi^T\bigl(\phi(q, u_n)\bigr) + \mu_n^T\bigr) \\
&= \alpha_n \phi(q, x_n) + \beta_n \phi(q, z_n) + \beta_n v_n^S \psi^S\bigl(\phi(q, z_n)\bigr) + \beta_n \mu_n^S \\
&\quad + \gamma_n \phi(q, u_n) + \gamma_n v_n^T \psi^T\bigl(\phi(q, u_n)\bigr) + \gamma_n \mu_n^T \\
&\le \alpha_n \phi(q, x_n) + \beta_n \phi(q, z_n) + \beta_n v_n^S \psi^S\bigl(\phi(q, x_n)\bigr) + \beta_n \mu_n^S \\
&\quad + \gamma_n \phi(q, u_n) + \gamma_n v_n^T \psi^T\bigl(\phi(q, x_n)\bigr) + \gamma_n \mu_n^T \\
&\le \alpha_n \phi(q, x_n) + \beta_n \phi(q, z_n) + \beta_n v_n \psi\bigl(\phi(q, x_n)\bigr) + \beta_n \mu_n \\
&\quad + \gamma_n \phi(q, u_n) + \gamma_n v_n \psi\bigl(\phi(q, x_n)\bigr) + \gamma_n \mu_n \\
&\le \alpha_n \phi(q, x_n) + \beta_n \phi(q, z_n) + \gamma_n \phi(q, u_n) + (1 - \alpha_n) v_n \psi\bigl(\phi(q, x_n)\bigr) + (1 - \alpha_n)\mu_n \\
&\le \alpha_n \phi(q, x_n) + \beta_n \phi(q, z_n) + \gamma_n \phi(q, u_n) + v_n \sup_{q \in F} \psi\bigl(\phi(q, x_n)\bigr) + \mu_n \\
&\le \alpha_n \phi(q, x_n) + \beta_n \phi(q, z_n) + \gamma_n \phi(q, u_n) + \zeta_n \\
&\le \alpha_n \phi(q, x_n) + \beta_n \phi(q, x_n) + \gamma_n \phi(q, u_n) + \zeta_n \\
&\le \alpha_n \phi(q, x_n) + \beta_n \phi(q, x_n) + \gamma_n \phi(q, K_{r_n} x_n) + \zeta_n \\
&\le \alpha_n \phi(q, x_n) + \beta_n \phi(q, x_n) + \gamma_n \phi(q, x_n) + \zeta_n \\
&\le \phi(q, x_n) + \zeta_n,
\end{aligned} \tag{3.2}$$

where $\zeta_n = v_n \sup_{q \in F} \psi(\phi(q, x_n)) + \mu_n$. This shows that $q \in C_{n+1}$, thus $F \subset C_{n+1}$. Hence, $F \subset C_n$ for all $n \ge 1$. This implies that the sequence $\{x_n\}$ is well defined.

We show that $\lim_{n\to\infty} x_n = p$. From the definition of C_{n+1} with $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, it follows that

$$\phi(x_n, x_1) \le \phi(x_{n+1}, x_1), \quad \forall n \ge 1. \tag{3.3}$$

By Lemma 2.4, we get

$$\begin{aligned}\phi(x_n,x_1) &= \phi(\Pi_{C_n}x_1,x_1)\\ &\leq \phi(q,x_1)-\phi(q,x_n)\\ &\leq \phi(q,x_1),\quad \forall q\in F.\end{aligned}\tag{3.4}$$

From (3.3) and (3.4), we have that $\lim_{n\to\infty}\phi(x_n,x_1)$ exists. In particular, it follows from (1.6) that the sequence $\{x_n\}$ is bounded and so are $\{z_n\}$, $\{u_n\}$, and $\{y_n\}$. Since $x_n\in C_n\subset E$ and E is reflexive, the sequence $\{x_n\}$ converges weakly to an element of E, we assume that $x_n\rightharpoonup p$. Note that C_n is closed and convex and $x_n\in C_n$. We have that $p\in C_n$, that is,

$$x_n\rightharpoonup p\in C_n\quad \text{as } n\to\infty.\tag{3.5}$$

For $p\in C_n$, we have

$$\begin{aligned}\liminf_{n\to\infty}\phi(x_n,x_1) &= \liminf_{n\to\infty}\bigl\{\|x_n\|^2-2\langle x_n,Jx_1\rangle+\|x_1\|^2\bigr\}\\ &\geq \|p\|^2-2\langle p,Jx_1\rangle+\|x_1\|^2\\ &= \phi(p,x_1).\end{aligned}$$

On the other hand, $x_n=\Pi_{C_n}x_1$, we have

$$\phi(x_n,x_1)\leq\phi(p,x_1),\quad \forall p\in C_n.$$

It follows that

$$\phi(p,x_1)\leq\liminf_{n\to\infty}\phi(x_n,x_1)\leq\limsup_{n\to\infty}\phi(x_n,x_1)\leq\phi(p,x_1).$$

This implies that $\lim_{n\to\infty}\phi(x_n,x_1)=\phi(p,x_1)$. Hence, we get

$$\|x_n\|\to\|p\|\quad \text{as } n\to\infty.\tag{3.6}$$

From (3.5), (3.6), and the Kadec-Klee property of E, we have

$$\lim_{n\to\infty}x_n=p.\tag{3.7}$$

Therefore,

$$\lim_{n\to\infty}\zeta_n=\lim_{n\to\infty}\nu_n\sup_{q\in\mathcal{F}}\psi\bigl(\phi(q,x_n)\bigr)+\mu_n=0.\tag{3.8}$$

From (3.7), it follows that

$$\lim_{n\to\infty}\|x_n-x_{n+1}\|=0\tag{3.9}$$

and hence

$$\lim_{n\to\infty} \|Jx_n - Jx_{n+1}\| = 0. \tag{3.10}$$

We show that $p \in F(S) \cap F(T) \cap A^{-1}0 \cap EP(f)$.

Now, we show that $p \in EP(f)$. For $x_{n+1} \in C_{n+1} \subset C_n$ and $x_n = \Pi_{C_n}x_1$, it follows that

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n}x_1) \\ &\leq \phi(x_{n+1}, x_1) - \phi(\Pi_{C_n}x_1, x_1) \\ &= \phi(x_{n+1}, x_1) - \phi(x_n, x_1).\end{aligned}$$

Since $\lim_{n\to\infty} \phi(x_n, x_1)$ exists, we have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.11}$$

Since $x_{n+1} \subset C_n$ and the definition of C_{n+1}, we have $\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + \zeta_n$. From (3.11), we also have

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{3.12}$$

From (1.6) and (3.7), it follows that

$$\|y_n\| \to \|p\| \quad \text{as } n \to \infty, \tag{3.13}$$

and hence

$$\|Jy_n\| \to \|Jp\| \quad \text{as } n \to \infty. \tag{3.14}$$

This implies that $\{\|Jy_n\|\}$ is bounded. Note that E is reflexive and E^* is also reflexive, we can assume that $Jy_n \rightharpoonup x^* \in E^*$. Since E is reflexive, we see that $J(E) = E^*$. Hence, there exists $x \in E$ such that $Jx = x^*$ and we have

$$\begin{aligned}\phi(x_{n+1}, y_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|y_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|Jy_n\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above, in view of the weak lower semicontinuity of the norm $\|\cdot\|$, it follows that

$$\begin{aligned}0 &\geq \|p\|^2 - 2\langle p, x^*\rangle + \|x^*\|^2 \\ &= \|p\|^2 - 2\langle p, Jx\rangle + \|Jx\|^2 \\ &= \|p\|^2 - 2\langle p, Jx\rangle + \|x\|^2 \\ &= \phi(p, x).\end{aligned}$$

From Remark 2.2, we have $p = x$, which implies that$Jy_n \rightharpoonup Jp$ as $n \to \infty$. From the Kadec-Klee property of E^*, we obtain that

$$Jy_n \to Jp \quad \text{as } n \to \infty. \tag{3.15}$$

Note that $J^{-1} : E^* \to E$ is demicontinuous, that is, $y_n \rightharpoonup p$ as $n \to \infty$. From the Kadec-Klee property of E, it follows that

$$\lim_{n\to\infty} y_n = p. \tag{3.16}$$

From (3.2), (3.7), and (3.16), it follows that $\lim_{n\to\infty} \phi(q, u_n) = \phi(q, p)$. Since $u_n = K_{r_n}x_n$ and from Lemma 2.8, we have

$$\phi(u_n, x_n) = \phi(K_{r_n}x_n, x_n) \le \phi(q, x_n) - \phi(q, K_{r_n}x_n) = \phi(q, x_n) - \phi(q, z_n) \to 0 \quad \text{as } n \to \infty.$$

From (1.6), it follows that

$$\|u_n\| \to \|p\| \quad \text{as } n \to \infty. \tag{3.17}$$

Since $\{u_n\}$ is bounded and E is also reflexive, we can assume that $u_n \rightharpoonup u \in E$ and we have

$$\phi(u_n, x_n) = \|u_n\|^2 - 2\langle u_n, Jx_n\rangle + \|x_n\|^2.$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above, in view of the weak lower semicontinuity of the norm $\|\cdot\|$, it follows that

$$\begin{aligned} 0 &\ge \|u\|^2 - 2\langle u, Jp\rangle + \|p\|^2 \\ &= \phi(u, p). \end{aligned}$$

From Remark 2.2, we have $u = p$, that is, $u_n \rightharpoonup p$ as $n \to \infty$. From the Kadec-Klee property of E, we obtain that

$$\lim_{n\to\infty} u_n = p. \tag{3.18}$$

Since $\lim_{n\to\infty} u_n = p$ and $\lim_{n\to\infty} x_n = p$, we have that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.19}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \|Ju_n - Jx_n\| = 0.$$

From $r_n > 0$, we have $\frac{\|Ju_n - Jx_n\|}{r_n} \to 0$ as $n \to \infty$ and

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \ge 0, \quad \forall y \in C.$$

By (A2),

$$\begin{aligned}\|y-u_n\|\frac{\|Ju_n-Jx_n\|}{r_n} &\geq \frac{1}{r_n}\langle y-u_n, Ju_n-Jx_n\rangle\\ &\geq -f(u_n,y)\\ &\geq f(y,u_n), \quad \forall y\in C,\end{aligned}$$

and $u_n \to p$, we get $f(y,p) \leq 0$ for all $y \in C$. For $0 < t < 1$, define $y_t = ty + (1-t)p$. Then $y_t \in C$, which implies that $f(y_t,p) \leq 0$. From (A1), we obtain that

$$0 = f(y_t,y_t) \leq tf(y_t,y) + (1-t)f(y_t,p) \leq tf(y_t,y).$$

Thus $f(y_t,y) \geq 0$. From (A3), we have $f(p,y) \geq 0$ for all $y \in C$. Hence, $p \in EP(f)$.

Next, we show that $p \in A^{-1}0$. From (3.2), (3.7), (3.16), and (3.18), it follows that $\lim_{n\to\infty}\phi(q,z_n) = \phi(q,p)$. Since $z_n = J_{r_n}x_n$ and from Lemma 2.10, we have

$$\phi(z_n,x_n) = \phi(J_{r_n}x_n,x_n) \leq \phi(q,x_n) - \phi(q,J_{r_n}x_n) = \phi(q,x_n) - \phi(q,z_n) \to 0 \quad \text{as } n\to\infty.$$

From (1.6), it follows that

$$\|z_n\| \to \|p\| \quad \text{as } n\to\infty. \tag{3.20}$$

Since $\{u_n\}$ is bounded and E is also reflexive, we can assume that $z_n \rightharpoonup z \in E$ and we have

$$\phi(z_n,x_n) = \|z_n\|^2 - 2\langle z_n, Jx_n\rangle + \|x_n\|^2.$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above, in view of the weak lower semicontinuity of the norm $\|\cdot\|$, it follows that

$$\begin{aligned}0 &\geq \|z\|^2 - 2\langle z, Jp\rangle + \|p\|^2\\ &= \phi(z,p).\end{aligned}$$

From Remark 2.2, we have $z = p$, that is, $u_n \rightharpoonup p$ as $n\to\infty$. From the Kadec-Klee property of E, we obtain that

$$\lim_{n\to\infty} z_n = p. \tag{3.21}$$

Since $\lim_{n\to\infty} z_n = p$ and $\lim_{n\to\infty} x_n = p$, we have that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0, \tag{3.22}$$

and hence

$$\lim_{n\to\infty}\|Jx_n - Jz_n\| = 0.$$

From the condition $\{r_n\} \subset [d, \infty)$ for some $d > 0$, we have

$$\lim_{n\to\infty} \frac{1}{r_n} \|Jx_n - Jz_n\| = 0.$$

Thus, since $z_n = J_{r_n} x_n$, we have

$$\lim_{n\to\infty} \|A_{r_n} x_n\| = \lim_{n\to\infty} \frac{1}{r_n} \|Jx_n - Jz_n\| = 0.$$

For any $(w, w^*) \in G(A)$, it follows from the monotonicity of A that $\langle w - z_n, w^* - A_{r_n} x_n \rangle \geq 0$ for all $n \geq 0$. Letting $n \to \infty$, we get $\langle w - p, w^* \rangle \geq 0$. Therefore, since A is maximal monotone, we obtain $p \in A^{-1}0$.

On the other hand, we have

$$\begin{aligned}\phi(q, x_n) - \phi(q, y_n) &= \|x_n\|^2 - \|y_n\|^2 - 2\langle q, Jx_n - Ju_n \rangle \\ &\leq \|x_n - y_n\| \big(\|x_n + y_n\|\big) + 2\|q\| \|Jx_n - Jy_n\|.\end{aligned}$$

In view of $\|x_n - y_n\| \to 0$ and $\|Jx_n - Jy_n\| \to 0$ as $n \to \infty$, we obtain that

$$\phi(q, x_n) - \phi(q, y_n) \to 0 \quad \text{as } n \to \infty. \tag{3.23}$$

From Lemma 2.9, we have

$$\begin{aligned}\phi(q, y_n) &= \phi\big(q, J^{-1}\alpha_n Jx_n + \beta_n JS^n z_n + \gamma_n JT^n u_n\big) \\ &\leq \|q\|^2 - 2\big\langle q, \alpha_n Jx_n + \beta_n JS^n z_n + \gamma_n JT^n u_n \big\rangle + \big\|\alpha_n Jx_n + \beta_n JS^n z_n + \gamma_n JT^n u_n \rangle\big\|^2 \\ &\quad - \alpha_n \beta_n g\big(\|Jx_n - JS^n z_n\|\big) \\ &= \alpha_n \phi(q, x_n) + \beta_n \phi\big(q, S^n z_n\big) + \gamma_n \phi\big(q, T^n u_n\big) - \alpha_n \beta_n g\big(\|Jx_n - JS^n z_n\|\big) \\ &\leq \phi(q, x_n) + \zeta_n - \alpha_n \beta_n g\big(\|Jx_n - JS^n z_n\|\big).\end{aligned} \tag{3.24}$$

It follows from $\liminf_{n\to\infty} \alpha_n \beta_n > 0$, (3.23), (3.8), and the property of g that

$$\lim_{n\to\infty} \big\|Jx_n - JS^n z_n\big\| = 0.$$

Since $x_n \to p$ as $n \to \infty$ and J is uniformly continuous, it yields that $Jx_n \to Jp$, we have

$$JS^n z_n \to Jp. \tag{3.25}$$

Since J^{-1} is demicontinuous, we also have

$$S^n z_n \rightharpoonup p. \tag{3.26}$$

On the other hand, we observe that

$$\big|\|S^n z_n\| - \|p\|\big| = \big|\|J\big(S^n z_n\big)\| - \|Jp\|\big| \leq \big\|J\big(S^n z_n\big) - Jp\big\|,$$

we obtain that $\|S^n z_n\| \to \|p\|$. Since E has the Kadee-Klee property, we get

$$\lim_{n\to\infty} S^n z_n = p. \tag{3.27}$$

By the assumption that S is uniformly L-Lipschitz continuous, we have

$$\begin{aligned}\left\|S^{n+1}z_n - S^n z_n\right\| &\le \left\|S^{n+1}z_n - S^{n+1}z_{n+1}\right\| + \left\|S^{n+1}z_{n+1} - z_{n+1}\right\| + \|z_{n+1} - z_n\| + \left\|z_n - S^n z_n\right\| \\ &\le (L+1)\|z_{n+1} - z_n\| + \left\|S^{n+1}z_{n+1} - z_{n+1}\right\| + \left\|z_n - S^n z_n\right\|. \end{aligned} \tag{3.28}$$

Since $\lim_{n\to\infty} z_n = p$ and $\lim_{n\to\infty} S^n z_n = p$, it yields that $\|S^{n+1}z_n - S^n z_n\| \to 0$, $n \to \infty$. From $S^n z_n \to p$, we get $S^{n+1}z_n \to p$, that is, $SS^n z_n \to p$. In view of the closeness of S, we have $Sp = p$. This implies that $p \in F(S)$. By the same way, we have that $p \in F(T)$.

We show that $p = \Pi_F x_1$. From $x_n = \Pi_{C_n} x_1$, we have $\langle Jx_1 - Jx_n, x_n - v\rangle \ge 0$, $\forall v \in C_n$. Since $F \subset C_n$, we also have

$$\langle Jx_1 - Jx_n, x_n - y\rangle \ge 0, \quad \forall y \in F.$$

By taking limit $n \to \infty$, we obtain that

$$\langle Jx_1 - Jp, p - y\rangle \ge 0, \quad \forall y \in F.$$

By Lemma 2.3, we can conclude that $p = \Pi_F x_1$ and $x_n \to p$ as $n \to \infty$. The proof is completed. □

Let A be a continuous and monotone operator of C into E^*. Then we can find a solution of $VI(A, C)$ in a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property by using the following lemma.

Lemma 3.2 (Zegeye and Shahzad [32]) *Let C be a nonempty closed convex subset of a uniformly smooth strictly convex real Banach space E. Let $A : C \to E^*$ be a continuous monotone mapping. For any $r > 0$, define a mapping $W_r : E \to C$ as follows*:

$$W_r x = \left\{z \in C : \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \forall y \in C\right\}$$

for all $x \in C$. Then the following hold:

(1) *W_r is single-valued*;
(2) $F(W_r) = VI(A, C)$;
(3) *$VI(A, C)$ is a closed and convex subset of C*;
(4) *$\phi(q, W_r x) + \phi(W_r x, x) \le \phi(q, x)$ for all $q \in F(W_r)$.*

Corrollary 3.3 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly uniformly convex Banach space E with the Kadec-Klee property. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions* (A1)-(A4) *and let A be a continuous and monotone operator of C into E^*. Let $S : C \to C$ be a closed and total quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequences ν_n^S, μ_n^S with $\nu_n^S \to 0$, $\mu_n^S \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi^S : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi^S(0) = 0$. Let*

$T : C \to C$ be a closed and total quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequences ν_n^T, μ_n^T with $\nu_n^T \to 0$, $\mu_n^T \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi^T : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi^T(0) = 0$. Assume that S and T are uniformly L-Lipschitz continuous and $F = F(S) \cap F(T) \cap EP(f) \cap VI(A,C) \neq \emptyset$. For an initial point $x_1 \in E$, $C_1 = C$, define the sequence $\{x_n\}$ by

$$\begin{cases} z_n = W_{r_n} x_n, \\ u_n = K_{r_n} x_n, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JS^n z_n + \gamma_n JT^n u_n), \\ C_{n+1} = \{\nu \in C_n : \phi(\nu, y_n) \leq \phi(\nu, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.29}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\{r_n\} \subset [d,\infty)$ for some $d > 0$ $\mu_n = \sup\{\mu_n^S, \mu_n^T\}$, $\nu_n = \sup\{\nu_n^S, \nu_n^T\}$, $\psi = \sup\{\psi^S, \psi^T\}$ for all $n \geq 1$, $\zeta = \nu_n \sup_{q\in\mathcal{F}} \psi(\phi(q, x_n)) + \mu_n$. If $\lim_{n\to\infty} \alpha_n\beta_n = 0$ and $\liminf_{n\to\infty} \alpha_n\gamma_n < 1$, then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Proof From the proof of Theorem 3.1, we known that $\lim_{n\to\infty} z_n = p$ and $\lim_{n\to\infty} x_n = p$. We obtain that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0, \tag{3.30}$$

and hence

$$\lim_{n\to\infty} \|Jx_n - Jz_n\| = 0.$$

Since $\{r_n\} \subset [d,\infty)$ for some $d > 0$, we have

$$\lim_{n\to\infty} \frac{1}{r_n} \|Jx_n - Jz_n\| = 0. \tag{3.31}$$

From the definition of W, it follows that

$$\langle y - z_n, A_n z_n\rangle + \frac{1}{r_n}\langle y - z_n, Jz_n - Jx_n\rangle \geq 0, \quad \forall y \in C.$$

For $0 < t < 1$, define $y_t = ty + (1-t)p$, then $y_t \in C$. We have

$$\langle y_t - z_n, A_n z_n\rangle + \frac{1}{r_n}\langle y_t - z_n, Jz_n - Jx_n\rangle \geq 0, \quad \forall y_t \in C. \tag{3.32}$$

It follows that

$$\begin{aligned} \langle y_t - z_n, A_n y_t\rangle &\geq \langle y_t - z_n, A_n y_t\rangle - \langle y_t - z_n, A_n z_n\rangle + \frac{1}{r_n}\langle y_t - z_n, Jz_n - Jx_n\rangle \\ &= \langle y_t - z_n, A_n y_t - A_n z_n\rangle + \left\langle y_t - z_n, \frac{Jz_n - Jx_n}{r_n}\right\rangle. \end{aligned}$$

Since A_n are continuous and monotone mappings, we have $\langle y_t - z_n, A_n y_t - A_n z_n\rangle \geq 0$. From (3.31) and (3.32), it follows that $\langle y_t - z_n, A_n y_t\rangle \geq 0$. Take the limit as $n \to \infty$ and $z_n \to p$. We get $\langle y_t - p, A_n y_t\rangle \geq 0$ for all $y_t \in C$. Therefore, $\langle y - p, A_n y_t\rangle \geq 0$ for all $y \in C$. If $t \to 0$, we have that $\langle y - p, A_n p\rangle \geq 0$ for all $y \in C$. Hence, $p \in VI(A, C)$. The proof is completed. □

Corrollary 3.4 *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly uniformly convex Banach space E with the Kadec-Klee property. Let A be a continuous and monotone operator of C into E^* and let $B \subset E \times E^*$ be a maximal monotone operator satisfying $D(B) \subset C$ and $J_{r_n} = (J + r_n B)^{-1} J$ for all $r_n > 0$. Let $S : C \to C$ be a closed and total quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequences ν_n^S, μ_n^S with $\nu_n^S \to 0$, $\mu_n^S \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi^S : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi^S(0) = 0$. Let $T : C \to C$ be a closed and total quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequences ν_n^T, μ_n^T with $\nu_n^T \to 0$, $\mu_n^T \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi^T : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi^T(0) = 0$. Assume that S and T are uniformly L-Lipschitz continuous and $F = F(S) \cap F(T) \cap VI(A, C) \cap B^{-1}0 \neq \emptyset$. For an initial point $x_1 \in E$, $C_1 = C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} z_n = J_{r_n} x_n, \\ u_n = W_{r_n} x_n, \\ y_n = J^{-1}(\alpha_n J x_n + \beta_n J S^n z_n + \gamma_n J T^n u_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.33}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\{r_n\} \subset [d, \infty)$ for some $d > 0$ $\mu_n = \sup\{\mu_n^S, \mu_n^T\}$, $\nu_n = \sup\{\nu_n^S, \nu_n^T\}$, $\psi = \sup\{\psi^S, \psi^T\}$ for all $n \geq 1$, $\zeta = \nu_n \sup_{q \in \mathcal{F}} \psi(\phi(q, x_n)) + \mu_n$. If $\lim_{n\to\infty} \alpha_n \beta_n = 0$ and $\liminf_{n\to\infty} \alpha_n \gamma_n < 1$, then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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